Advanced Clustering with Frequency Sweeping (ACFS) Methodology for the Stability Analysis of Multiple Time-Delay Systems

Ismail Ilker Delice & Rifat Sipahi

Abstract—A novel methodology, Advanced Clustering with Frequency Sweeping (ACFS), is introduced for the stability analysis of the most general class of linear time-invariant (LTI) time-delay systems (TDS) with multiple delays. Different from the literature, ACFS does not impose any restrictions in system order, the number of delays and the ranks of the system matrices in the LTI-TDS considered. An unmatched strength of ACFS is that it can achieve to directly extract the 2D cross-sections of the stability views in the domain of any of the two delays. ACFS owes this superiority to an elegant way of cross-fertilizing the resultant theory, frequency sweeping technique, and the root clustering paradigms. ACFS also proposes a new formula that can compute the precise lower and upper bounds of the only parameter, the frequency, that it sweeps. Case studies that are prohibitive to analyze with the existing methods are presented to demonstrate the strengths of ACFS.

Keywords: Multiple Time-Delay System, Stability, Stability Switching Curves, Rekasius Substitution, Resultant Theory.

I. INTRODUCTION

In this paper, one of the most important and unresolved problems of time-delay systems (TDS) is studied: the asymptotic stability of linear time-invariant (LTI) multiple time-delay system (MTDS) with respect to delays $\tau_i$. The system is expressed in state space form as,

$$
\frac{d\vec{x}(t)}{dt} = A \vec{x}(t) + \sum_{\ell=1}^{L} B_{\ell} \vec{x}(t - \tau_{\ell}),
$$

where $A \in \mathbb{R}^{N \times N}$, $B_{\ell} \in \mathbb{R}^{N \times N}$ are constant matrices; $\vec{x}(t) \in \mathbb{R}^{N \times 1}$ is the state vector; $\tau_{\ell}$ are the nonnegative delays. Different than the literature cited below, no restriction is imposed here on the system order $N$, the ranks of $A$ and $B_{\ell}$ matrices as well as the number of delays $L$ considered.

Presence of delays leads to an infinite spectrum in (1) making the stability assessment of (1) in delay parameter space a non-trivial task. The display of the stability with respect to delay parameter is called as ‘stability map’ or ‘stability chart’ [1], where this map is a 1D nonnegative delay axis along which stable and unstable delay intervals are marked [2], [3], [4], [5]. It is crucial to surface all these intervals with their precise lower and upper bounds for the necessary and sufficient conditions of asymptotic stability. In the case with $L = 2$, stability maps are the displays of 2D stability/instability regions on the plane of two delays [1], [6], [7], [8].

The main objective in 2D stability analysis is to construct all the potential stability switching curves (PSSC) which partition the delay space into stable and unstable regions. Obviously, the accuracy and completeness of the analysis strongly depends on finding all the existing PSSC without any approximations. To the best of our knowledge, the first attempts in analyzing stability for $L = 2$ delays are found in [1], [5], [6], [9]. The most recent methods along this line start to arrive from 2002 on, with the work of [10], [11]. Needless to say, the cited works are implemented on case specific problems, limiting their extensions to general treatment of two delay problems. This gap was bridged in 2005 by two different methods, [7] and [8]. Three new techniques are observed after these publications, where in [12] and [13], the stability problem is initially formulated differently, but leads to the computation of generalized eigenvalues of a matrix pencil, and in [14], the authors identify PSSC by using the ‘Building Block’ concept.

In the case of three delays, $L = 3$, there are only a handful number of studies in the literature [13], [15], [16], [17], [18], and see [15] for the case with arbitrarily large number of delays. These advancements are case-specific and there still exists no method to treat the stability of the most general system in (1). As recognized in [13], [15], the limitations in the existing methodologies can be summarized as follows: (a) they require exponentially increasing computation times as they perform multiple parameter sweeping in nested loops when extracting the potential stability switching hypersurfaces (PSSH) of the $L$-dimensional stability maps, (b) they are case-specific, (c) they cannot extract the 2D or 3D cross sections of an $L$ dimensional stability map and therefore they are limited to treat $L \leq 3$ problems. One exception to (c) is our recent work [15] which still falls short to treat the stability of (1), which has relaxed rank conditions here on $B_{i}$.

