Quadratic Multi-Dimensional Signaling Games and Affine Equilibria

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Abstract

This paper studies the decentralized signaling problem when the encoder and the decoder, viewed as two decision makers, have misaligned objective functions. In particular, the study investigates extensions of the quadratic cheap talk and signaling game problem, which has been introduced in the economics literature. Two main contributions of this study are the extension of Crawford and Sobel’s cheap talk formulation to multi-dimensional sources, and the extension to noisy channel setups as a signaling game problem. We show that, in the presence of misalignment, the quantized nature of all equilibrium policies holds for any scalar random source. It is shown that for multi-dimensional setups, unlike the scalar case, equilibrium policies may be of non-quantized nature, and even linear. In the noisy setup, a Gaussian source is to be transmitted over an additive Gaussian channel. The goals of the encoder and the decoder are misaligned by a bias term and encoder’s cost also includes a power term scaled by a multiplier. Conditions for the existence of affine equilibrium policies as well as general informative equilibria are presented for both the scalar and multi-dimensional setups. Our findings provide further conditions on when affine policies may be optimal in decentralized multi-criteria control problems and lead to conditions for the presence of active information transmission in strategic environments.

Index Terms

Information theory, game theory, signaling games, cheap talk, quantization.

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I. INTRODUCTION

Team theory is concerned with the interaction dynamics among decentralized decision makers with identical objective functions. On the other hand, game theory deals with setups with misaligned objective functions. Each player chooses a strategy from its own set to maximize its own utility which is determined by the joint strategies chosen by all players.

Information transmission in team problems is well-understood. Despite the difficulty to obtain solutions under general information structures, it is evident in team problems that more information provided to any of the decision makers does not hurt the system performance. However, for game problems, informational aspects are very challenging to address; more information can hurt some or even all of the players in a system, see e.g. [1]. On the other hand, for zero-sum games, as in teams, more information to a player has a positive effect on payoffs; further intricacies on informational aspects in competitive setups have been discussed in [2] and [3].

Signaling games and cheap talk are concerned with a class of Bayesian games where an informed decision maker transmits information to another decision maker and the information transmission policy can be viewed as the action of this decision maker. Unlike a team setup, however, the goals of the agents are misaligned. Beside the differences between the goals, there may be differences between the information sets of the agents. Signaling game and its specific case cheap talk can be viewed in the classical communications theory setup with the encoder as the sender and the decoder as the receiver taking part in a team decision problem where the encoder and the decoder have aligned objectives. In this case, the fields of information theory and estimation theory have studied such problems extensively. Namely, the sender and receiver reach the most informative equilibrium.

Such problems find applications in network control systems when a communication channel/network is present among competitive and non-cooperative decision makers [4]. One may consider a utility company which wishes to inform users regarding pricing information; if the utility company and the users engage in selfish behaviour, it may be beneficial for the utility company to hide information and the users to be strategic about how they interpret the given information. Another application area is smart-grids; there may be corrupted sensors in the system [5] and goals of the sensors in the system may become misaligned. Furthermore, the security of the smart-grid infrastructure can be analyzed by considering the adversarial nature of the interaction between an attacker and a defender [6], [7], and a game theoretic setup would be
appropriate to analyze such interactions. One further area of application is recommender systems (as in rating agencies) [8].

Identifying when optimal policies are linear or affine for decentralized systems involving Gaussian variables under quadratic criteria is a recurring problem in control theory, starting perhaps from the seminal work of Witsenhausen [9], where suboptimality of linear policies for such problems under non-classical information structures is presented. The reader is referred to Chapters 3 and 11 of [10] for a detailed discussion on when affine policies are and are not optimal. These include the problem of communicating a Gaussian source over a Gaussian channel, variations of Witsenhausen’s counterexample [11]; and game theoretic variations of such problems. For example if the noise variable is viewed as the maximizer and the encoders/decoders (or the controllers) act as the minimizer, then affine policies may be optimal for a class of settings, see [12], [13], [14], [15] and [16]. [16] also provides a review on LQG problems under nonclassical information including Witsenhausen’s counterexample. Our study provides further conditions on when affine policies may constitute equilibria for such decentralized quadratic Gaussian optimization problems.

This study investigates extensions of the quadratic cheap talk and signaling game problem studied by Crawford and Sobel [17] in the economics literature. In this literature; there have been a number of related contributions, which we briefly review in the following: In [18], Krishna and Morgan study the setup in [17] with two senders and show that if senders send the messages sequentially once, then the equilibrium is always quantized and if senders send the messages simultaneously and their biases are either both positive or both negative, then a fully revealed equilibrium is possible. In [19], Shintaro studies an unbounded state space setup and shows that if senders send the messages sequentially and their biases have opposite signs, then a fully revealed equilibrium is possible. Shintaro also works on unbounded two-dimensional real space and shows that a fully revealed equilibrium occurs if and only if the experts have perfectly opposing biases. Moreover, multi-dimensional cheap talk with multiple senders is analyzed in [20] and [21] with unbounded and restricted state spaces, respectively. Levy and Razin [22] study multi-dimensional cheap talk with single and multiple senders where the senders have lexicographic preferences over the receiver’s actions. In [23], multi-dimensional cheap talk is adjusted to the valuation and recommendation games over preferences. Andreas et.al. [24] add uniform noise between the sender and receiver, and show that there may be infinitely many actions (countable or uncountable) induced in equilibrium even though all equilibria are interval
partitions in the noiseless case \cite{17}. Furthermore, there are some contributions which modify
the information structure in Crawford and Sobel’s game setup. In \cite{25}, the sender knows that
the receiver has partial information about his/her private information; whereas the sender doesn’t
know this in \cite{26}, \cite{27}. Golosov et al. \cite{28} study Crawford and Sobel’s game setup in a finite
horizon environment where, in each period, a privately informed sender sends a message and a
receiver takes an action.

In the control community, recently, there have been few studies: \cite{29} considered a Gaussian
cheap talk game with quadratic cost functions where the analysis considers Stackelberg equilibria,
for a class of single- and multi-terminal setups and where linear equilibria have been studied. For
the setup of Crawford and Sobel but when the source admits an exponentially distributed real
random variable, \cite{30} establishes the discrete-nature of equilibria, and obtains the equilibrium
bins with \textit{finite upper bounds on the number of bins} under any equilibrium in addition to some
structural results on informative equilibria for general sources.

A. Contributions in this Study

The main contributions of this study are the extension of Crawford and Sobel’s cheap talk
formulation to multi-dimensional sources, and the extension to noisy channel setups (as a
signaling game) where a Gaussian source and a Gaussian channel are assumed. We show that
for multi-dimensional setups, unlike the scalar case, equilibrium policies may be non-quantized
and even linear. In the noisy setup, a Gaussian source is to be transmitted over an additive
Gaussian channel. The goals of the encoder and the decoder are misaligned by a bias term and
encoder’s cost also includes a power term scaled by a multiplier. Conditions for the existence
of affine equilibrium policies as well as general informative equilibria are presented for both
the scalar and multi-dimensional setups. We compare the results with socially optimal costs and
information theoretic lower bounds, and discuss the effects of the bias term.

II. Problem Definition

Let there be two decision makers (DMs): An encoder (DM 1) and a decoder (DM 2). DM
1 wishes to encode the $\mathbb{M}$-valued random variable $M$ to DM 2. Let $X$ denote the $\mathbb{X}$-valued
random variable which is transmitted to DM 2. DM 2, upon receiving $X$, generates its optimal
decision $U$ which we also take to be $\mathbb{M}$-valued. We allow for randomized decisions, therefore,
we let the policy space of DM 1 be the set of all stochastic kernels from \( M \) to \( X \). Let \( \Gamma^e \) denote the set of all such policies. We let the policy space of DM 2 be the set of all stochastic kernels from \( X \) to \( U \). Let \( \Gamma^d \) denote the set of all such stochastic kernels.

Given \( \gamma^e \in \Gamma^e \) and \( \gamma^d \in \Gamma^d \), the goal in the classical communications theory is to minimize the expectation

\[
J(\gamma^e, \gamma^d) = \int c(m, u)\gamma^e(dx|m)\gamma^d(du|x)P(dm),
\]

where \( c \) is some cost function. Typical applications have \( c(m, u) = |m - u|^2 \). It is well-known that for such problems the extreme solutions are the most informative ones: Optimal encoders and decoders are deterministic with as high information rate as possible.

Recall that a collection of decision makers who have an agreement on the probabilistic description of a system and a cost function to be minimized, but who may have different on-line information is said to be a team (see, e.g. [10]). Hence, the classical communications setup may be viewed as a team of an encoder and a decoder.

In many applications (in networked systems, recommendation systems, and applications in economics) the objectives of the encoder and the decoder may not be aligned. For example, DM 1 may aim to minimize

\[
J^e(\gamma^e, \gamma^d) = \int c^e(m, u)\gamma^e(dx|m)\gamma^d(du|x)P(dm), \tag{1}
\]

whereas DM 2 may aim to minimize

\[
J^d(\gamma^e, \gamma^d) = \int c^d(m, u)\gamma^e(dx|m)\gamma^d(du|x)P(dm). \tag{2}
\]

Such a problem is known in the economics literature as cheap talk. A more general formulation would be the case when the transmitted signal is also an explicit part of the cost function \( c^e \) or \( c^d \); in that case, the setup is called a signaling game. We will consider a noisy communication setup, where the problem may be viewed as a signaling game, rather than cheap talk, later in this study.

Since the goals are not aligned, such a problem is studied under the tools and concepts provided by game theory. A pair of policies \( \gamma^e, \gamma^d \) is said to be a Nash equilibrium if

\[
J^e(\gamma^e, \gamma^d) \leq J^e(\gamma^*, \gamma^d) \quad \forall \gamma^e \in \Gamma^e \quad \text{and} \quad J^d(\gamma^e, \gamma^d) \leq J^d(\gamma^e, \gamma^d) \quad \forall \gamma^d \in \Gamma^d. \tag{3}
\]

1Recall that \( P \) is a stochastic kernel from \( M \) to \( X \) if \( P(\cdot|X = x) \) is a probability measure on \( B(X) \) for every \( x \in M \) and for every Borel \( A \), \( P(A|X = x) \) is a measurable function of \( x \).
We note that when $c^e = c^d$ the setup is a traditional communication theoretic setup. If $c^e = -c^d$, that is, if the setup is a zero-sum game, then an equilibrium is achieved when $\gamma^e$ is non-informative (e.g., a kernel with actions statistically independent of the source) and $\gamma^d$ uses only the prior information (since the received information is non-informative). We call such an equilibrium a non-informative equilibrium. The following can be shown:

**Proposition 2.1:** A non-informative equilibrium always exists for the cheap talk game.

Crawford and Sobel [17] have made foundational contributions to the study of cheap talk with misaligned objectives where the cost functions $c^e$ and $c^d$ satisfy certain monotonicity and differentiability properties but there is a bias term in the cost functions for a uniform source. Their result is that the number of bins in an equilibrium is upper bounded by a function which is negatively correlated to the bias.

