Parametric Classes of Generalized Conjunction and Disjunction Operations for Fuzzy Modeling

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Abstract—It is argued that inference procedures of fuzzy models do not always require commutativity and associativity of the operations used. This raises the possibility of considering nonassociative and noncommutative conjunction and disjunction operations. Such operations are investigated in this paper and different methods for their generation are proposed. A number of new types of conjunction operations that are simpler than the known parametric classes of T-norms are given and, as an application example, the approximation of a function by a fuzzy inference system is considered.

Index Terms—Associativity, commutativity, conjunction, disjunction, fuzzy modeling, T-conorm, T-norm.

I. INTRODUCTION

AFTER the publication of the seminal paper of Zadeh [39], many different definitions of conjunction and disjunction operations have been introduced and studied [2], [6], [8]–[10], [12]–[15], [17], [21], [22], [27], [28], [34], [36]. In [6], the axiomatization of fuzzy operations is constructed by generalizing the properties of operations on usual sets and operations of binary logic. From these axioms it follows that min and max operations are the unique, possible conjunction and disjunction operations, respectively (see also [33]). As is shown in [2] and [34], this is because of the restrictions put on the possible forms of conjunction and disjunction operation by the distributivity property. The elimination of the latter from the set of axioms breaks the uniqueness of min and max operations and gives rise to the possibility of constructing a wide range of fuzzy connectives, based on the concepts of T-norm and T-conorm [2], [15], [19], [21], [27], [34]. The distributivity property is very important in fuzzy logic because it gives possibility to making equivalent transformations of logical forms from disjunctive to conjunctive forms and vice versa. In many practical applications of fuzzy models, however, such transformations are not needed and for this reason the distributivity property may be deleted from the system of axioms of fuzzy connectives.

The concepts of T-norm and T-conorm have come into the theory of fuzzy sets from the theories of functional equations and probabilistic metric spaces [1], [30]. The related axioms give possibility to build an infinite number of logical connectives. If the associativity property in the definition of T-norms and T-conorms is considered as a functional equation, then T-norms and T-conorms will be solutions of this equation and they may be generated by functions with one argument. A large amount of parametric classes of T-norms and T-conorms have been introduced in this way and they play an important role in fuzzy logic. But, from another point of view, the associativity property also appears as restrictive, because, in the construction of optimal fuzzy models, the parametric classes of T-norms and T-conorms constructed in such ways are generally far too complicated to render an easy hardware realization and optimization of the parameters [7], [37]. In this paper, it is proposed that as is done to the distributivity property, the commutativity and the associativity properties are also eliminated from the set of axioms of fuzzy connectives. This has the goal of reducing the limitations on the form of possible parametric classes of fuzzy connectives that may be used in fuzzy models. The properties of associativity and commutativity of fuzzy connectives are important, for example, in fuzzy models of multicriteria decision-making because one of the reasonable requirements laid on decision-making procedures is their nondependence on the order of consideration of criteria. But for fuzzy inference systems, these properties are not always necessary, especially when positions of variables in fuzzy rules are fixed.

The simplest fuzzy inference systems that have wide applications are based on the rules such as

\[ R_i; \text{IF } x \text{ is } A_i \text{ AND } y \text{ is } B_i \text{ THEN } z \text{ is } C_i, \]

\[ R_j; \text{IF } x \text{ is } A_j \text{ AND } y \text{ is } B_j \text{ THEN } z = f(x, y) \]

where \( A_i, B_i, C_i \) are some fuzzy sets, and \( f \) is some function [16], [18], [19], [23]–[25], [29]. For given crisp values of \( x^* \) and \( y^* \) the firing value \( \alpha \) of the rules is calculated as \( \alpha = T_1(\mu_{A_i}(x^*), \mu_{B_i}(y^*)) \), where \( T_1 \) is some T-norm representing connective AND and \( \mu_{A_i}(x^*), \mu_{B_i}(y^*) \) are the membership values. A conclusion of rules may then be calculated as \( \mu_{C_i}(z) = T_2(\alpha, \mu_{C_j}(z)) \), and \( z^* = T_2(\alpha, f(x^*, y^*)) \), respectively, where \( T_2 \) is a T-norm perhaps different from \( T_1 \). For the aggregation of conclusions of rules some disjunction operation (often some T-conorm) may be used. The construction of optimal fuzzy models is traditionally based on a tuning of membership functions of fuzzy sets used in the rules. When these membership functions are given parametrically then such tuning may be based on the optimization of these parameters.
The influence of fuzzy logic operations on the behavior of fuzzy systems and optimization of parameters of operations of fuzzy models have been studied theoretically and experimentally in many papers [3], [7], [16], [20], [26], [32], [35], [38]. The tuning or optimization of parametric operations may be done in addition to or instead of tuning of membership functions. But this approach would turn out to be somewhat complicated because of the complexity of existing parametric classes of \( T \)-norms and \( T \)-conorms used as conjunction and disjunction operations. Also, hardware realization of such complicated operations is not easy [37]. From this point of view it appears that simpler parametric classes of conjunction and disjunction operators would have many advantages.