In this paper, a new methodology called Advanced Clustering with Frequency Sweeping (ACFS) is introduced to remove the complications listed in (a)-(c) above. ACFS is a unique and nontrivial cross fertilization of resultant theory [19], Cluster Treatment of Characteristic Roots (CTCR) [7] and frequency sweeping techniques [8], [15], [20], [21], and it can efficiently extract the 2D cross sections of $L$-D stability maps. Arising from ACFS’s theoretical construct are the following new results: (i) the maximum number of points generating the kernel points associated with PSSC can be computed as a function of the ranks of system matrices, and (ii) necessary and sufficient conditions which yield the exact lower and upper bounds of the crossing frequency...

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set (CFS) can be formulated via an automated sequential formula. These bounds are crucial as they determine the sweeping range of the only parameter, the frequency, that ACFS sweeps.

Notations used in the text are standard. We use bold face font for matrices and sets, and an arrow over the symbol for vectors. We use $s \in \mathbb{C}$ for the Laplace variable; $\mathbb{R}(s)$ for the real part of $s$ and $\Im(s)$ for the imaginary part of $s$. $\mathbb{R}_+$, $\mathbb{R}_0^+, \mathbb{Z}_+$ and $\mathbb{N}$ denote the set of positive real numbers, nonnegative real numbers, positive integer numbers, and natural numbers, respectively. $\sup$ is the supremum of a set; $\deg$ is the degree of a polynomial and $\bullet$ indicates a fixed value of a variable. $\bar{\tau} = (\tau_1, \ldots, \tau_L)$ is the delay vector and $\bar{T} = (T_\ell)_{\ell=1}^L$ is the pseudo-delay vector. $R_{T_\ell}(p_1, p_2)$ denotes the resultant of bivariate polynomials $p_1(T_1, T_2)$ and $p_2(T_1, T_2)$ with eliminating $T_1$, $\ell = 1, 2$. We omit the arguments when no confusion occurs.

II. Problem Formulation and Preliminary Facts

The stability of (1) is studied over its characteristic function given by:
\[ f(s, \bar{\tau}) = \sum_{k=0}^K P_k(s) e^{-s \sum_{\ell=1}^L z_{k\ell} \tau_\ell}, \tag{2} \]
where $P_k$ are polynomials in terms of $s$ with real coefficients, $K \in \mathbb{Z}_+$ and $z_{k\ell} \in \mathbb{N}$. MTDS in (1) is a \textit{retarded} class LTI-TDSs as the highest order derivative of the state is not influenced by delays. This corresponds to the case where $P_0$ does not multiply any terms carrying delays, $\{z_{0\ell}\}_{\ell=1}^L = 0$, and $P_0$ has the highest power of $s$ in (2).

Due to the presence of transcendental terms, the characteristic function (2) possesses infinitely many roots for a given set of delays, $\bar{\tau}_1, \ldots, \bar{\tau}_L$ with at least one delay being nonzero. The LTI-MTDS in (1) is asymptotically stable for a given $\bar{\tau} = \bar{\tau}$ if and only if the measure $\alpha(\bar{\tau}) = \sup \{\Re(s) \mid f(s, \bar{\tau}) = 0\} = 0$ is negative for $\bar{\tau}$, $\alpha(\bar{\tau}) < 0$ [22]. Furthermore, the continuity of $\alpha$ holds with respect to the imaginary axis [23] and with this knowledge \textit{stability transitions} of the dynamics can be studied via $\alpha(\bar{\tau}) = 0$.