We will first consider the scalar setting by taking the cost functions as

$$c^e(m,u) = (m - u - b)^2, \quad c^d(m,u) = (m - u)^2$$

where $b$ denotes the bias term. The motivation for such functions stems from the fields of information theory, communication theory and LQG control. Recall that for the $b = 0$ case, the cost functions simply reduce to those for a minimum mean-square estimation (MMSE) problem.

The formulation considered in this study focuses on setups where there is not necessarily a commitment: If the encoder and the decoder take part in a repeated game and are committed to their announced policies, they should play according to a policy that minimizes the total cost:

$$J^e(\gamma^e, \gamma^d) + J^d(\gamma^e, \gamma^d);$$

which results in $2 \inf_{\gamma^e, \gamma^d} E[(m - u - \frac{b}{2})^2] + \frac{b^2}{2}$. However, a lack of commitment may lead one of the players to deviate from their strategy and pick another function, hence the need for a game theoretic solution arises.

Among the equilibria, one particular interest is on the most informative ones: The (socially optimal) equilibrium with the smallest $J^e(\gamma^{*,e}, \gamma^{*,d}) + J^d(\gamma^{*,e}, \gamma^{*,d})$. As is common in game theoretic problems, a socially optimal solution which minimizes the sum $J^e(\gamma^e, \gamma^d) + J^d(\gamma^e, \gamma^d)$ is typically not achieved in any equilibrium, as we will observe later in the manuscript.

### III. Scalar Setup

As before, let the cost functions be defined as $c^d(m,u) = (m - u)^2$ and $c^e(m,u) = (m - u - b)^2$ where $b$ is the bias term. Some existence and deterministic properties of the equilibrium policies of the encoder and the decoder are stated in [30] and [10, Chp.4].
Theorem 3.1: \[30\] (i) For any $\gamma^e$, there exists an optimal $\gamma^d$, which is deterministic. (ii) For any $\gamma^d$, any randomized encoding policy can be replaced with a deterministic $\gamma^e$ without any loss to DM 1. (iii) Suppose $\gamma^e$ is an $M$-cell quantizer, then there exists an optimal $\gamma^d$, which is the conditional expectation of the respective bin.

The following builds on [17, Lem.1], which considers uniform scalar sources. We note that the analysis here applies to arbitrary scalar valued random variables.

Theorem 3.2: An equilibrium policy has to be quantized (or is equivalent to a quantized policy) for the encoder cost function $c^e(m,u) = (m - u - b)^2$ and the decoder cost function $c^d(m,u) = (m - u)^2$ where $m$ is any scalar random source and $b \neq 0$.

Proof: Let there be an equilibrium in the game (with possibly uncountably infinitely many bins, countably many bins or finitely many bins). Let two bins be $B^\alpha$ and $B^\beta$. Also let $m^\alpha$ indicate any point in $B^\alpha$; i.e., $m^\alpha \in B^\alpha$. Similarly, let $m^\beta$ indicate any point in $B^\beta$; i.e., $m^\beta \in B^\beta$. The decoder chooses the action $u^\alpha = E[m|m \in B^\alpha]$ when the encoder sends $m^\alpha \in B^\alpha$ and action $u^\beta = E[m|m \in B^\beta]$ when the encoder sends $m^\beta \in B^\beta$ in order to minimize its total cost. Without loss of generality, we can assume that $m^\alpha < m^\beta$, hence $u^\alpha < u^\beta$. Because of the equilibrium definitions from the view of the encoder; $(m^\alpha - u^\alpha - b)^2 < (m^\alpha - u^\beta - b)^2$ and $(m^\beta - u^\beta - b)^2 < (m^\beta - u^\alpha - b)^2$. Hence, $\exists \bar{m}$ that satisfies $(\bar{m} - u^\alpha - b)^2 = (\bar{m} - u^\beta - b)^2$ which reduces to $\bar{m} = \frac{u^\alpha + u^\beta}{2} + b$. Recall that $m^\alpha < \bar{m} < m^\beta$ due to the continuity of the cost functions and this implies $u^\alpha < \bar{m} < u^\beta$. It then follows that $\bar{m} = \frac{u^\alpha + u^\beta}{2} + b \Rightarrow (\bar{m} - u^\alpha) = (u^\beta - \bar{m}) + 2b$ and $u^\beta - u^\alpha = (u^\beta - \bar{m}) + (\bar{m} - u^\alpha)$. These imply that $2(u^\beta - \bar{m}) + 2b > 2b$ and that $2(\bar{m} - u^\alpha) - 2b > -2b$ and finally that $u^\beta - u^\alpha > 2|b|$. Thus, there must be at least $2|b|$ distance between the equilibrium points (decoder’s actions, centroids of the bins), which guarantees that the equilibrium policy must be discrete.

Recall again that for the case when the source is uniform, Crawford and Sobel established the discrete nature of the equilibrium policies. For the case when the source is exponential, \[30\] established the discrete-nature, and obtained the equilibrium bins with finite upper bounds on the number of bins in any equilibrium. For the Gaussian source case, obtaining an analytical solution appears to be difficult due to the complicated integrations involved, but the quantized nature can easily be established: The Gaussian case is important because if the costs are aligned, optimal encoder and decoder policies are always linear in a team-theoretic setup. When $b$ is
non-zero, this may not be true. We will revisit this topic later.

The theorem below is due to [17] and is valid for $m \sim U(0, 1)$.

**Theorem 3.3:** [17]

1) In order to achieve an equilibrium in the case of two levels of quantization, $|b|$ must be upper bounded by $1/4$. Otherwise there cannot be more than one quantization level, which corresponds to a completely non-informative equilibrium.

2) The relation between the number of bins $N$ and bias $b$ in the 1-dimensional case in the equilibrium can be characterized by $|b| < \frac{1}{2N(N-1)}$.

3) The socially optimal cost for any number of bins $N$ is given by $J_c(\gamma^*, e, \gamma^*, d) + J_d(\gamma^*, e, \gamma^*, d) = \left(\frac{1}{12N^2} + \frac{b^2(N^2-1)}{3} + b^2\right) + \left(\frac{1}{12N^2} + \frac{b^2(N^2-1)}{3}\right)$.

4) The most informative equilibrium is reached with the maximum possible number of bins; i.e., if there are two different equilibria with $M$ and $N$ bins for a constant $b$ where $N > M$, the equilibrium with $N$ bins is more informative.

**IV. Multi-Dimensional Cheap Talk**

Let the source be uniform on $[0, 1] \times [0, 1]$ and the cost function of the encoder be defined by $c^e(\vec{m}, \vec{u}) = \|\vec{m} - \vec{u} - \vec{b}\|^2$ where the lengths of the vectors are defined in $L_2$ norm.

**Theorem 4.1:** An equilibrium policy, unlike the scalar case, can be non-discrete and in fact linear.

**Proof:** It suffices to provide an example. Consider $\vec{b} = [0.2 \ 0]$. Then, as a limit case of the equilibrium in Fig. 2a, the following encoder and decoder policies form an equilibrium:

$$
\gamma^e(m_1, m_2) = (x_1, x_2) =
\begin{cases}
(C^1, m_2) & \text{if } m_1 \in [0.0, 0.9] \\
(C^2, m_2) & \text{if } m_1 \in (0.9, 1.0)
\end{cases}
$$

$$
\gamma^d(x_1, x_2) = (u_1, u_2) =
\begin{cases}
(0.45, m_2) & \text{if } x_1 = C^1 \\
(0.95, m_2) & \text{if } x_1 = C^2
\end{cases}
$$

where $C^1$ and $C^2$ are any two constants. Here, the scalar setup is applied on the $x$-dimension with two quantization bins (recall that $u_1 = E[m_1|x_1 = C^1]$ or $u_1 = E[m_1|x_1 = C^2]$), and a fully-informative equilibrium exists on the $y$-dimension since there is no bias on that dimension. It is observed that the encoder and decoder have linear policies due to the unbiased property of the $y$-dimension.
In the linear equilibrium, the equilibria lead to convex bins (codecells) which are characterized by a set of hyperplanes parallel to the direction of $\vec{b}$. However, the linearity is a rare occurrence for an equilibrium. Suppose that in the equilibrium two bins $R_1$ and $R_2$ are given such that
\[
\bar{m} \in R_1 \Rightarrow \| \bar{m} - \bar{u}^1 - \vec{b} \|^2 < \| \bar{m} - \bar{u}^2 - \vec{b} \|^2
\]
\[
\bar{m} \in R_2 \Rightarrow \| \bar{m} - \bar{u}^2 - \vec{b} \|^2 < \| \bar{m} - \bar{u}^1 - \vec{b} \|^2
\]
where $\bar{u}^1 = E [\bar{m} | \bar{m} \in R_1]$ and $\bar{u}^2 = E [\bar{m} | \bar{m} \in R_2]$. Then, $\exists$ a boundary line such that
\[
\| \bar{z} - \bar{u}^1 - \vec{b} \|^2 = \| \bar{z} - \bar{u}^2 - \vec{b} \|^2
\]
\[
\Rightarrow (z_2 - b_2) - \frac{u_2^1 + u_2^2}{2} = \frac{u_2^1 - u_2^1}{u_2^1 - u_2^2} \left( (z_1 - b_1) - \frac{u_1^1 + u_2^2}{2} \right)
\]
(4)
The boundary line in (4) is a shifted version of the perpendicular bisector of $\bar{u}^1$ and $\bar{u}^2$ and the shift amount is equal to $\vec{b}$. As can be deduced from the pictorial analysis in Fig. 1 for a linear equilibrium certain symmetry conditions need to hold (in the case depicted, the boundary line and the perpendicular bisector must be aligned). Thus, the presence of linear equilibria is indeed a rare occurrence.

Fig. 1: Suppose point A is $(0, 0)$ and AF and AG are the medians of the triangles ADE and ABC, respectively. The centroids of the regions ADE and DBCE lie on AG. Bias vector $\vec{b}$ must be parallel to the perpendicular bisector of the centroids of the corresponding regions and the boundary line between the corresponding regions. This implies $\vec{b}$ must be in $[-1 1]$ direction for the linear equilibrium; otherwise the linear equilibrium cannot be achieved.

