LETTER

On Robust Approximate Feedback Linearization: A Nonlinear Control Approach

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SUMMARY In this letter, we consider a problem of global stabilization of a class of approximately feedback linearized systems. We propose a new nonlinear control approach which includes a nonlinear controller and a Lyapunov-based design method. Our new nonlinear control approach broadens the class of systems under consideration over the existing results.
key words: *approximate feedback linearization, global exponential stabilization, nonlinear approach*

1. Introduction

We consider a global exponential stabilization problem of a class of approximately feedback linearized systems. The approximate feedback linearization was originated in [5]. Since then, a large number of related studies have been reported [1]–[4], [7]–[9]. These existing results often assume some particular forms on the perturbed nonlinearity. In [4], [7], they propose control methods under the triangular form, which is also a linear growth condition. On the other hand, some feedforward forms are considered in [8], [9]. Then, in [1], a unified control approach is proposed to handle both triangular and feedforward forms by utilizing the gain factor. After that, some successive results are reported in [2] and [3], respectively. In [2], the main assumption of [1] is extended to an LMI condition so that the controller design becomes more flexible. In [3], a new analysis using a scaling factor is introduced to show that some mixture of triangular and feedforward forms are allowed.

Although there are certain improvements in [2] and [3] over [1], they basically provide some improved analysis and explore more hidden features of the controller in [1]. In essence, they basically use the same controller with different analysis. Thus, the fundamental limitations of the method of [1] such as the linear growth conditions have not been overcome, yet. In this letter, we propose a new nonlinear controller along with a Lyapunov-based design method so that we can actively treat some high-order nonlinearity, which are not done in [1]–[4], [7]. Moreover, all merits of [1]–[3] are still retained in our control method.

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2. Preliminaries

We consider a class of single-input nonlinear systems as

$$\dot{x} = Ax + Bu + \delta(t, x, u) \tag{1}$$

where $x \in R^n$ and $u \in R$ are the system state and input, respectively. The system matrices (A, B) are in a Brunovsky canonical pair and the nonlinearity is $\delta(t, x, u) = [\delta_1(t, x, u), \cdots, \delta_n(t, x, u)]^T$. First, we briefly address the relating results of [1]–[3]. The common feedback controller of [1]–[3] takes the following form.

$$u = K(\epsilon)x,\ K(\epsilon) = \left[\frac{k_1}{\epsilon^n}, \cdots, \frac{k_n}{\epsilon}\right],\ \epsilon > 0 \tag{2}$$

In [1], it is stated that the system (1) can be always globally exponentially stabilized with the controller (2) when $\delta(t, x, u)$ belongs to one of the following linear growth conditions.

Triangular form: For $i = 1, \cdots, n$, there exists a constant $L \geq 0$ such that $\delta_i(t, x, u) \leq L(|x_1| + \cdots + |x_i|)$.
Feedforward form: For $i = 1, \cdots, n-2$, there exists a constant $L \geq 0$ such that $\delta_i(t, x, u) \leq L(|x_{i+2}| + \cdots + |x_n|)$ with $\delta_{n-1}(t, x, u) = \delta_n(t, x, u) = 0$.

After [1], there have been analytical progresses which explore more features of the controller (2) in [2] and [3], respectively. However, all results of [1]–[3] are basically limited to the linear growth conditions. These limitations come from the fact that the controller (2) is a linear controller after all. In the next, we propose a new nonlinear controller with a Lyapunov-based design method to tackle some high-order nonlinearities, which generalizes the results of [1]–[3].

3. Main Result

First, we propose a nonlinear controller as

$$u = s(x)K(\epsilon)x \tag{3}$$

where $s(x) = 1+\alpha(x)$ and $\alpha(x) \geq 0, \forall x$, is a smooth function. Here, we define some notations to be used for convenience and simplicity.
Notations: $E_\epsilon = \mathrm{diag}[1, \epsilon, \cdots, \epsilon^{n-1}]$, $K = K(1)$, $A_K = A + BK$, $A_{K,x} = A + s(x)BK$, $A_{K,x,\epsilon} = A + s(x)BK(\epsilon)$, $X_\epsilon = E_\epsilon x$, I is an identity matrix, and $\star$ denotes appropriate entries for symmetric matrices.

We provide a Lyapunov-based design method as addressed in the following steps.

Design steps:

1. Select K such that A_K is Hurwitz.
2. Obtain P from $A_K^T P + PA_K = -I$.
3. Utilizing the relation of $A_K^T P + PA_K = -I$, select $P_x = P_x^T > 0$ from P such that $A_{K,x}^T P_x + P_x A_{K,x} = -\Gamma_x$ where $\Gamma_x = \Gamma_x^T > 0, \forall x$.
4. Choose $V(x) = x^T P_{x,\epsilon} x$ where $P_{x,\epsilon} = E_\epsilon P_x E_\epsilon$.