This requires to investigate the imaginary roots $s = j\omega$ of (2), where $\omega \in \mathbb{R}_0^+$, without loss of generality. All $\omega$ values, where $s = j\omega$ is a root of (2) for some nonnegative delays, define the crossing frequency set,
\[ \Omega = \{\omega \in \mathbb{R}_0^+ \mid f(j\omega, \bar{\tau}) = 0, \text{ for some } \bar{\tau} \in \mathbb{R}_0^L \}, \tag{3} \]
and $\bar{\omega} \in \Omega$ maps to at least a point $\bar{\tau}$ as well as to all the infinitely many solutions of (2),
\[ (\bar{\tau}_1, \ldots, \bar{\tau}_L) + (\eta_1, \ldots, \eta_L) \cdot \frac{2\pi}{\omega}, \{\eta_\ell\}_{\ell=1}^L \in \mathbb{N}^L, \tag{4} \]
where $(\bar{\tau}_1, \ldots, \bar{\tau}_L)$ are the minimum nonnegative delays in (4) without loss of generality. The solutions in (4), considering all $\omega \in \Omega$, lie on the $L$-dimensional PSSH aforementioned earlier. PSSH denoted by $\varphi$ is defined as
\[ \varphi = \{\bar{\tau} \in \mathbb{R}_0^L \mid f(j\omega, \bar{\tau}) = 0, \forall \omega \in \Omega\}. \tag{5} \]

Among all the PSSH, there exists a special subset which constitutes the kernel hypersurfaces, defined by $\varphi_{\text{kernel}} = \{\varphi \mid \{\eta_\ell\}_{\ell=1}^L = 0\}$. It is easy to see that given $\bar{\omega} \in \Omega$ and a point $\{\bar{\tau}\}_{\ell=1}^L \in \varphi_{\text{kernel}}$, one can generate the remaining infinitely many solutions by incrementing the counter $\eta_\ell$ in (4). In other words, kernel hypersurfaces are the generators of infinitely many hypersurfaces called the offspring and defined as $\varphi_{\text{offspring}} = \varphi \setminus \varphi_{\text{kernel}}$ [24].

A. Review of CTCR Methodology

The first step of CTCR methodology constructs $\Omega$ and $\varphi_{\text{kernel}}$ starting from (2) [7]. Construction is done as in the following.

1) Identification of PSSH and Crossing Frequency Set: Replace the exponential terms in (2) with the Rekasius transformation [25],
\[ e^{-s \tau_\ell s} \rightarrow \frac{1 - T_\ell s}{1 + T_\ell s}, s = j\omega, T_\ell \in \mathbb{R}, \ell = 1, \ldots, L. \tag{6} \]
Transformation (6) is \textit{exact} for imaginary roots $s = j\omega$ when the following back transformation rule found by developing the phase conditions on both sides of (6) holds,
\[ \tau_\ell = \frac{2}{\omega} \left( \tan^{-1}(\omega T_\ell) + \pi \eta_\ell \right), \ell = 1, \ldots, L, \tag{7} \]
where $0 \leq \tan^{-1}(.) < \pi$ [14], $\omega \in \Omega$, and the counters $\eta_\ell$ are defined in (4). Since the Rekasius transformation (6) along with (7) is exact for $s = j\omega$, it proves to be convenient for solving $s = j\omega$ roots of (2). Upon substitution of (6) into (2) and with the following manipulation, transformed characteristic function is obtained as,
\[ g(s, \bar{T}) = \left. \left( f(s, \bar{\tau}) \right\vert_{\text{e}^{-s \tau_\ell s} = \frac{1 - T_\ell s}{1 + T_\ell s}} \right\vert_{\ell = 1, \ldots, L} \right\langle \prod_{\ell=1}^L (1 + T_\ell s)^{c_\ell} = \sum_{m=0}^M Q_m(\bar{T}) s^m, \tag{8} \]
where $Q_m(\bar{T})$ are multinomials only in terms of agent parameters (pseudo-delays) $T_1, \ldots, T_L$; $c_\ell = \text{rank}(B_\ell) \leq N$ and $M = N + \sum_{\ell=1}^L c_\ell \leq N(L + 1)$. Define now a similar set as in (3), but over equation (8),
\[ \bar{\Omega} = \{\omega \in \mathbb{R}_0^+ \mid g(j\omega, \bar{T}) = 0, \text{ for some } \bar{T} \in \mathbb{R}^L \}. \tag{9} \]

Corollary 1 ([7]): The identity $\Omega \equiv \bar{\Omega}$ holds.