Besides linear equilibria, there may be multiple (hence, non-unique) quantized equilibria with finite regions in the multi-dimensional case as illustrated in Fig. 2b.
(a) There are 2 quantization levels on the $x$-dimension and 200 quantization levels on the $y$-dimension. The number of quantization levels on the $y$-dimension can be arbitrarily chosen (since $\vec{b}$ is orthogonal to that dimension). As the number of levels goes to infinity, this construction converges to the structure of a linear equilibrium.

(b) Sample finite equilibria in 2D with $\vec{b}_x = 0.1$ and $\vec{b}_y = 0.2$ where the crosses indicate the centroids of the bins, the star indicates the middle point and the square indicates the shifted middle point.

Fig. 2: Sample equilibria in 2D

**Remark 4.1:** From the discussion above, it can be deduced that if $\vec{b}$ is orthogonal to the basis vectors or satisfies certain symmetry conditions as in Fig. 1, then linear equilibria exist. This approach applies also to an $n$-dimensional setup for any $n \in \mathbb{N}$.

V. Quadratic Signaling Game: Scalar Case

The noisy game setup is similar to the noiseless case except that there exists an additive Gaussian noise channel between the encoder and decoder, as shown in Fig. 3 and the encoder has a soft power constraint.

![General system model](image)

Fig. 3: General system model.

The encoder (DM 1) encodes a zero-mean Gaussian random variable $M$ and sends the real-valued random variable $X$. During the transmission, the zero mean Gaussian noise with a variance
of $\sigma^2$ is added to $X$; hence, the decoder (DM 2) receives $Y = X + W$. The policy space of DM 1, $\Gamma^e$, is similarly defined as the policy space in the noiseless case: the set of stochastic kernels from $\mathbb{R}$ to $\mathbb{R}$ (this can be viewed as the measurable subset of the space of all product measures involving $M, X$ with a fixed input marginal, under the weak topology). The policy space of DM 2, $\Gamma^d$, is the set of stochastic kernels from $Y$ to $U$. The cost functions of the encoder and the decoder are also slightly modified as follows: DM 1 aims to minimize

$$J^e(\gamma^e, \gamma^d) = \int c^e(m, u)\gamma^e(dx|m)\gamma^d(dy|u)P(dy|x)P(dm),$$

whereas DM 2 aims to minimize

$$J^d(\gamma^e, \gamma^d) = \int c^d(m, u)\gamma^e(dx|m)\gamma^d(dy|u)P(dy|x)P(dm),$$

where $P(dy|x) = P(W \in dy - x)$ with $W \sim \mathcal{N}(0, \sigma^2)$. The cost functions are modified as

$$c^e(m, x, u) = (m - u - b)^2 + \lambda x^2 \quad \text{and} \quad c^d(m, u) = (m - u)^2$$

Note that a power constraint with an associated multiplier is appended to the cost function of the encoder, which corresponds to power limitation for transmitters in practice. If $\lambda = 0$, this corresponds to the setup with no power constraint at the encoder.

A. A Supporting Result

Suppose that there is an equilibrium with an arbitrary policy leading to finite (at least two), countably infinite or uncountably infinite equilibrium bins. Let two of these bins be $B^\alpha$ and $B^\beta$. Also let $m^\alpha$ indicate any point in $B^\alpha$; i.e., $m^\alpha \in B^\alpha$; and the encoder encodes $m^\alpha$ to $x^\alpha$ and sends to the decoder. Similarly, let $m^\beta$ indicate any point in $B^\beta$; i.e., $m^\beta \in B^\beta$; and the encoder encodes $m^\beta$ to $x^\beta$ and sends to the decoder. Without any loss of generality, we can assume that $m^\alpha < m^\beta$. The decoder chooses the action $u = E[m|y]$ (MMSE rule). Let $F(m, x)$ be the encoder cost when message $m$ is encoded as $x$; i.e.,

$$F(m, x) = \int_y p(y|\gamma^d(y) = u|\gamma^e(m) = x) \left((m - u - b)^2 + \lambda x^2\right)dy$$

Then the equilibrium definitions from the view of the encoder requires $F(m^\alpha, x^\alpha) \leq F(m^\alpha, x^\beta)$ and $F(m^\beta, x^\beta) \leq F(m^\beta, x^\alpha)$. Now let $G(m) = F(m, x^\alpha) - F(m, x^\beta)$. If it can be shown that $G(m)$ is a continuous function of $m$ on the interval $[m^\alpha, m^\beta]$, then it can be deduced that $\exists \bar{m} \in [m^\alpha, m^\beta]$ such that $G(\bar{m}) = 0$ by the Mean Value Theorem since $G(m^\alpha) \leq 0$ and $G(m^\beta) \geq 0$. 

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Proposition 5.1: \( G(m) \) is a continuous function of \( m \) on the interval \([m^\alpha, m^\beta]\).

**Proof:** It suffices to show that \( F(m, x) \) is continuous in \( m \). Let \( \{m_n\} \) be a sequence which converges to \( m \). Recall that \((m_n - u - b)^2 \leq 2m_n^2 + 2(u + b)^2 < \infty \) since \( m \) is bounded from above and below \((m \in [m^\alpha, m^\beta])\), \( b \) is a finite bias and \( E[u^2] = E[(\gamma^d(y))^2] < \infty \). Then, by the dominated convergence theorem,

\[
\lim_{n \to \infty} F(m_n, x) = \lim_{n \to \infty} \int_y p(\gamma^d(y) = x|\gamma^e(m_n) = x) \left((m_n - u - b)^2 + \lambda x^2\right) dy
\]

\[= \int_y p(\gamma^d(y) = x|\gamma^e(m) = x) \left((m - u - b)^2 + \lambda x^2\right) dy = F(m, x)\]

which shows the continuity of \( F(\cdot, x) \) in the interval \((m^\alpha, m^\beta)\).

From Proposition 5.1, \( \exists \overline{m} \in [m^\alpha, m^\beta] \) such that \( G(\overline{m}) = 0 \) which implies \( F(\overline{m}, x^\alpha) = F(\overline{m}, x^\beta) \). Then

\[
\int_y p(\gamma^d(y) = x|\gamma^e(\overline{m}) = x^\alpha) ((\overline{m} - u - b)^2 + \lambda(x^\alpha)^2) dy
\]

\[= \int_y p(\gamma^d(y) = x|\gamma^e(\overline{m}) = x^\beta) ((\overline{m} - u - b)^2 + \lambda(x^\beta)^2) dy.
\]

Also recall that

\[
\int_y p(\gamma^d(y) = x|\gamma^e(\overline{m}) = x^\alpha) u dy = E\left[\gamma^d(y)\mid x^\alpha\right],
\]

\[
\int_y p(\gamma^d(y) = x|\gamma^e(\overline{m}) = x^\beta) u dy = E\left[\gamma^d(y)\mid x^\beta\right],
\]

\[
\int_y p(\gamma^d(y) = x|\gamma^e(\overline{m}) = x^\alpha) u^2 dy = E\left[(\gamma^d(y))^2\mid x^\alpha\right],
\]

\[
\int_y p(\gamma^d(y) = x|\gamma^e(\overline{m}) = x^\beta) u^2 dy = E\left[(\gamma^d(y))^2\mid x^\beta\right].
\]

As a result

\[
\overline{m} = \frac{E[(\gamma^d(y))^2\mid x^\beta] - E[(\gamma^d(y))^2\mid x^\alpha] + \lambda ((x^\beta)^2 - (x^\alpha)^2)}{2(E[\gamma^d(y)\mid x^\beta] - E[\gamma^d(y)\mid x^\alpha])} + b \tag{6}
\]

Recall that the arguments in Theorem 3.2 cannot be applied here because of the presence of noise. However, when there is noise in a communication channel, the relation between \( E[u\mid x] \), \( E[u^2\mid x] \) and \( \overline{m} \) can be constructed as in (6).

B. Existence and Uniqueness of Informative Equilibria and Affine Equilibria

We first note that Proposition 2.1 is valid also in the noisy formulation; i.e. a non-informative equilibrium is an equilibrium for the noisy signaling game, since the appended power constraint is always positive. The following holds:

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Theorem 5.1:

1) Let \( 0 < \lambda < \frac{E[m^2]}{E[w^2]} \). For any \( b \in \mathbb{R} \), there exists a unique informative affine equilibrium.

2) If \( \lambda \geq \frac{E[m^2]}{E[w^2]} \), there does not exist an informative (affine or non-linear) equilibrium. The only equilibrium is the non-informative one.

3) If \( \lambda = 0 \), there exists no informative equilibrium with affine policies.

Proof:

1) If the encoder is linear (affine), the decoder, as an MMSE decoder for a Gaussian source over a Gaussian channel, is linear (affine); this follows from the property of the conditional expectation for jointly Gaussian random variables. Suppose on the other hand that the decoder is affine so that \( u = \gamma^d(y) = Ky + L \) and the encoder policy is \( x = \gamma^e(m) \). We will show that the encoder is also affine in this case: With \( y = \gamma^e(m) + w \), it follows that \( u = K\gamma^e(m) + Kw + L \). By completing the squares, the optimal cost of the encoder can be written as

\[
J^{*,e} = \min_{x=\gamma^e(m)} E[(m - u - b)^2 + \lambda x^2]
\]

\[
= \min_{\gamma^e(m)} E[(m - K\gamma^e(m) - Kw - L - b)^2 + \lambda(\gamma^e(m))^2]
\]

\[
= \min_{\gamma^e(m)} (K^2 + \lambda) E\left[\left(\gamma^e(m) - \frac{(m - L - b)K}{K^2 + \lambda}\right)^2\right]
\]

\[
+ \frac{\lambda}{K^2 + \lambda} \left(E[m^2] + (L + b)^2\right) + K^2 E[w^2] \tag{7}
\]

Hence, the optimal \( \gamma^e(m) \) can be chosen as

\[
\gamma^{*,e}(m) = \frac{(m - L - b)K}{K^2 + \lambda} = \frac{(m - L - b)}{K + \frac{\lambda}{K}} \tag{8}
\]

and the minimum encoder cost is obtained as

\[
J^{*,e} = \frac{\lambda}{K^2 + \lambda} \left(E[m^2] + (L + b)^2\right) + K^2 E[w^2] \tag{9}
\]

Recall that (8) implies that an optimal encoder policy for a Gaussian source over a Gaussian channel is an affine policy if the decoder policy is chosen as affine.