In design step 3, it may not be clear how to actually select P_x. Thus, we provide a particular guideline on the design of P_x in the following.

Case study on design step 3: We begin with a second-order case for easy understanding. For $n = 2$, we have

$$A_K = \begin{bmatrix} 0 & 1 \\ k_1 & k_2 \end{bmatrix}, \; P = \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} > 0 \tag{4}$$

From $A_K^T P + PA_K = -I$, we have

$$\begin{bmatrix} 2P_2 k_1 & P_1 + P_2 k_2 + P_3 k_1 \\ \star & 2(P_2 + P_3 k_2) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{5}$$

Then, we may select P_x from P with $s(x)$ as

$$P_x = \begin{bmatrix} s(x)P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \tag{6}$$

With a simple algebraic manipulation and using (5), we obtain

$$\begin{aligned} & A_{K,x}^T P_x + P_x A_{K,x} \\ & \quad = \begin{bmatrix} -s(x) & 0 \\ 0 & -1 + 2\alpha(x) P_3 k_2 \end{bmatrix} = -\Gamma_x \end{aligned} \tag{7}$$

where $P_3 k_2 < 0$. Thus, $\Gamma_x > 0, \forall x$.

Similarly, for $n = 3$, we may select P_x as

$$P_x = \begin{bmatrix} s(x)P_1 & s(x)P_2 & P_4 \\ s(x)P_2 & s(x)(P_3 + P_4) & P_5 \\ P_4 & P_5 & P_6 \end{bmatrix} \tag{8}$$

which results in

$$\Gamma_x = \begin{bmatrix} s(x) & 0 & 0 \\ 0 & s(x) & -P_4 \\ 0 & -P_4 & 1 - 2\alpha(x) P_6 k_3 \end{bmatrix} \tag{9}$$

where $P_6 k_3 < 0$. Thus, $\Gamma_x > 0, \forall x$.

Finally, for $n = 4$, we may select P_x as

$$P_x = \begin{bmatrix} s(x)P_1 & s(x)P_2 & s(x)P_4 & P_7 \\ s(x)P_2 & s(x)P_3 & s(x)P_5 & P_8 \\ s(x)P_4 & s(x)P_5 & s(x)(P_6 + P_8) & P_9 \\ P_7 & P_8 & P_9 & P_{10} \end{bmatrix} \tag{10}$$

We can easily check that we obtain $\Gamma_x > 0, \forall x$ with P_x in (10). From (6), (8), and (10), we can observe a pattern in designing P_x. Thus, we can design P_x analogously for $n \geq 5$ cases.

Remark 1. *Besides the guideline shown in the case study, there are other possibilities in selecting P_x. For example, a more general way for $n = 2$ case is to set as*

$$\begin{aligned} A_{K,x} &= \begin{bmatrix} 0 & 1 \\ s(x)k_1 & s(x)k_2 \end{bmatrix}, \\ P_x &= \begin{bmatrix} a_1(x)P_1 & a_2(x)P_2 \\ a_2(x)P_2 & a_3(x)P_3 \end{bmatrix} \end{aligned} \tag{11}$$

Then, one can choose different P_x by trying various combinations of $a_1(x)$ through $a_3(x)$.

Note that the actually engaged controller (3) contains a factor ϵ besides $s(x)$. Thus, for the stability analysis, we need the following lemma.

Lemma 1. *The following Lyapunov equation holds*

$$A_{K,x,\epsilon}^T P_{x,\epsilon} + P_{x,\epsilon} A_{K,x,\epsilon} = -\epsilon^{-1} E_\epsilon \Gamma_x E_\epsilon \tag{12}$$

if K and P_x are selected as in the design steps.

Proof. We already have

$$A_{K,x}^T P_x + P_x A_{K,x} = -\Gamma_x \tag{13}$$

Between $A_{K,x}$ and $A_{K,x,\epsilon}$, the following relation holds.

$$A_{K,x} = \epsilon E_\epsilon A_{K,x,\epsilon} E_\epsilon^{-1} \tag{14}$$

Then, substituting (14) into (13), we can easily obtain (12). □

On the perturbed nonlinearity $\delta(t, x, u)$, we assume the following.

Assumption 1. *There exist matrices $M_x = M_x^T$ and $N_x = N_x^T$ such that*

$$2x^T P_{x,\epsilon} \delta(t, x, u) \leq X_\epsilon^T M_x X_\epsilon \tag{15}$$

$$x^T \dot{P}_{x,\epsilon} x \leq X_\epsilon^T N_x X_\epsilon \tag{16}$$

with a suitable choice of $s(x)$.

Now, we state the main theorem.