Corollary 1 indicates that instead of finding $\Omega$ from the infinite dimensional equation (2), alternatively one can obtain $\Omega$ by finding $\bar{\Omega}$ from the algebraic equation (8). In the pursuit of finding $\bar{\Omega}$, CTCR builds a Routh’s array using the coefficients $Q_m(\bar{T})$. The entries of this array are parameters of $L$ different pseudo-delays $T_\ell$, and by exploiting the standard rules of the array, one can express the $s = j\omega$ roots of equation (8) in terms of $T_\ell$. PSSH in delay domain $\tau_\ell$ can then be computed using $\omega \in \Omega$ and $\bar{T}$ in (7).
2) Stability Analysis: Delay values found from (7) constitute the PSSH in $L$-dimensional delay space, $\vec{\tau} \in \mathcal{P}$. PSSH decomposes this delay space into regions where the system in (1) is either stable or unstable. Characterization of the stability regions stems from the well-known $\tau$-decomposition theorem [2], and can be easily implemented to detect which regions of the delay space are stable [7], as long as all the existing $\varphi$ are determined precisely and exhaustively.

III. ACFS Methodology

The objective of ACFS is stated as follows: compute the projections of PSSH, which are PSSC on any 2D delay plane, when the remaining delays are numerically fixed. To respond to this open problem, ACFS starts similar to CTCR, however, for the fixed delays it does not require the Rekasius substitution. This innocuous choice when combined with frequency sweeping and the resultant theory offers unmatched strength in revealing the PSSH, as demonstrated below.

A. Theoretical Construct of ACFS Methodology

Assumptions in our theoretical development are as follows:

1) It is assumed that $\omega = 0$ is not a root of (11). Degeneracy cases arising by relaxing this assumption can easily be adapted to the general framework of ACFS by following the work in [14].

2) Delays $\tau_1 = \bar{\tau}_1, \ldots, \tau_L = \bar{\tau}_L$ are given.

3) Frequency $\omega \in \mathbb{R}_+$ is a given sweep parameter.

4) ACFS presented below extracts the projections of PSSH on the $(\tau_1, \tau_2)$ domain by assuming, without loss of generality, $c_2 = \text{rank}(\mathbf{B}_2) \leq c_1 = \text{rank}(\mathbf{B}_1)$.

In light of the discussions and assumptions, the new characteristic function to be studied becomes,

\[
h(j\omega, T_1, T_2, e^{-j\bar{\tau}_1\omega}, \ldots, e^{-j\bar{\tau}_L\omega}) = \left( f(j\omega, \vec{\tau}) \right)^{c_2} \frac{1}{1+jT_{1,\omega}^2} \prod_{\ell=1}^{2}(1+jT_{\ell,\omega}^2)^{c_\ell}.
\]

where $h_R = \Re(h), h_\Im = \Im(h)$. For $\bar{\omega}$ to be a zero of (11), both $h_R$ and $h_\Im$ should be concurrently zero for some $(T_1, T_2)$. We can investigate those $(T_1, T_2)$ solutions from

\[
h_R = \sum_{i=0}^{c_2} a_i(T_1) T_{2i} = 0, \quad a_{c_2} \neq 0,
\]

and

\[
h_\Im = \sum_{i=0}^{c_2} b_i(T_1) T_{2i} = 0, \quad b_{c_2} \neq 0,
\]

where all $a_i$’s and $b_i$’s are real polynomials with respect to $T_1$.

We next define the resultant and the discriminant of two polynomials, and provide the resultant theorem for multivariable polynomials. We will see that these definitions are fundamental for ACFS construction.