We now wish to see if these sets of policies satisfy a fixed point equation. If the decoder has an affine policy, it is proved that the optimal policy of the encoder is also affine:

\[
\gamma^e(m) = Am + C = \left(\frac{1}{K + \frac{\lambda}{K}}\right)m + \left(\frac{-L - b}{K + \frac{\lambda}{K}}\right) \tag{10}
\]
Recall that if $A$ is a nonzero fixed point, then $\lambda E = \lambda E$ implies $E = E$ by assumption $A \neq 0$. Hence, the optimal decoder policy would be

$$\gamma^d(y) = Ky + L = \frac{AE[m^2]}{A^2E[m^2] + E[w^2]}(y - C).$$

By combining these, we obtain $(K^2 + \lambda)^2 E[w^2] = \lambda E[m^2]$ by assuming $A \neq 0$; which implies $K^2 = \sqrt{\frac{\lambda E[m^2]}{E[w^2]}} - \lambda$. If we combine the equations above by using $A$, we obtain

$$A = \frac{AE[m^2]}{A^2E[m^2] + E[w^2]} = \frac{A}{A^2 + E[w^2]}/E[m^2] + \lambda.$$  \hspace{1cm} (11)

Note now that

$$A \geq 1 \Rightarrow \frac{A}{A^2 + E[w^2]}/E[m^2] < 1 \Rightarrow \frac{A}{A^2 + E[w^2]}/E[m^2] + \lambda < \frac{1}{\lambda}$$

$$A < 1 \Rightarrow \frac{A}{A^2 + E[w^2]}/E[m^2] < \frac{E[m^2]}{E[w^2]} \Rightarrow \frac{A}{A^2 + E[w^2]}/E[m^2] + \lambda < \frac{E[m^2]}{E[w^2]} + \lambda$$

which implies that the mapping defined by

$$T(A) = \frac{A}{A^2 + E[w^2]}/E[m^2] + \lambda$$

can be viewed as a continuous function mapping the compact convex set $[0, \max(E[m^2]/E[w^2], 1)/\lambda]$ to itself. Therefore, by Brouwer’s fixed point theorem [31], there exists $A = T(A)$. Indeed, we can find nonzero $A$ for every $0 < \lambda < \frac{E[m^2]}{E[w^2]}$ after finding $A$, the values for $K$, $C$ and $L$ can also be obtained based on the equilibrium equations in (10) and (11). For the uniqueness of an informative fixed point, suppose that there are two different nonzero fixed points: $A_1 = T(A_1)$ and $A_2 = T(A_2)$ and let $\gamma = E[w^2]/E[m^2]$ for simplicity. Then $A_1/T(A_1) = A_2/T(A_2)$ implies

$$\frac{A_1^2}{A_1^2 + \gamma} + \lambda(A_1^2 + \gamma) = \frac{A_2^2}{A_2^2 + \gamma} + \lambda(A_2^2 + \gamma) \Rightarrow (A_1^2 - A_2^2)\left(\frac{\gamma}{A_1^2 + \gamma} - \frac{A_2^2}{A_2^2 + \gamma}\right) + \lambda = 0$$

Hence, $|A_1| = |A_2|$ is obtained, and since the mapping is defined from $[0, \max(E[m^2]/E[w^2], 1)/\lambda]$ to itself, the nonzero fixed point is unique. Then the encoder may choose the nonzero

\footnote{Recall that if $A \neq 0$ and $0 < \lambda < \frac{E[m^2]}{E[w^2]}$, we have $K^2 = \sqrt{\frac{\lambda E[m^2]}{E[w^2]}} - \lambda$, which implies $A = \frac{1}{\lambda + \sqrt{\lambda^2 - \lambda E[w^2]/E[m^2]}}$.}
fixed point for the informative equilibrium if it results in a lower cost than the non-informative equilibrium (due to the cost of communication, an informative equilibrium is not always beneficial to the encoder compared to the non-informative one).

2) Let $\lambda \geq E[m^2]/E[w^2]$ and suppose that we are in an equilibrium. Then, the encoder cost $J^e = E[(m - u - b)^2 + \lambda x^2]$ reduces to $J^e = E[(m - u)^2] + \lambda E[x^2] + b^2$, and since the decoder in an equilibrium always chooses $u = E[m|y]$, through $P = E[x^2]$, the following analysis leads to a lower bound on the encoder cost:

$$J^e = b^2 + \lambda E[x^2] + E[(m - u)^2] \geq b^2 + \lambda P + E[m^2]e^{-2\sup I(X;Y)}$$

(13)

$$= b^2 + \lambda P + E[m^2]e^{-2\frac{1}{\lambda} \log \left(1 + \frac{P}{E[w^2]}\right)} = b^2 + \lambda P + \frac{E[m^2]}{1 + P/E[w^2]}.$$  

(14)

Here, (13) follows from a rate-distortion theoretic bound through the data-processing inequality (see for example p. 96 of [10]). However, it follows that when $\lambda \geq E[m^2]/E[w^2]$, (14) is minimized at $P = 0$; that is, the encoder does not signal any output. Hence, the encoder engages in an non-informative equilibrium and the minimum cost becomes $E[m^2] + b^2$ at this non-informative equilibrium.

3) It is proved that an optimal encoder is affine such that $x = \gamma^e(m) = Am + C$ when the decoder is affine, that is, $u = \gamma^d(y) = Ky + L$. Then, by inserting $\lambda = 0$ to (6), $\overline{m}$ is obtained as

$$\overline{m} = KA \frac{(m^\alpha + m^\beta)}{2} + KC + L + b.$$

The above holds for all $m^\alpha$ and $m^\beta$ with $m^\alpha \leq \overline{m} \leq m^\beta$. Thus, if the distance between $m^\alpha$ and $m^\beta$ is made arbitrarily small, then it must be that $KA = 1$ and $KC + L + b = 0$. On the other hand, it was shown that an optimal decoder policy is affine if an encoder is affine in (11). By combining $KA = 1$ and $K = \frac{AE[m^2]}{A^2E[m^2] + E[w^2]}$, it follows that a real solution does not exist for any given affine coding parameter.

**Remark 5.1:** Note that, from (10) and (11), we have $A = \frac{1}{K + \lambda/K}$, $K = \frac{AE[m^2]}{A^2E[m^2] + E[w^2]}$, $L = -KC$ and $Ab = (AK - 1)C$. From these equalities, we observe the following.

1) when $\lambda = 0$, it is shown in Theorem 5.1 that there is not any fixed point solution to (12). However, if there is not a noisy channel between the encoder and the decoder; i.e., the noise variance is zero ($E[w^2] = 0$), then (12) has a fixed point solution. Even when (12) has a fixed point solution $A$, (10) and (11) cannot hold together unless $b = 0$. 

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2) when the noise variance is zero \((E[w^2] = 0)\), there is not any fixed point solution to (12) unless \(\lambda = 0\). Even when (12) has a fixed point solution \(A\), (10) and (11) cannot hold together unless \(b = 0\).

3) when \(\lambda = 0\) and the noise variance is zero \((E[w^2] = 0)\); the consistency of (10) and (11) can be satisfied if only if \(b = 0\). Hence, if \(b \neq 0\), there cannot be a affine equilibrium; the equilibrium has to be discrete due to Theorem 3.2.

Thus, if either \(\lambda\) or \(E[w^2]\) is 0, an affine equilibrium exists only if \(\lambda, E[w^2]\) and \(b\) are all 0.

### C. Price of Anarchy and Comparison with Socially Optimal Cost

In a game theoretic setup, the encoder and the decoder try to minimize their individual costs, thus the game theoretic cost can be found as \(\min_{\gamma_e} J^e + \min_{\gamma_d} J^d\). If the encoder and the decoder work together to minimize the total cost, then the problem can be regarded as a team problem and the resulting cost is a socially optimal cost, which is \(\min_{\gamma_e, \gamma_d} (J^e + J^d)\). In the game theoretic setup, because of the selfish behavior of the players, there is some loss from the socially optimal cost, which is named as the price of anarchy. In this part, it will be shown that the game theoretic cost is higher than the socially optimal cost as expected, and the information theoretic lower bounds on the costs and their achievability will be discussed.

**Theorem 5.2:** There always exists a price of anarchy in the game setup in the sense that the sum of the costs under any equilibria is always larger than the socially optimal cost.

**Proof:**

1) **Game Theoretic Cost Analysis:** Note from (10) and (11) that we have \(A = \frac{1}{K + \lambda/K}, K = \frac{AE[m^2]}{K^2 E[m^2] + E[w^2]}, L = -KC\) and \(Ab = C(AK - 1)\). Also we have \((K^2 + \lambda)E[w^2] = \lambda E[m^2]\) which implies \(K^2 = \sqrt{\frac{\lambda E[m^2]}{E[w^2]}} - \lambda\) and \(\lambda < E[m^2]/E[w^2]\) for nonzero \(A\). Recall that if \(\lambda \geq E[m^2]/E[w^2]\), then \(A = C = K = L = 0\), which implies the non-existence of the informative linear (also affine) equilibrium. Thus, for \(\lambda < E[m^2]/E[w^2]\), by using \(K^2 = \sqrt{\frac{\lambda E[m^2]}{E[w^2]}} - \lambda, A = \frac{1}{K + \lambda/K}\), \(C = \frac{Ab}{AK - 1}\) and \(L + b = -\frac{C}{A}\) in (9), we have

\[
J^*,e = 2\sqrt{\lambda E[m^2]E[w^2]} + b^2 \sqrt{\frac{E[m^2]}{\lambda E[w^2]}} - \lambda E[w^2]
\]

Now recall that the optimal decoder policy is \(u^* = E[m](y = Am + C + w) = \frac{AE[m^2]}{K^2 E[m^2] + E[w^2]}(y - C)\), and we have \(\sigma_e^2 = \sigma_x^2 - \frac{\sigma_y^2}{\sigma_y^2}\) where \(e = x - E[x|y]\). In this case, \(x \to m, y \to y, \sigma_x^2 \to E[m^2]\),
\( \sigma_{xy} \rightarrow AE[m^2] \) and \( \sigma_{y}^2 \rightarrow A^2E[m^2] + E[w^2] \). Thus, we have

\[
J^{*,d} = \min_{u=\gamma^{e}(y)} E[(m - u)^2] = E[(m - E[m|y])^2] = \sigma_m^2 - \frac{\sigma_{xy}^2}{\sigma_y^2}
\]

\[
= E[m^2] - \frac{A^2(E[m^2])^2}{A^2E[m^2] + E[w^2]} = \sqrt{\lambda E[m^2]E[w^2]}
\]

As a result, the game theoretic cost at the equilibrium is found as

\[
J^{*,g} = 3\sqrt{\lambda E[m^2]E[w^2]} + b^2 \sqrt{\frac{E[m^2]}{\lambda E[w^2]}} - \lambda E[w^2]
\] (15)

Recall that, if \( \lambda \geq E[m^2]/E[w^2] \), then \( J^{*,e} = E[m^2] + b^2 \) and \( J^{*,d} = E[m^2] \); hence, \( J^{*,g} = 2E[m^2] + b^2 \). If there were no cost of communication (consider the cheap talk; i.e., remove \( \lambda x^2 \) from the encoder cost function), then one could say that the informative equilibria would always be beneficial to both the encoder and the decoder; however, due to the cost of communication, an informative equilibrium is not always beneficial to the encoder when compared with the non-informative one (i.e., for \( \lambda < E[m^2]/E[w^2] \), it does not always hold that \( 2\sqrt{\lambda E[m^2]E[w^2]} + b^2 \sqrt{\frac{E[m^2]}{\lambda E[w^2]}} - \lambda E[w^2] < E[m^2] + b^2 \)). For the receiver, however, information never hurts the performance and the informative equilibria are more desirable (i.e., for \( \lambda < E[m^2]/E[w^2] \), the inequality \( \sqrt{\lambda E[m^2]E[w^2]} < E[m^2] \) always holds).

