Spectral density of a non-linear single-degree-of-freedom system’s response to a white-noise random excitation: A unique case of an exact solution

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Abstract—A SDOF vibro-impact system with a one-sided rigid barrier is considered for the case of a perfectly elastic impact and stationary zero-mean Gaussian white-noise excitation. Special piecewise-linear transformation of the system’s state variables is applied, which effectively reduces the original problem to one without impacts. Moreover, for the special case, where position of the barrier coincides with that of the system’s static equilibrium, the transformed equation of motion is found to be linear, implying Gaussian transition probability densities of the transformed state variables. Thus, the non-linear random vibration problem is reduced to one with “inertialless” non-linearity only. The above transformation is used to obtain an exact explicit expression for the response autocorrelation function, thus leading to quadrature expression for spectral density of the response.

Difficulties encountered in analytical studies of non-linear systems’ response to random excitation are well recognized. Rigorous method for analysis, based on a theory of Markov processes, leads to the well-known Fokker–Planck–Kolmogorov (FPK) partial differential equation for a joint probability density of the state variables of the system, excited by white-noise force(s). Certain analytical solutions of these FPK equations are known for non-linear systems; however, only for unconditional stationary probability densities, which correspond to steady-state responses. On the other hand, if the spectral density of the response is desired, then the transitional probability density should be available within the framework of this approach. Such analytical solutions are not known for higher-than-first-order systems, except for the case where the system is linear and the well-known Gaussian response probability density is obtained.

Thus, attempts have been made to develop various approximate procedures for analysis of the response spectral density (e.g. see a review of these studies in [4]). The simplest well-established procedures for mean-square response analysis, such as those based on statistical linearization and/or certain moment closure schemes, are usually found to be inadequate for predicting spectral density of the response; for example, they cannot describe the effect of broadening of the spectral density peak due to non-linearity in the restoring force of a single-degree-of-freedom system with a white-noise excitation—an effect which is well known to experimenters [3]. More advanced methods, such as those developed in [4, 5], are still approximate, and thus may need some independent confirmation.

The aim of this note is to present an exact solution for the spectral density of the response of a certain specific non-linear single-degree-of-freedom system, excited by a white-noise force. Specifically, a single-mass vibro-impact system with a linear spring and one-sided rigid barrier is considered, so that the motion between impacts is governed by a linear second-order stochastic differential equation, whereas the impact itself is assumed to be perfectly elastic. Using a certain piecewise-linear transformation of the response variable this system, with impact-type non-linearity (velocity jump), is reduced to a system with “regular” (piecewise-type) non-linearity in the restoring force [3]. Moreover, in a special case where the original system’s equilibrium position is exactly at the barrier, the transformed system is found to be perfectly linear. Its state variable is then Gaussian, and an exact solution for the autocorrelation function of this transformed variable can be easily...
found in a closed form. From these data the autocorrelation function of the original state variable is obtained easily, since this non-linear transformation is “inertialless”. Numerical results for the response spectral density are presented, which illustrate peaks at both main resonant peak and its harmonics. All peaks are shown to have the same relative bandwidth, which is quite natural in view of the fact that the original system’s natural period does not depend on the amplitude of oscillations.

It should be admitted that the system—which permits the exact solution for the response spectral density—may be considered as a rather special one, so that the results obtained may be regarded as being important mostly for comparison purposes, for checking various approximate methods. However, there also exists a certain class of systems which may satisfy the condition of equilibrium position being situated at the barrier: moored bodies with inextensible mooring cables. For such systems, any slack of the cable may not be permitted, whereas initial preload may be found to be small compared with the level of dynamic loads during motions excited by ocean waves.

Thus, consider the second-order equation

\[ m(t) + 2\alpha y(t) + \Omega^2 y(t) = \zeta(t), \quad y > 0, \]  

which governs the system’s motion between impacts at the barrier at \( y = 0 \); here \( \zeta(t) \) is a zero-mean Gaussian “physical” (in Stratonovich sense) white noise with intensity \( D \). The impact condition at \( y = 0 \) may be written as

\[ \dot{y}_+ = -\dot{y}_-, \quad y(t_\pm) = 0, \]

where subscripts plus and minus denote values of velocity immediately after and before the impact, respectively, whereas time instant \( t_\pm \) of the impact is defined by the condition \( y(t_\pm) = 0 \).

We introduce now in (1) the following basic change of state variable [3]

\[ y = |x| \]

supplemented with a condition

\[ \dot{y} = \dot{x} \text{sgn}(x), \quad \text{sgn}(x) = \begin{cases} +1, & x > 0; \\ 0, & x = 0; \\ -1, & x < 0. \end{cases} \]

Thus, the motion is mapped on to the whole phase plane \((x, \dot{x})\). Moreover, it is seen from (3) and (4) that the impact condition (2) is reduced to the condition \( \dot{x}_+ = \dot{x}_- \) at \( x = 0 \) for the new state variable \( x(t) \), so that the latter has continuous time derivative at the instant of the impact.