Definition 1: Let $F_1$ and $F_2$ be two polynomials both in terms of $\mu$ and $\nu$. Resultant $R_{\nu} (R_{\mu})$ of $F_1$ and $F_2$ is computed via the determinant of Sylvester (or Bézout) matrix obtained by eliminating $\mu$ (or $\nu$). Then, the resultant of $R_{\mu}$ and $\partial R_{\mu}/\partial \nu$ (or $R_{\nu}$ and $\partial R_{\nu}/\partial \mu$) is called the discriminant of the polynomials $F_1$ and $F_2$ with respect to $\nu$ (or $\mu$) [26].

Theorem 1 ([27], [28]): If $(T_1, T_2)$ is a common zero of (12)-(13), then $R_{T_2}(h_R, h_\Im) = 0$. Conversely, if $R_{T_3}(h_R, h_\Im) = 0$, then at least one of the following four conditions holds: (i) $a_0(T_1) = \cdots = a_{c_2}(T_1) = 0$, (ii) $b_0(T_1) = \cdots = b_{c_2}(T_1) = 0$, (iii) $a_{c_2}(T_1) = b_{c_2}(T_1) = 0$, (iv) For some $(T_1, T_2)$ is a common zero of both (12) and (13).

Notice that solving the common zeros of (12)-(13) can be achieved through the resultant explained in Theorem 1. In the theorem, Case 2(iii) are degeneracy cases. Case 2(iv) is the general case to which we focus in the following. Once we finalize the coverage of ACFS, it will become clear that handling Case 2(i)-(iii) is trivial.

Theorem 2: In the general case Case 2(iv) in Theorem 1 and given $\bar{\omega} \in \Omega$, for the general control system (1), the number of points generating the kernel points on $\tau_1 - \tau_2$ plane is bounded by $2c_1c_2^2$: twice the product of larger commensurate degree and square of smaller commensurate degree associated with the delays defining the 2D delay plane.

Proof: Exploit the resultant theory to eliminate $T_2$ from (12) and (13). $R_{T_2}(h_R, h_\Im) = (a_{c_2})^{c_2} (b_{c_2})^{c_2} \prod_{i,k} (\beta_i - \gamma_k)$ where $\beta_i$ and $\gamma_k$ are the zeros of $h_R$ and $h_\Im$, respectively [19, pg. 398]. Since the maximum degree of $T_1$ in both $a_{c_2}$ and $b_{c_2}$ is $c_1$, we have $\deg(R_{T_2}(h_R, h_\Im)) = 2c_1c_2$ indicating that there can be at most $2c_1c_2$ number of $T_1$ solutions [29, pg. 147]. For each $T_1$ solution, $h_R$ and $h_\Im$ admit at most $c_2$ number of $T_2$ solutions, thus the maximum number of points generating the kernel points is $2c_1c_2^2$ with the fact that each $T \in \mathbb{R}^L$ solution point generates one solution as per (7) and the definition of $\mathcal{P}_{\text{kernel}}$. ■

Remark 1: For the case of $c_1 < c_2$, one can obtain the maximum number of kernel points as $2c_2c_1^2$. The measure $2c_1c_2^2$ ($c_2 \leq c_1$) defines how intricate the geometry of the stability maps can be. In other words, it is expected
that larger commensurate degrees conduce larger number of kernel curves, which may lead to more intricate stability maps. Notice that this complexity measure is not directly related to the system order $N$.

**Corollary 2:** Let all the $T_1$ roots of the resultant be represented by $\mathbf{V} = \{T_1 \in \mathbb{C} | R_{T_2}(h_R, h_\Omega) = 0, \forall \omega \in \Omega\}$, and let all $(T_1, T_2) \in \mathbb{R}^2$ roots of $h(j\omega, T_1, T_2)$ be defined by $\mathbf{V} = \{(T_1, T_2) \in \mathbb{R}^2 | h = 0, \forall \omega \in \Omega\}$. The projection of all the points in $\mathbf{V}$ onto real $T_1$ axis is a subset of $\mathbf{V}$, but not vice versa.