2) Socially Optimal Cost Analysis: The part below aims to construct the socially optimal affine setup. In this part, \( J^{*,t} \) represents the team cost minimized over the encoder policies for a given decoder policy, \( J^{d,t} \) represents the team cost minimized over the decoder policies for a given encoder policy, and \( J^{*,t} \) represents the optimum team cost; i.e., minimization over all affine encoding and decoding policies. In order to see the effect of the price of the anarchy, the cost of the team setup must be calculated.

\[
J^{*,t} = \min_{x=\gamma^{e}(m), u=\gamma^{d}(y)} E[(m - u - b)^2 + \lambda x^2 + (m - u)^2]
\]

Similar to the game theoretic analysis above, with the given affine encoding analysis policy \( x = \gamma^{e}(m) = Am + C \) (then \( y = x + w = Am + C + w \)), the optimal decoder policy can be found as follows (by completing the squares):

\[
J^{d,t} = \min_{u=\gamma^{d}(y)} E[(m - u - b)^2 + \lambda x^2 + (m - u)^2] = \min_{u=\gamma^{d}(y)} 2E\left[(m - u - \frac{b}{2})^2 + \frac{b^2}{4} + \lambda \frac{x^2}{2}\right]
\]

Hence the optimal decoder policy can be chosen as \( \gamma^{d,t}(y) = E\left[m - \frac{b}{2}\right] y \). Due to the joint Gaussianity of \( m \) and \( y \), the minimizer decoder policy is affine:

\[
\gamma^{d,t}(y) = Ky + L = \frac{AE[m^2]}{A^2E[m^2] + E[w^2]} (y - C) - \frac{b}{2}
\] (16)
Similar to the game theoretic analysis above, for any affine decoder policy \( \gamma^d(y) = Ky + L \), the optimal encoder policy for the team setup can be obtained as follows (by completing the squares):

\[
J^{e,t} = \min_{x = \gamma^e(m)} E[(m - u - b)^2 + \lambda x^2 + (m - u)^2]
\]

\[
= \min_{\gamma^e(m)} E[(m - K\gamma^e(m) - Kw - L - b)^2 + \lambda(\gamma^e(m))^2 + (m - K\gamma^e(m) - Kw - L)^2]
\]

\[
= \min_{\gamma^e(m)} (2K^2 + \lambda)E\left[ \left(\gamma^e(m) - \frac{(2m - 2L - b)K}{2K^2 + \lambda}\right)^2 \right] + \frac{b^2K^2 + \lambda(2E[m^2] + (L + b)^2 + L^2)}{2K^2 + \lambda} + 2K^2E[w^2]
\]

Hence, the optimal encoder \( \gamma^e(m) \) is

\[
\gamma^{e,t}(m) = Am + C = \frac{(2m - 2L - b)}{2K + \lambda/K}
\]  \hspace{1cm} (17)

and the minimum team cost is obtained as

\[
J^{*,t} = \frac{b^2K^2 + \lambda(2E[m^2] + (L + b)^2 + L^2)}{2K^2 + \lambda} + 2K^2E[w^2]
\]  \hspace{1cm} (18)

This implies that, in the team setup, an optimal encoder policy for a Gaussian source over a Gaussian channel is an affine policy if the decoder policy is chosen as affine.

In order to achieve the socially optimal cost \( J^{*,d} \), the optimal encoder policy \( \gamma^{e*,t}(m) \) and the optimal decoder policy \( \gamma^{d*,t}(y) \) must satisfy the following equalities by (16) and (17):

\[
A = \frac{2}{2K + \lambda/K}\quad C = \frac{A}{2}(-2L - b) = -AL - \frac{Ab}{2}\quad K = \frac{AE[m^2]}{A^2E[m^2] + E[w^2]}\quad L = -KC - \frac{b}{2}
\]

\[
\Rightarrow C = -AL - \frac{Ab}{2} = -A\left(-KC - \frac{b}{2}\right) - \frac{Ab}{2} = AKC
\]

Here, either \( AK = 1 \) or \( C = 0 \). If \( AK = 1 \), then \( E[w^2] = 0 \) which contradicts with the noise assumption. Then \( C = 0 \) and \( L = -b/2 \). By using the equalities for \( A \) and \( K \) above, one can obtain \( 2(K^2 + \lambda/2)E[w^2] = \lambda E[m^2] \) by assuming \( A \neq 0 \); which implies \( K^2 = \sqrt{\frac{\lambda E[m^2]}{2E[w^2]}} - \frac{\lambda}{2} \).

Since \( K^2 \) is positive, \( \lambda \) cannot be greater than \( \frac{2E[m^2]}{E[w^2]} \); otherwise, because of our assumption, \( A \) must be equal to 0 which implies that \( K = 0 \), and there does not exist an informative affine team setup. Then \( K^2 = \sqrt{\frac{\lambda E[m^2]}{2E[w^2]}} - \frac{\lambda}{2}, A = \frac{2K}{2K^2 + \lambda}, C = 0 \) and \( L = -\frac{b}{2} \) in (18), we have

\[
J^{*,t} = 2\sqrt{2\lambda E[m^2]E[w^2]} \frac{b^2}{2} - \lambda E[w^2]
\]  \hspace{1cm} (19)

Recall that, if \( \lambda \geq \frac{2E[m^2]}{E[w^2]} \), then \( J^{*,t} = 2E[m^2] + \frac{b^2}{2} \).
By comparing $J^{\ast,g}$ and $J^{\ast,t}$, one can observe that $J^{\ast,g} > J^{\ast,t}$ always holds (both in the informative and non-informative equilibria), which shows that the price of anarchy is present.

In the following, we discuss information theoretic lower bounds on the performance of equilibria and socially optimal strategies.

**Theorem 5.3:**

1) For the game setup, if $\lambda \geq \frac{E[m^2]}{E[w^2]}$ (i.e., non-informative equilibria), the information theoretic lower bounds on the costs are achievable.

2) For the game setup, if $\lambda < \frac{E[m^2]}{E[w^2]}$ and $b = 0$, then the information theoretic lower bounds on the costs are achievable by linear policies.

3) For the game setup, if $\lambda < \frac{E[m^2]}{E[w^2]}$ and $b \neq 0$, the information theoretic lower bounds on the costs are not achievable by affine policies.

4) For the team setup, the information theoretic lower bounds on the costs are always (both in the informative and non-informative equilibria) achievable by affine policies.

**Proof:**

1) Recall that the encoder cost is $J^e = E[(m - u - b)^2 + \lambda x^2]$ and we know that this reduces to $J^e = E[(m - u)^2] + \lambda E[x^2] + b^2$ since the decoder always chooses $u = E[m|y]$. Representing the power by $P = E[x^2]$, we again, as in (14), obtain the following bound:

$$J^e = b^2 + \lambda E[x^2] + E[(m - u)^2] \geq b^2 + \lambda P + \frac{E[m^2]}{1 + P/E[w^2]}.$$  \hfill (20)

This bound is tight when the encoder and the decoder use linear policies leading to jointly Gaussian random variables. For $\lambda < \frac{E[m^2]}{E[w^2]}$, a minimizer of this cost is $P^* = \sqrt{\frac{E[m^2]E[w^2]}{\lambda}} - E[w^2]$. If we insert this value into the encoder cost, we have

$$J^e \geq b^2 + \sqrt{\lambda E[m^2]E[w^2]} - \lambda E[w^2] + \sqrt{\lambda E[m^2]E[w^2]} = 2\sqrt{\lambda E[m^2]E[w^2]} + b^2 - \lambda E[w^2]$$

By the same reasoning above, we also have

$$J^d = E[(m - u)^2] \geq \frac{E[m^2]}{1 + \frac{P}{E[w^2]} \geq \sqrt{\lambda E[m^2]E[w^2]}$$

Hence, the information theoretic lower bound on the game cost $J^g = J^e + J^d$ is found as

$$J^g \geq 3\sqrt{\lambda E[m^2]E[w^2]} + b^2 - \lambda E[w^2]$$ \hfill (21)

Through an analysis similar to the one in [10], one can see that when $\lambda \geq \frac{E[m^2]}{E[w^2]}$, (20) is minimized at $P = 0$ (the encoder does not signal any output); thus we obtain a non-informative equilibrium: The encoder and the decoder do not engage in communications;
i.e., $A = 0$ and $K = 0$ is an equilibrium. In this case the encoder may be considered to be linear, but this is a degenerate coding policy. This implies $J^g \geq 2E[m^2] + b^2$, and remember that $J^{*,g} = 2E[m^2] + b^2$ when $\lambda \geq E[m^2]/E[w^2]$, hence the information theoretic lower bound is achievable in the non-informative equilibria.

2) From (15) and (21), it can be deduced that when $b = 0$, the lower bound of the encoder cost is achievable by linear policies; i.e., $C = 0$ and $L = 0$. When $b = 0$, the problem corresponds to what is known as a soft-constrained version of the quadratic signaling problem where we append the constraint to the cost functional (see page 96 of [10]).

3) If $b \neq 0$, then, from (15) and (21), one can observe that the lower bound becomes unachievable by affine policies since the power constraint related part of the cost function, $\lambda x^2$, contains $b^2$ related parameters (recall $C = \frac{Ab}{K - 1}$). In this case, by modifying the power from $P$ to $P - C^2$ (which must be positive) in the information theoretic inequalities; i.e., $J^e \geq b^2 + \lambda P + \frac{E[m^2]}{1 + (P - C^2)/E[w^2]}$, then the minimum game cost is obtained as $J^g \geq 3\sqrt{\lambda E[m^2]E[w^2]} + b^2 \sqrt{\frac{E[m^2]}{\lambda E[w^2]} - \lambda E[w^2]}$ which is the same cost that is achieved by affine policies.