The above change of variables (3) and (4) is used now in equation (1) of motion between the impacts. Since \( (d/dx)(\text{sgn} x) = 0 \) for \( x \neq 0 \), we have \( \dot{y} = \dot{x} \text{sgn} x \) for \( x \neq 0 \). Then, multiplying the resulting equation for \( x(t) \) by \( \text{sgn} x \) and using the identity \( |x| = x \text{sgn}(x) \) yields the following stochastic equation for the new state variable:

\[ \ddot{x}(t) + 2\alpha \dot{x}(t) + \Omega^2 x(t) = \text{sgn}(x)\zeta(t). \]

Thus, the equation of motion itself is found to be linear, though it still contains a non-linearity in the right-hand side due to the \( \text{sgn} x \) factor. This factor indeed presents a major complication in the case of the excitation with a finite correlation time. However, in the case of a zero-mean Gaussian white noise \( \zeta(t) \) this right-hand side will also be a white noise with the same intensity factor \( D \), which is obvious from the expression for its mean square integral, and it is also Gaussian according to Levy’s characterization theorem [6]. Gaussian probability density (p.d.f) of \( x(t) \) can also be established from the solutions to the FPK equation, as obtained separately for \( x > 0 \) and \( x < 0 \). These solutions match each other along \( x = 0 \), both in p.d.f itself and in “probability flow”. Thus, the original non-linear problem is reduced to a “common” linear random vibration problem, its steady-state stationary solution for the autocorrelation function \( K_{xx}(r) \) of the (transformed) response
variable $x(t)$ being presented in numerous textbooks (e.g. [3]):

$$K_{xx}(t) = \sigma^2 R_{xx}(t)$$
$$R_{xx}(t) = e^{-\sigma^2 |t|} \left( \cos \omega_d t + \frac{\alpha}{\omega_d} \sin \omega_d |t| \right)$$
$$\omega_d = \sqrt{\Omega^2 - \sigma^2}$$
$$\sigma^2 = \frac{D}{4\pi}$$

Here $\sigma^2$ is clearly seen to be the r.m.s. value of the zero-mean Gaussian process $x(t)$. We may now return to the original response variable $y(t) = |x(t)|$ using normal joint probability density $p(x, x_t)$ of steady-state values $x(t)$ and $x(t + \tau)$:

$$p(x, x_t) = \frac{1}{2\pi\sigma^2 \sqrt{1 - R_{xx}^2}} e^{\frac{1}{2\sigma^2} \left(1 - R_{xx}^2\right) \left(x^2 - 2R_{xx} x x_t + x_t^2\right)}.$$  (7)

The autocorrelation function $K_{yy}(t)$ of the original response variable $y(t)$ is obtained now as

$$K_{yy}(t) = \frac{2\sigma^2}{\pi} \left[ R_{xx}(t) \arcsin R_{xx}(t) + \sqrt{1 - R_{xx}^2} \right].$$  (8)

It is seen that $K_{yy}(t)$ does not approach zero as $t$ approaches $\infty$. In fact, it approaches a limiting value $K_{yy}(\infty) = 2\sigma^2/\pi = m_y^2$, where $m_y$ is the expected value of $y(t)$. Deducting this value from $K_{yy}(t)$, we obtain the autocorrelation function $K_{yy}(t)$ of the zero-mean part of the response, and Fourier transform of the latter yields the desired power spectral density of the response:

$$\Phi_{yy}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[ K_{yy}(t) - K_{yy}(\infty) \right] e^{-i\omega t} dt$$
$$= \frac{2\sigma^2}{\pi^2} \int_{0}^{\infty} \left[ R_{xx}(t) \arcsin K_{xx}(t) + \sqrt{1 - R_{xx}^2(t)} - 1 \right] \cos \omega t dt.$$  (9)

Fig. 1. The response spectral density $\Phi_{yy}(\omega)$ for two different values of damping ratio $\omega_d$. 
Figure 1 shows graphs of the response spectral density $\Phi_{y,Y}(\omega)$ for two different values of damping ratio $\alpha/\Omega$, as obtained by numerical integration according to (9) with $R_{xx}(\tau)$ as defined by equation (6). Whilst two peaks only are seen here, at $\omega = 0$ and at the doubled natural frequency $\omega = 2\Omega$, higher-order peaks were obtained also at even integer multiples of the natural frequency $4\Omega, 6\Omega$, etc.; all these progressively reduced peaks, however, were found to be too small for reproduction in Fig. 1. Of course, the spectral density of the velocity response $\omega^2\Phi_{x,x}(\omega)$ has more pronounced peaks at the higher harmonics of $\Omega$. Expansion of expression (8) for $K_{xx}(\tau)$ in a power series of $R_{xx}(\tau)$ shows that in view of equation (6), all peaks of the response spectral density have the same relative bandwidth $\alpha/\Omega$. This may be explained by the fact that the natural period of the specific non-linear system considered does not depend on the amplitude (or energy) of motion.

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