**Proof:** Proof follows from the fact that vanishing of $R_{T_2}(h_R, h_\Omega)$ is a necessary condition for $h_R$ and $h_\Omega$ to have common zeros.

Via Corollary 2, Theorem 2 and Definition 1, we can now present the theorem proving the non-conservative lower $\omega$ and upper bounds of $\Omega$ in (9) or equivalently $\Omega$ in (3).

**Theorem 3:** Minimum and maximum positive real roots of the discriminant of $h_R$ and $h_\Omega$ with respect to $\omega$, that correspond to $(T_1, T_2) \in \mathbb{R}^2$ solutions in (11) yield the exact lower and upper bounds of the crossing frequency set.

**Proof:** For the delay-dependent case, the existence of lower $\omega$ and upper $\bar{\omega}$ bounds of the CFS is well-known [1]. To find the global maximum $\bar{\omega}$ / global minimum $\omega$, maximize / minimize $\omega$ with respect to $\tau_1$, $\ell \in 1, 2$ by studying $\partial \omega / \partial \tau_1 = 0$, which is identical to studying

$$
\frac{\partial \omega}{\partial \tau_1} = \frac{\partial \omega}{\partial T_1} \frac{\partial T_1}{\partial \tau_1} = 0,
$$

where $\partial T_1 / \partial \tau_1 = 0.5 (1 + \omega^2 T_1^2) \neq 0$ as per (7). $R_{T_2}$ in $T_1$-space covers the common $\omega$ solutions of $h_R$ and $h_\Omega$ in $(T_1, T_2)$-space including endpoints, lower and upper bounds of the CFS [30]. Therefore, it is necessary and sufficient to maximize / minimize $\omega$ only with respect to $T_1$ by using $R_{T_2}$.

In light of Corollary 2, one then should study the following by invoking the implicit function theorem [31],

$$
\frac{\partial \omega}{\partial T_1} = \frac{-\partial R_{T_2}(h_R, h_\Omega)/\partial T_1}{\partial R_{T_2}(h_R, h_\Omega)/\partial \omega} = 0,
$$

which leads to $\partial R_{T_2}(h_R, h_\Omega)/\partial T_1 = 0$, assuming $\partial R_{T_2}(h_R, h_\Omega)/\partial \omega \neq 0$. Thus, one has two equations, $R_{T_2}$ and $\partial R_{T_2}/\partial T_1$, and they should be simultaneously zero. This requires to study the zeros of the resultant of these two equations, particularly by eliminating $T_1$. The resultant of $R_{T_2}$ and $\partial R_{T_2}/\partial T_1$ becomes only a function of $\omega$,

$$
Z(\omega) = R_{T_1}(R_{T_2}, \partial R_{T_2}/\partial T_1) = 0,
$$

which is the discriminant by Definition 1. The minimum and maximum positive real zeros of $Z(\omega)$ that correspond to $(T_1, T_2) \in \mathbb{R}^2$ solutions in (11) are the exact lower and upper bounds of the CFS, respectively.

**Remark 2:** In addition to $R_{T_2}(h_R, h_\Omega) = 0$ and $\partial R_{T_2}(h_R, h_\Omega)/\partial T_1 = 0$ conditions; if $\partial R_{T_2}(h_R, h_\Omega)/\partial \omega$ is also zero in (14), “singular points” occur. These types of singularities including double points, cusps, isolated points [30, 32] should be separately treated following the main spirit of solving (12)-(13) concurrently.

**Remark 3:** $\omega$ and $\bar{\omega}$ are the lower and upper bounds of the CFS on $\tau_1 - \tau_2$ domain since (10) is only a function of $T_1$ and $T_2$. We state that computation of precise lower and upper bounds of the CFS will significantly advance the frequency sweeping practice that has been increasingly preferred in the recent years. Currently, the methods based on frequency sweeping are conservative in the sense that they sweep $\omega$ to large numbers, and they are graphical-based in order to avoid missing out any $\omega$ solutions, see for instance [8], [15], [17] and the references therein. With the availability of the exact lower and upper bounds of the CFS, these computationally involved efforts will be reduced substantially.