4) By following a similar approach to that above for finding the lower bound on the socially optimal cost, we can obtain:

$$J^t = E[(m - u - b)^2 + \lambda x^2 + (m - u)^2] = E[2(m - u)^2 - 2(m - u)b + b^2 + \lambda x^2]$$

$$= \frac{b^2}{2} + \lambda E[x^2] + 2E\left[\left(m - u - \frac{b}{2}\right)^2\right] \geq \frac{b^2}{2} + \lambda P + \frac{2E[m^2]}{1 + P/E[w^2]}$$

Here (a) holds since the decoder chooses $u = E[m - \frac{b}{2}|y]$ and shifting does not affect the differential entropy. Similar to the previous analysis, a minimizer of this cost is $P^* = \sqrt{\frac{2E[m^2]E[w^2]}{\lambda} - E[w^2]}$ for $\lambda < 2E[m^2]/E[w^2]$. If we insert this value into the total cost, we have

$$J^t \geq 2\sqrt{2\lambda E[m^2]E[w^2]} + \frac{b^2}{2} - \lambda E[w^2].$$

(22)

Recall that, if $\lambda \geq 2E[m^2]/E[w^2]$, then $P = 0$ becomes the minimizer, hence $J^t \geq 2E[m^2] + \frac{b^2}{2}$ in the non-informative equilibrium. Remember that $J^{*,t} = 2E[m^2] + \frac{b^2}{2}$ in this case, thus the information theoretic lower bound is achievable in the non-informative equilibria. In addition, from (19) and (22), for $\lambda < 2E[m^2]/E[w^2]$ (which implies the informative equilibria), it can easily be seen that the information theoretic lower bound is
achievable by affine policies (actually the encoder policy is linear and the decoder policy is affine).

We state the following summary.

1) If $\lambda < E[m^2]/E[w^2]$ and $b = 0$, then the information theoretic lower bound on the game cost is achievable by the linear policies.

2) If $\lambda < E[m^2]/E[w^2]$ and $b \neq 0$, then the information theoretic lower bounds on the game cost are not achievable by the affine policies; but they become achievable after slight modification on the power parameter in the information theoretic inequality.

3) The team cost $J^{*,t}$ in the affine equilibrium is always equal to the information theoretic lower bound on the team cost.

4) There is always the price of anarchy: The socially optimal cost is always lower than the cost in any equilibrium.

5) In the game setup, the non-informative equilibrium may be preferred over the informative equilibrium by the encoder due to the cost of the signal $\lambda x^2$.

VI. Quadratic Signaling Game: Multi-Dimensional Gaussian Noisy Case

The scalar setup considered in Section V can be extended to the multi-dimensional Gaussian noisy signaling game problem setup as follows. The encoder (DM 1) encodes an $n$-dimensional zero-mean Gaussian random variable $\vec{M}$ and sends the real-valued $n$-dimensional random variable $\vec{X}$. During the transmission, the $n$-dimensional zero-mean Gaussian noise with the covariance matrix $\Sigma_{\vec{W}}$ is added to $\vec{X}$ and the decoder (DM 2) receives $\vec{Y} = \vec{X} + \vec{W}$. The policy space of DM 1, $\Gamma^e$, is the set of stochastic kernels from $\mathbb{R}^n$ to $\mathbb{R}^n$. The policy space of DM 2, $\Gamma^d$, is the set of stochastic kernels from $\vec{Y}$ to $\vec{U}$. The cost functions of the encoder and the decoder are as follows: DM 1 aims to minimize

$$J^e(\gamma^e, \gamma^d) = \int c^e(\vec{m}, \vec{u}) \gamma^e(d\vec{x}|\vec{m}) \gamma^d(d\vec{u}|\vec{y}) P(d\vec{y}|\vec{x}) P(d\vec{m}),$$

whereas DM 2 aims to minimize

$$J^d(\gamma^e, \gamma^d) = \int c^d(\vec{m}, \vec{u}) \gamma^e(d\vec{x}|\vec{m}) \gamma^d(d\vec{u}|\vec{y}) P(d\vec{y}|\vec{x}) P(d\vec{m}),$$

where $P(d\vec{y}|\vec{x}) = P(\vec{W} \in d\vec{y} - \vec{x})$ with $\vec{W} \sim \mathcal{N}(0, \Sigma_{\vec{W}})$. The cost functions are

$$c^e(\vec{m}, \vec{x}, \vec{u}) = ||\vec{m} - \vec{u} - \vec{b}||^2 + \lambda ||\vec{x}||^2$$

and

$$c^d(\vec{m}, \vec{u}) = ||\vec{m} - \vec{u}||^2.$$ 

Note that we have appended a power constraint and an associated multiplier. If $\lambda = 0$, this corresponds to the setup with no power constraint at the encoder.
A. Affine Equilibria

**Theorem 6.1:**

1) If the encoder is linear (affine), the decoder, as an MMSE decoder for a Gaussian source over a Gaussian channel, is linear (affine).

2) If the decoder is linear (affine), then an optimal encoder policy for a multi-dimensional Gaussian source over a multi-dimensional Gaussian channel is an affine policy.

3) For $\lambda > 0$, there exists an affine equilibrium in the multi-dimensional Gaussian noisy signaling game.

**Proof:**

1) Let the affine encoding policy be $\tilde{x} = \gamma^e(\tilde{m}) = A\tilde{m} + \tilde{C}$ where $A$ is an $n \times n$ matrix and $\tilde{C}$ is an $n \times 1$ vector. Then $\tilde{y} = \tilde{x} + \tilde{w} = A\tilde{m} + \tilde{C} + \tilde{w}$. The optimal cost of the decoder, by the law of the iterated expectations, can be expressed as

$$J^{*,d} = \min_{\tilde{u} = \gamma^d(\tilde{y})} E\left[\|\tilde{m} - \tilde{u}\|^2 | \tilde{y}\right]$$

Hence, a minimizer policy of the decoder is $\tilde{u} = \gamma^{*,d}(\tilde{y}) = E[\tilde{m}|\tilde{y}]$. Since both $\tilde{m}$ and $\tilde{y}$ are Gaussian, then the optimal decoder is

$$E[\tilde{m}|\tilde{y}] = E[\tilde{m}] + \Sigma_{\tilde{m}Y} \Sigma_{YY}^{-1}(\tilde{y} - E[\tilde{y}]) = \Sigma_{\tilde{m}} A^T (A \Sigma_{\tilde{m}} + \Sigma_{\tilde{w}})^{-1} (\tilde{y} - \tilde{C})$$  \hspace{1cm} (23)

2) Let the affine decoding policy be $\tilde{u} = \gamma^d(\tilde{y}) = K\tilde{y} + \tilde{L}$ where $K$ is an $n \times n$ matrix and $\tilde{L}$ is an $n \times 1$ vector. Then $\tilde{u} = K\tilde{y} + \tilde{L} = K(\tilde{x} + \tilde{w}) + \tilde{L} = K\gamma^e(\tilde{m}) + K\tilde{w} + \tilde{L}$. By using the completion of the squares method, the optimal cost is

$$J^{*,e} = \min_{\tilde{x} = \gamma^e(\tilde{m})} E\left[\|\tilde{m} - \tilde{u} - \tilde{b}\|^2 + \lambda\|x\|^2\right]$$

$$= \min_{\tilde{x} = \gamma^e(\tilde{m})} E\left[\|\tilde{m} - \tilde{u} - \tilde{b}\|^2 + \lambda\|x\|^2| \tilde{m}\right]$$

$$= \min_{\gamma^e(\tilde{m})} E\left[\|\tilde{m} - K\gamma^e(\tilde{m}) - K\tilde{w} - \tilde{L} - \tilde{b}\|^2 + \lambda\|\gamma^e(\tilde{m})\|^2| \tilde{m}\right]$$

$$= \min_{\gamma^e(\tilde{m})} E\left[(\tilde{m} - \tilde{L} - \tilde{b})^T (\tilde{m} - \tilde{L} - \tilde{b}) - 2(\tilde{m} - \tilde{L} - \tilde{b})^T K\gamma^e(\tilde{m}) + \gamma^e(\tilde{m})^T K^T K \gamma^e(\tilde{m})\right]$$

$$+ \left((K^T K + \lambda I)\gamma^e(\tilde{m}) - K^T (\tilde{m} - \tilde{L} - \tilde{b})\right)^T \left(K^T K + \lambda I\right)^{-1}$$

$$\left((K^T K + \lambda I)\gamma^e(\tilde{m}) - K^T (\tilde{m} - \tilde{L} - \tilde{b})\right) + (\tilde{m} - \tilde{L} - \tilde{b})^T K^T (\tilde{m} - \tilde{L} - \tilde{b})^T$$

$$\left(1 - K(K^T K + \lambda I)^{-1} K^T\right)(\tilde{m} - \tilde{L} - \tilde{b})^T + E[\tilde{w}^T K^T K \tilde{w}]$$  \hspace{1cm} (24)
Hence, the optimal $\gamma_e(m)$ can be chosen as follows:

$$
\gamma^e_m = A\tilde{m} + \tilde{C} = \left(K^T K + \lambda I\right)^{-1} K^T \left(\tilde{m} - \tilde{L} - \tilde{b}\right) 
$$  \hspace{1cm} (25)

3) We have $K = \Sigma M^T (A\Sigma M^T + \Sigma W)^{-1}$ and $A = \left(K^T K + \lambda I\right)^{-1} K^T$ by (23) and (25). By combining these, we obtain the following:

$$
A = T(A) = (FF^T + \lambda I)^{-1} F 
$$  \hspace{1cm} (26)

where $F = (A\Sigma M^T + \Sigma W)^{-1} A\Sigma M$. (26) implies a mapping and this mapping is denoted by $T(A)$. Since $FF^T$ is a real and symmetric matrix, it is diagonalizable and can be written as $FF^T = \sum v_i \left(v_i + \lambda\right)^{-1}$ for a diagonal $\sum$. Now consider $\|T(A)\|_F$:

$$
\|T(A)\|_F = \text{tr} \left( (FF^T + \lambda I)^{-1} F \right) \left( (FF^T + \lambda I)^{-1} F \right)^T 
$$

$$
= \text{tr} \left( (Q\gamma Q^{-1} + \lambda I)^{-1} Q\gamma Q^{-1} (Q\gamma Q^{-1} + \lambda I)^{-1} \right) 
$$

$$
= \text{tr} \left( (\gamma + \lambda I)^{-1} \gamma (\gamma + \lambda I)^{-1} \right) = \sum_{i=1}^{n} \frac{v_i}{(v_i + \lambda)^2} 
$$  \hspace{1cm} (27)

where $v_i, i = 1, \ldots, n$ are the eigenvalues of $FF^T$ and since $FF^T$ is positive semi-definite, all these eigenvalues are nonnegative. Since $\lambda > 0$, observe the following:

$$
\frac{v_i}{(v_i + \lambda)^2} < \frac{1}{\lambda^2} \quad \text{and} \quad \frac{v_i}{(v_i + \lambda)^2} < \frac{v_i}{v_i^2} = \frac{1}{v_i} < 1 
$$  \hspace{1cm} (28)

Hence, $v_i/(v_i + \lambda)^2 < \max(1, 1/\lambda^2)$ always holds. Then, by (27), we have $\|T(A)\|_F < n \max(1, 1/\lambda^2)$, which implies that $T(A)$ can be viewed as a continuous function mapping the compact convex set $\|A\|_F \in [0, n \max(1, 1/\lambda^2)]$ to itself. Therefore, by Brouwer’s fixed point theorem [31], there exists $A = T(A)$.