Theorem 1. *Select K and P_x as shown in the design steps. Suppose that there exist $s(x)$ and ϵ such that $\epsilon^{-1}\Gamma_x > M_x + N_x$, $\forall x$ under Assumption 1. Then, with the controller (3), the origin of the system (1) is globally exponentially stable.*

Proof. Applying (3) to (1), the closed-loop system is

$$\dot{x} = A_{K,x,\epsilon} x + \delta(t, x, u) \tag{17}$$

Set $V(x) = x^T P_{x,\epsilon} x$. Then, along the trajectory of (17), we obtain the following inequality using Lemma 1 and Assumption 1.

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T P_{x,\epsilon} x + x^T P_{x,\epsilon} \dot{x} + x^T \dot{P}_{x,\epsilon} x \\ &= -\epsilon^{-1} x^T (E_\epsilon \Gamma_x E_\epsilon) x + 2x^T P_{x,\epsilon} \delta(t, x, u) \\ &\quad + x^T \dot{P}_{x,\epsilon} x \\ &\leq -X_\epsilon^T (\epsilon^{-1}\Gamma_x - M_x - N_x) X_\epsilon \end{aligned} \tag{18}$$

The global exponential stability is followed because of the quadratic Lyapunov function [6]. □

Remark 2. *If we simply set $\alpha(x) = 0$, then the controller (3) reduces to the controller (2). Thus, all the results of [1]–[3] are naturally contained in our result.*

4. Illustrative Example

Consider a system given by

$$\begin{aligned}\dot{x}_1 &= x_2 + x_1 \sin x_2^2 \\ \dot{x}_2 &= u + \theta(t) x_1^3\end{aligned} \tag{19}$$

where $\theta(t) \leq 1$, $\forall t$. The considered system (19) contains an uncertain nonlinear term which violates the linear growth condition in $\delta_2(t, x, u)$. Moreover, it does not belong to the feedforward forms in [8] and [9], respectively. Thus, the results of [1]–[4], [7]–[9] cannot treat the system (19). We apply our proposed method to the system (19) systematically as follows.

(i) With $K = [-1, -2]$, $P = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$.

(ii) With P_x as in (6),

$$\epsilon^{-1}\Gamma_x = \begin{bmatrix} \epsilon^{-1}(1 + \alpha(x)) & 0 \\ 0 & \epsilon^{-1}(1 + 2\alpha(x)) \end{bmatrix} \tag{20}$$

(iii) For (15), $2x^T P_{x,\epsilon}\delta(t, x, u) = 3s(x)x_1^2 \sin x_2^2 + \epsilon\theta(t)x_1^4 + \epsilon x_1 x_2 \sin x_2^2 + \epsilon^2\theta(t)x_1^3 x_2$, which results in

$$M_x = \begin{bmatrix} M_x(1,1) & M_x(1,2) \\ \star & 0 \end{bmatrix} \tag{21}$$

where $M_x(1,1) = 3s(x)\sin x_2^2 + \epsilon\theta(t)x_1^2$ and $M_x(2,1) = (\sin x_2^2 + \epsilon\theta(t)x_1^2)/2$.

(iv) Select $s(x) = 1 + \alpha(x) = 1 + 0.5x_1^2$.

(v) Then, for (16), $x^T \dot{P}_{x,\epsilon} x = x_1^3 x_2 + \epsilon x_1^3 x_2 \sin x_2^2$, which results in

$$N_x = \begin{bmatrix} 0 & 0.5(\epsilon^{-1}x_1^2 + x_1^2 \sin x_2^2) \\ \star & 0 \end{bmatrix} \tag{22}$$

(vi) Finally, we can check that there exists ϵ^* such that $\epsilon^{-1}\Gamma_x > M_x + N_x$, $\forall x$ for $0 < \epsilon < \epsilon^*$. Choosing $\epsilon = 0.25$ completes the controller design. The simulation result is shown in Fig. 1.

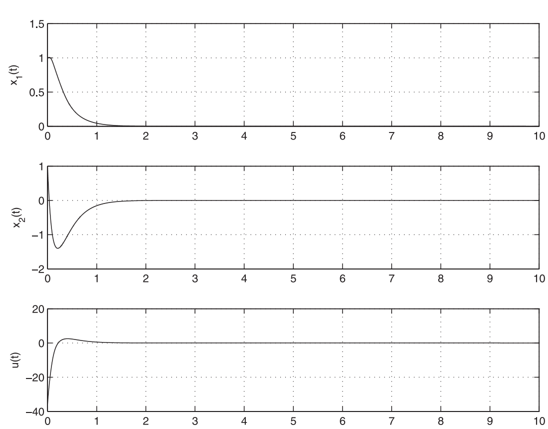

Fig. 1 System trajectories.

5. Conclusions

We have proposed a new nonlinear controller along with a Lyapunov-based design method. Through the analysis and example, we show that the proposed method is improved and generalized over the existing results. In particular, the restriction of the linear growth condition is now relaxed. Moreover, all merits of [1]–[3] are still retained.

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