We note that ACFS uniquely stands out from the existing approaches as it is able to accommodate fixed delays in its theoretical construct, as presented above. This ultimately allows one to extract the cross sectional views PSSC of the PSSH without the need for computing the PSSH. We finally note that ACFS can be seen as a nontrivial extension of the previous work in [5]. ACFS is able to achieve (with $L > 3$) nontrivially what the work in [5] achieved (with $L = 2$) by analyzing the stability of a MTDS along $\tau_1$ via fixing $\tau_2$.

**B. Algorithmic Construct of ACFS Methodology**

In the sequel, our new methodology ACFS is presented step by step. Notice that ACFS methodology does not impose any restrictions on $c_1, c_2, N$ and $L$. The steps of ACFS are presented in view of Assumption 4, and ACFS only requires frequency sweeping from the precise lower bound $\omega$ to the precise upper bound $\bar{\omega}$ that ACFS identifies via Theorem 3. For each $\omega \in [\omega, \bar{\omega}]$ with an appropriately chosen step size, perform the following steps:

1. Solve the polynomial equation $R_{T_2}(h_R, h_\Omega) = 0$ for $T_1 \in \mathbb{R}$ values.
2. For each $T_1 \in \mathbb{R}$ found from above, if $T_2 \in \mathbb{R}$ values exist satisfying $h_R = 0$ and $h_\Omega = 0$. Then proceed to the next step, otherwise increase $\omega$ by an amount of the step size, and restart from the step above.
3. Via (4) and (7), calculate the delay values $(\tau_1, \tau_2)$ corresponding to $(T_1, T_2) \in \mathbb{R}^2$ pairs, and restart from Step 1 increasing $\omega$ by an amount of the step size.

**Remark 4:** Existence of $T_2$ values in Step 2 depends on the roots of the greatest common divisor (gcd) of $h_R = 0$ and $h_\Omega = 0$ [29, pg. 162]. If any real root of gcd of $h_R = 0$ and $h_\Omega = 0$ for the computed $T_1 \in \mathbb{R}$ values exists, this root gives rise to admissible $T_2$ solutions, $T_2 \in \mathbb{R}$. There exists no real or complex common $T_2$ solutions when gcd is 1.

**Remark 5:** For the case when either all $a_i$’s or $b_i$’s are zero, a modification is needed in the above algorithm. This can be done by simply analyzing the $(T_1, T_2) \in \mathbb{R}^2$ solutions of $b_i$’s or $a_i$’s. Similarly, if $a_{c_2}$ (or $b_{c_2}$) becomes zero, then there exists a $T_2 \to \infty$ (or a $T_1 \to \infty$), and the treatment of this degeneracy follows from [33]. Next, step 3 of ACFS is to be revisited for finding the corresponding PSSC. These modification cover the degeneracies in Case 2(i)-(iii) of Theorem 1.

**Remark 6:** ACFS methodology can extract 3D stability maps by additionally sweeping $\tau_3$, [15].
IV. Case Studies

We first study a simpler yet nontrivial stability problem with \( N = 2 \) and \( L = 3 \). Then, a complicated case study with \( N = 4 \), \( L = 4 \), \( c_1 = 4 \) and \( c_2 = 2 \) is studied in order to demonstrate the capabilities of ACFS.

**Case 1:** Let the state matrices in (1) be

\[
A = \begin{bmatrix}
0 & 1 \\
-20.91 & -9.2
\end{bmatrix},
B_1 = \begin{bmatrix}
-0.968 & 0.01 \\
3.1 & -2.6
\end{bmatrix},
B_2 = \begin{bmatrix}
0 & 0 \\
0.127 & 5.86
\end{bmatrix},
B_3 = \begin{bmatrix}
0 & 0.26 \\
0.28 & -2.7
\end{bmatrix},
\]

where \( N = 2 \), \( L = 3 \), and the ranks of \( B_1 \) and \( B_2 \) are \( c_1 = 2 \) and \( c_2 = 1 \), respectively. Next, we implement the ACFS for arbitrarily chosen two \( \tau_3 \) delay values, \( \tau_3 = 1.5 \) and \( \tau_3 = 4.0 \). Following Theorem 3, corresponding frequency ranges are found as \( [\omega, \bar{\omega}]_{1.5} = [1.4774, 6.5821] \) and \( [\omega, \bar{\omega}]_{4.0} = [2.0241, 8.5652] \), which are computed in approximately 1 second. Upon sweeping \( \omega \) in these ranges, the PSSC are extracted, see Fig. 1 and Fig. 2.