We note, however, that there always exist a non-informative equilibrium (see Proposition 2.1, which also applies to the signaling game discussed in this section). However, there exist games with informative affine equilibria as we state in the following (see Theorem 6.2).

**Proposition 6.1:** If either $\lambda$ or $\Sigma W$ is zero, an informative affine equilibrium exists only if $\lambda$, $\Sigma W$, and $\tilde{b}$ are all zero.

**Proof:** Note that, from (23) and (25), we have $A = \left(K^T K + \lambda I\right)^{-1} K^T$, $A\tilde{b} = (AK - I)\tilde{C}$, $K = \Sigma M^T (A\Sigma M^T + \Sigma W)^{-1}$ and $\tilde{L} = -K\tilde{C}$. From these equalities, we can analyze the equilibrium as in the scalar case:
1) when $\lambda = 0$ and the noise is zero ($\Sigma_{\tilde{W}} = 0$), then $A = K^{-1}$ and $K = A^{-1}$ are obtained. Then $A\tilde{b} = (AK - I)\tilde{C} = 0$, thus the consistency of the equalities can be satisfied if only if $b = 0$. Hence, if $b \neq 0$, there cannot exist an informative affine equilibrium. Recall that in the multi-dimensional noiseless cheap talk, the linearity of the equilibrium is shown for the uniform source; here the source is Gaussian.

2) when $\lambda = 0$, $A = K^{-1}$ and $A\Sigma_{\tilde{M}}A^T + \Sigma_{\tilde{W}} = K^{-1}\Sigma_{\tilde{M}}A^T$ are obtained. There does not exist a solution to (12) unless the noise is zero ($\Sigma_{\tilde{W}} = 0$). Even when (12) has a fixed point solution $A$, (10) and (11) cannot hold together unless $b = 0$.

3) when the noise is zero ($\Sigma_{\tilde{W}} = 0$), $K = A^{-1}$ and $K^TK + \lambda I = K^TA^{-1}$ are obtained. There does not exist a solution to (12) unless $\lambda = 0$. Even when (12) has a fixed point solution $A$, (10) and (11) cannot hold together unless $b = 0$.

**Remark 6.1:** In the multi-dimensional case, fixed points may not be unique: with $\lambda = 1.0311$ and

$$
\Sigma_{\tilde{M}} = \begin{bmatrix}
1.6421 & 0.1299 & 0.5713 & 0.2305 \\
0.1299 & 1.4803 & 0.6810 & 0.4749 \\
0.5713 & 0.6810 & 1.7312 & 0.4292 \\
0.2305 & 0.4749 & 0.4292 & 1.3515
\end{bmatrix}
\quad
\Sigma_{\tilde{W}} = \begin{bmatrix}
1.2742 & 0.1868 & 0.2318 & 0.0559 \\
0.1868 & 1.8266 & 0.5955 & 0.3091 \\
0.2318 & 0.5955 & 1.2377 & 0.4951 \\
0.0559 & 0.3091 & 0.4951 & 1.5336
\end{bmatrix}
$$

we can obtain two fixed points with different absolute-valued elements as follows (recall that if $A$ is a fixed point, $-A$ is also a fixed point):

$$
A = \begin{bmatrix}
-0.1543 & 0.1762 & 0.0606 & 0.1117 \\
0.1602 & 0.0159 & 0.1036 & 0.0279 \\
-0.2000 & -0.1879 & -0.2700 & -0.1565 \\
0.0603 & 0.1052 & 0.1221 & 0.0824
\end{bmatrix}
\quad
A = \begin{bmatrix}
-0.2431 & 0.0738 & -0.0752 & 0.0285 \\
0.0293 & -0.1351 & -0.0966 & -0.0948 \\
0.1520 & 0.2181 & 0.2682 & 0.1735 \\
-0.1003 & -0.0801 & -0.1236 & -0.0683
\end{bmatrix}
$$

**Theorem 6.2:** Let source $\tilde{M}$ be a zero-mean $n$-dimensional Gaussian random variable with covariance matrix $\Sigma_{\tilde{M}} = \text{diag}\{\sigma_{m_1}^2, \ldots, \sigma_{m_n}^2\}$ where $\text{diag}$ indicates a diagonal matrix, and noise $\tilde{W}$ be a zero-mean $n$-dimensional Gaussian random variable with covariance matrix $\Sigma_{\tilde{W}} = \text{diag}\{\sigma_{w_1}^2, \ldots, \sigma_{w_n}^2\}$. Then an informative affine equilibrium exists if $\lambda < \max\{\frac{\sigma_{m_1}^2}{\sigma_{w_1}^2}, \ldots, \frac{\sigma_{m_n}^2}{\sigma_{w_n}^2}\}$.

**Proof:** Since the source components are independent and the noise components are independent, the $n$-dimensional noisy signaling game problem turns into $n$ independent scalar noisy signaling game problems as follows:
1) If the decoder uses the channels independently; i.e., \( u_i = \gamma_d(y_i) \) for \( i = 1, \ldots, n \), then the optimal cost of the encoder will be

\[
J_{e}^{\ast} = \min_{\bar{x} = \gamma_e(\bar{m})} E \left[ \| \bar{m} - \bar{u} - \bar{b} \|^2 + \lambda \| x \|^2 \right] = \min_{\bar{x} = \gamma_e(\bar{m})} \sum_{i=1}^{n} E[(m_i - u_i - b_i)^2 + \lambda x_i^2]
\]

Since, \( y_i = x_i + w_i \) for each \( i = 1, \ldots, n \), the optimal encoder also uses the channels independently; i.e., \( x_i = \gamma_e(m_i) \) for \( i = 1, \ldots, n \).

2) Similarly, if the encoder uses the channels independently; i.e., \( x_i = \gamma_e(m_i) \) for \( i = 1, \ldots, n \), then the optimal cost of the decoder will be

\[
J_{d}^{\ast} = \min_{\bar{u} = \gamma_d(\bar{y})} E \left[ \| \bar{m} - \bar{u} \|^2 \right] = \sum_{i=1}^{n} \min_{\bar{u} = \gamma_d(\bar{y})} E[(m_i - u_i)^2]
\]

Since, \( y_i = \gamma_e(m_i) + w_i \) for each \( i = 1, \ldots, n \), the optimal decoder will also uses channels independently; i.e., \( u_i = \gamma_d(y_i) \) for \( i = 1, \ldots, n \).

Thus, we have, in each dimension \( i \) (\( i = 1, \ldots, n \));

- the source \( M_i \) is a zero-mean Gaussian with variance \( \sigma_{m_i}^2 \)
- the channel has the Gaussian noise \( W_i \) with zero-mean and variance \( \sigma_{w_i}^2 \)
- the encoder’s goal is to find the optimal policy which minimizes its cost as follows

\[
\min_{x_i = \gamma_e(m_i)} E[(m_i - u_i - b_i)^2 + \lambda x_i^2]
\]

- the decoder’s goal is to find the optimal policy which minimizes its cost as follows

\[
\min_{u_i = \gamma_d(y_i)} E[(m_i - u_i)^2]
\]

For each dimension, the informative affine equilibrium exists if \( \lambda < \sigma_{m_i}^2 / \sigma_{w_i}^2 \). For the multidimensional setup, the existence of the informative equilibrium in at least one dimension implies the existence of the informative equilibrium for the whole system. Hence, it is sufficient that the inequality \( \lambda < \sigma_{m_i}^2 / \sigma_{w_i}^2 \) is valid for at least one dimension. As a result, the condition for the existence of the informative affine equilibrium becomes \( \lambda < \max \{ \sigma_{m_1}^2 / \sigma_{w_1}^2, \ldots, \sigma_{m_n}^2 / \sigma_{w_n}^2 \} \).

Note that, from (23) and (25), by assuming \( |A| \neq 0 \), we have \( \lambda A \Sigma M A^T = K^T K \Sigma W \) which is equivalent to

\[
\lambda(A^T)^{-1} \Sigma M A^T = (K^T K + \lambda I)(K^T K + \lambda I) \Sigma W
\]
Remark 6.2: Assuming all channels are informative, i.e., $|A| \neq 0$, we make the following observations.

- If the source is i.i.d.; i.e., $\Sigma_{\tilde{M}} = \sigma^2_m I$, then (29) becomes
  \[
  \lambda (A^T)^{-1} \sigma^2_m I = (K^T K + \lambda I)(K^T K + \lambda I) \Sigma_{\tilde{W}} \Rightarrow \lambda \sigma^2_m (\Sigma_{\tilde{W}})^{-1} = (K^T K + \lambda I)(K^T K + \lambda I) \Rightarrow \lambda \sigma^2_m (\Sigma_{\tilde{W}})^{-1} \geq \lambda^2 I \Rightarrow \lambda \leq \sigma^2_m (\Sigma_{\tilde{W}})^{-1}
  \]
  This result implies that $\lambda$ must satisfy the inequality $\lambda \leq \sigma^2_m (\Sigma_{\tilde{W}})^{-1}$ for the i.i.d. source; otherwise, there must be at least one non-informative channel; i.e., $|A|$ must be 0.

- If the channel noise is i.i.d.; i.e., $\Sigma_{\tilde{W}} = \sigma^2_w I$, (since $\Sigma_{\tilde{M}}$ is real-symmetric, it has the eigenvalue decomposition as $\Sigma_{\tilde{M}} = Q \Lambda Q^T$), then (29) becomes
  \[
  \lambda (A^T)^{-1} \Sigma_{\tilde{M}} A^T = (K^T K + \lambda I)(K^T K + \lambda I) \sigma^2_w I \\
  \Rightarrow \lambda (A^T)^{-1} Q \Lambda Q^T A^T = (K^T K + \lambda I)(K^T K + \lambda I) \Rightarrow (A^T)^{-1} Q \Lambda Q^T A^T \geq \lambda \sigma^2_w
  \]
  This result implies that for each eigenvalue $\lambda_{\tilde{M}}$ of $\Sigma_{\tilde{M}}$, $\lambda$ must satisfy $\lambda \leq \lambda_{\tilde{M}} / \sigma^2_w$ for the i.i.d. channel noise; otherwise, there must be at least one non-informative channel; i.e., $|A|$ must be 0.

- For the general case, recall the Minkowski determinant theorem, $|A+B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}$ which holds for any non-negative $n \times n$ Hermitian matrix $A$ and $B$. This implies $|A+B| \geq |A| + |B|$. By using this inequality and (25),
  \[
  \frac{|A|}{|K|} \leq \frac{|K|}{|K^T K + \lambda I|} \leq \frac{|K|}{|K|^2 + \lambda^n}
  \]
  Assuming $|A| \neq 0$, recall the equality $\lambda A \Sigma_{\tilde{M}} A^T = K^T K \Sigma_{\tilde{W}}$. Taking the determinant of both sides,
  \[
  \lambda^n |A|^2 |\Sigma_{\tilde{M}}| = |K|^2 |\Sigma_{\tilde{W}}| \leq \lambda^n (\frac{|K|}{|K|^2 + \lambda^n})^2 |\Sigma_{\tilde{M}}| \leq \lambda^n (\frac{|K|^2}{\lambda^{2n}}) |\Sigma_{\tilde{W}}| \Rightarrow \lambda^n \left( \frac{|\Sigma_{\tilde{M}}|}{|\Sigma_{\tilde{W}}|} \right) \geq \lambda^n \leq \frac{|\Sigma_{\tilde{M}}|}{|\Sigma_{\tilde{W}}|} \quad (30)
  \]
  The result can be interpreted as follows: If $\lambda > \left( \frac{|\Sigma_{\tilde{M}}|}{|\Sigma_{\tilde{W}}|} \right)^{1/n}$, then $|A| = |K| = 0$ in the equilibrium; i.e., there must be at least one non-informative channel.

B. An information theoretic lower bound on the encoder cost and the existence of informative equilibria

We end the section with an information theoretic lower bound for the encoder cost; this serves us to obtain also condition for the existence of an informative equilibrium. Let $\tilde{e} = \tilde{m} - \tilde{u} = \tilde{u} - \tilde{m}$.
\[ \bar{m} - E[\bar{m}|\bar{y}], \] then we have \[ \Sigma_e = E[\bar{e}\bar{e}^T] = E[(\bar{m} - E[\bar{m}|\bar{y}])(\bar{m} - E[\bar{m}|\bar{y}])^T]. \] From the information theoretic inequalities;

\[ I(\bar{m}; \bar{y}) = H(\bar{m}) - H(\bar{m}|\bar{y}) = H(\bar{m}) - H(\bar{m} - E[\bar{m}|\bar{y}]|\bar{y}) \geq H(\bar{m}) - H(\bar{m} - E[\bar{m}|\bar{y}]) \]

\[ \geq H(\bar{m}) - \frac{1}{2} \log_2((2\pi e)^n|\Sigma_e|) = \frac{1}{2} \log_2((2\pi e)^n|\Sigma_{\bar{m}}|) - \frac{1}{2} \log_2((2\pi e)^n|\Sigma_e|) = \frac{1}{2} \log_2(|\Sigma_{\bar{m}}|/|\Sigma_e|) \]

Also from the rate-distortion theorem, the data processing theorem and the channel capacity theorem:

\[ R(D) \leq \min_{f(\bar{u}|\bar{m}): E[||\bar{m} - \bar{u}||^2] \leq D} I(\bar{m}; \bar{u}) \leq I(\bar{m}; \bar{y}) \leq \max_{f(\bar{x})} I(\bar{x}; \bar{y}) \leq C(P) \]

If we combine these, we obtain the following:

\[ |\Sigma_e| \geq |\Sigma_{\bar{m}}|^2 2^{-2R(D)} \geq |\Sigma_{\bar{m}}|^2 2^{-2I(\bar{m}; \bar{u})} \geq |\Sigma_{\bar{m}}|^2 2^{-2I(\bar{x}; \bar{y})} \geq |\Sigma_{\bar{m}}|^2 2^{-2C(P)} \quad (31) \]

Now consider the following:

\[ E[||\bar{m} - \bar{u}||^2] = E[||\bar{e}||^2] = \text{tr} \Sigma_e \geq n \prod_{i=1}^{n} \frac{\lambda_i}{\lambda_i} \geq n \left( |\Sigma_{\bar{m}}| \right)^{1/n} \geq n \left( |\Sigma_{\bar{m}}|^2 2^{-2C(P)} \right)^{1/n} \quad (32) \]

Here, (a) follows from the inequality for the arithmetic and geometric mean where \[ \Sigma_e(i, i) \] stands for \[ i \text{th diagonal element of } \Sigma_e, \] (b) follows from the Hadamard inequality (since \[ \Sigma_e \] is a positive semi-definite matrix), and (c) follows from (31). Now we will rewrite (32) Eq. (9.166)] which presents the capacity of the additive colored Gaussian noise channel with typo corrected:

\[ C(P) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \log_2 \left( 1 + \frac{\nu - \lambda_i}{\lambda_i} \right) \]

where \[ P = E[||\bar{x}||^2], \lambda_1, \lambda_2, \ldots, \lambda_n \text{ are the eigenvalues of } \Sigma_{\bar{u}} \text{ and } \nu \text{ is chosen so that } \sum_{i=1}^{n} \max(\nu - \lambda_i, 0) = nP. \] Then we can obtain the following:

\[ 2^{-2C(P)} = 2^{-\frac{1}{2} \sum_{i=1}^{n} \frac{1}{2} \log_2 \left( 1 + \frac{\max(\nu - \lambda_i, 0)}{\lambda_i} \right)} = \prod_{i=1}^{n} \left( 1 + \frac{\max(\nu - \lambda_i, 0)}{\lambda_i} \right)^{-1/n} = \prod_{i=1}^{n} \left( \frac{\max(\nu, \lambda_i)}{\lambda_i} \right)^{-1/n} \]

\[ \frac{\prod_{i=1}^{n} \lambda_i^{1/n}}{\prod_{i=1}^{n} \max(\nu, \lambda_i)^{1/n}} \geq \left( \frac{|\Sigma_{\bar{u}}|}{P + \sum_{i=1}^{n} \frac{\lambda_i}{n}} \right)^{1/n} \left( \frac{\text{tr } \Sigma_{\bar{u}}}{n} \right)^{-1/n} \]

Here, (a) holds, since our assumption \[ \sum_{i=1}^{n} \max(\nu - \lambda_i, 0) = nP \text{ implies } \sum_{i=1}^{n} \max(\nu, \lambda_i) = nP + \sum_{i=1}^{n} \lambda_i \text{ and } \left( \prod_{i=1}^{n} \max(\nu, \lambda_i) \right)^{1/n} \leq \sum_{i=1}^{n} \max(\nu, \lambda_i)/n = P + \sum_{i=1}^{n} \lambda_i/n \text{ holds by the inequality for the arithmetic and geometric mean.} \]

If we insert (33) to (32),

\[ E[||\bar{m} - \bar{u}||^2] \geq n \left( |\Sigma_{\bar{m}}| \right)^{1/n} \left( P + \frac{\text{tr } \Sigma_{\bar{u}}}{n} \right)^{-1/n} \]

\[ n \left( |\Sigma_{\bar{m}}| \right)^{1/n} \left( P + \frac{\text{tr } \Sigma_{\bar{u}}}{n} \right)^{-1/n} \]

\[ \text{(34)} \]
The encoder costs reduces to $J^e = E[||\tilde{m} - \tilde{u}||^2] + \lambda E[||\tilde{x}||^2] + ||\tilde{b}||^2$ since the decoder always chooses $\tilde{u} = E[|\tilde{m}|\tilde{y}]$. Then, by (34),

$$J^e = ||\tilde{b}||^2 + \lambda E[||\tilde{x}||^2] + E[||\tilde{m} - \tilde{u}||^2] \geq ||\tilde{b}||^2 + \lambda P + n(|\Sigma_m|)^{1/n}(|\Sigma_w|)^{1/n^2} \left(P + \frac{\text{tr } \Sigma_w}{n}\right)^{-1/n}$$

(35)

The minimizer of this function can be found by the local perturbation condition:

$$\lambda - (|\Sigma_m|)^{1/n}(|\Sigma_w|)^{1/n^2} \left(P + \frac{\text{tr } \Sigma_w}{n}\right)^{-\frac{1}{n}-1} = 0$$

Then,

$$\lambda = (|\Sigma_m|)^{1/n}(|\Sigma_w|)^{1/n^2} \left(P + \frac{\text{tr } \Sigma_w}{n}\right)^{-\frac{1}{n}-1}$$

$$(a) \leq (|\Sigma_m|)^{1/n}(|\Sigma_w|)^{1/n^2} \left(|\Sigma_w|\right)^{1/n} \left(\left(|\Sigma_w|\right)^{1/n}\right)^{-\frac{1}{n}-1} = (|\Sigma_m|)^{1/n}(|\Sigma_w|)^{-1/n}$$

Here, (a) follows from the nonnegativeness of $P$ and the inequality for the arithmetic and geometric mean and the Hadamard inequality, similar to (32). Hence, if $\lambda < (|\Sigma_m|)^{1/n}(|\Sigma_w|)^{-1/n}$, the lower bound is minimized at a nonzero $P$ value, but if $\lambda \geq (|\Sigma_m|)^{1/n}(|\Sigma_w|)^{-1/n}$, the minimizing $P$ becomes zero. Finally, if channels and source are assumed to be i.i.d., and the encoder and the decoder use linear policies, then (35) becomes tight and can be interpreted as follows: If $\lambda > (|\Sigma_m|)^{1/n}(|\Sigma_w|)^{-1/n} = E[m^2]/E[w^2]$, then (35) is minimized at $P = 0$; that is, the encoder does not signal any output. Hence, the encoder engages in an non-informative equilibrium and the minimum cost becomes $E[||\tilde{m}||^2] + ||\tilde{b}||^2$ at this non-informative equilibrium. Recall that this is analogous to the analysis in the scalar setup (14).

VII. CONCLUDING REMARKS

It has been shown that the equilibrium policies may be non-discrete and even linear for a multi-dimensional cheap talk problem which is different from the scalar case (the equilibria must be discrete for a nonzero bias for any scalar random source). The other is the extension to the noisy channel setups which have a signaling game flavour due to the communication constraints in the transmission. Conditions for the existence of affine equilibrium policies as well as general informative equilibria are presented for both the scalar and multi-dimensional setups. Our findings provide further conditions on when affine policies may be optimal in decentralized multi-criteria control problems and lead to conditions for the presence of active information transmission in strategic environments.
REFERENCES