It is worthy to note that identifying the PSSC in each one of these figures requires 28 seconds of computation time on average, and to the best of our knowledge, none of the existing techniques can extract the precise cross sections that ACFS can capture in Fig. 1 and Fig. 2.

The following step is the stability analysis, which commences with identifying the stability of the origin of the 2D delay space, \( \tau_1 = \tau_2 = 0 \). Using the technique in [4], the origin is found to be asymptotically stable independent of the values of \( \tau_3 \). This indicates that all the regions that can be connected to the origin with a continuous path without intersecting any PSSC are asymptotically stable. The stability features in the remaining regions are identified by computing the number of unstable roots \( NU \) of the system in these regions [7]. From these analyses, all the stable regions are identified and shaded; see Fig. 1 and Fig. 2.

**Remark 7:** There can also be multiple frequency ranges instead of a single range for MTDS, \( \Omega = \bigcup_{n=1}^{\infty} \Omega_n \). To the best of the authors' knowledge, situation \( n > 3 \) is observed for the first time in this example; \( n = 2 \) when \( \tau_3 = 1.5 \) and \( n = 5 \) when \( \tau_3 = 4.0 \), see Fig. 3. Moreover, in our case study, we observe that \( n \) increases as \( \tau_3 \) increases. Readers may consult [2] for studies on multiple number of admissible frequency ranges.

**Case 2:** The second example is with \( N = 4 \) and \( L = 4 \), where the state matrices in (1) are taken as

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-29.17 & -56 & -36.7 & -10.1
\end{bmatrix},
B_1 = \begin{bmatrix}
-1.55 & 1 & 0 & 0 \\
-1 & -0.3 & 0 & 0 \\
-0.7 & 0 & -0.34 & -2.6
\end{bmatrix},
B_2 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-0.33 & 0 & 0 & -1.1
\end{bmatrix},
B_3 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-0.08 & -0.7 & 0 & -1
\end{bmatrix},
\]

and \( B_4(4,3) = -3 \) with its remaining entries being zero; \( c_1 = 4 \) and \( c_2 = 2 \). Next, we arbitrarily choose \( \tau_3 = 0.169 \) and \( \tau_4 = 0.26 \), and implement the ACFS. We reveal that \( \omega \in [1.4004, 5.5849] \) from Theorem 3. Upon sweeping \( \omega \) in this range, the PSSC of the system is extracted, Fig. 4, where the kernel curve is shown in gray and the offspring curves are given in black (red and blue, respectively, when viewed in color). The stability region can be identified as the shaded region by means of [7]. We note that identifying the PSSC in Fig. 4 on average requires 40 seconds of computation time.

V. Concluding Remarks

A novel methodology, Advanced Clustering with Frequency Sweeping (ACFS), is proposed for studying the asymptotic stability of multiple time-delay systems (MTDS)...
in the parameter space of delays. Different from the existing methods, ACFS does not impose any restrictions on the number of delays and system dimension considered. By means of ACFS, potential stability switching curves (PSSC) on any 2D delay domain are extracted precisely and exhaustively. In addition to the asymptotic stability analysis of MTDS, maximum number of points generating the kernel points is studied, and a new formula which captures the precise lower and upper bounds of the crossing frequency set is presented. Case studies demonstrate the strengths and the capabilities of ACFS. Future work of this study involves the adaptation of ACFS methodology for the open problem of delay-independent stability test of MTDS [34], which is an extension of the single-delay cases studied in [35].

REFERENCES