It is easy to see that the associativity of conjunction operation is not required if only two variables in premises of rules and different conjunction operations \( T \) and \( S \) are used. In a more general case, when positions of variables in premises of rules and the procedure of calculation of firing values are fixed neither the commutativity nor the associativity of conjunction operations are needed. In this case, the conjunction of several arguments may be calculated sequentially in correspondence with given order of variables. Moreover, noncommutativity and nonassociativity may be desirable in some cases. For example, if \( x \) and \( y \) denote “error” and “change in error” correspondingly as in fuzzy control systems, then noncommutativity of conjunction operation may be used for taking into account different influences of these variables on the control process. So, if the commutativity of conjunction implies equality of rights of both operands, then the noncommutativity of conjunction with fixed positions of operands gives the possibility to build context dependent operations. We may propose also that parametric operations \( T_2 \) and \( T_1 \) may be “rule dependent” that gives possibility of separate tuning of parameters of these operations for rules related to different parts of control process, for example near points with maximal or zero errors and so on [29].

In Section II, the nomenclature is established by a review of the concepts of \( T \)-norm and \( T \)-conorm and the most popular examples of the parametric classes of \( T \)-norms are considered. In the following sections, the definition of noncommutative and nonassociative conjunction and disjunction operations is given and a number of different ways for generating new types of fuzzy connectives are proposed. Various examples of parametric classes of conjunction operations are then considered that are simpler than the parametric classes of \( T \)-norms reported in the literature. Finally, a simple example of fuzzy modeling based on the optimization of a parametric class of new operation is discussed.

II. \( T \)-NORMS AND \( T \)-CONORMS

\( T \)-norm and \( T \)-conorm are defined as functions \( T, S : [0, 1] \times [0, 1] \rightarrow [0, 1] \) satisfying the following properties:

(boundary condition)

\[
T(x, 1) = x \\
S(x, 0) = x
\]

(monotonicity)

\[
T(x, y) \leq T(u, v) \quad \text{and} \quad S(x, y) \leq S(u, v)
\]

if \( x \leq u, y \leq v \)

(commutativity)

\[
T(x, y) = T(y, x) \\
S(x, y) = S(y, x)
\]

(associativity)

\[
T(T(x, y), z) = T(x, T(y, z)) \\
S(S(x, y), z) = S(x, S(y, z))
\]

A negation is defined [31] as a function \( \overline{N} : [0, 1] \rightarrow [0, 1] \) satisfying the properties: \( \overline{N}(0) = 1, \overline{N}(1) = 0, \) and \( \overline{N}(x) \leq \overline{N}(y) \) if \( y \leq x \). A negation is called an involution if on \( [0, 1] \) it is fulfilled involutivity property \( \overline{N}(N(x)) = x \). Parametric class of Sugeno involutive negations has the form: \( \overline{N}(x) = (1 - x)/(1 + \lambda x), \) \( \lambda > -1 \). When \( \lambda = 0 \) we obtain the negation of Zadeh [39]: \( \overline{N}(x) = 1 - x \). Noninvolutive negations are studied in [4], [5], and [11].

\( T \)-norms and \( T \)-conorms can be obtained one from another as follows:

\[
T(x, y) = N(T(N(x), N(y))) \\
S(x, y) = N(S(N(x), N(y)))
\]

where \( N \) is an involution. The simplest examples of \( T \)-norms and \( T \)-conorms mutually related by these relations for \( N(x) = 1 - x \) are the following:

\[
T_c(x, y) = \min\{x, y\} \\
S_c(x, y) = \max\{x, y\} \\
T_p(x, y) = xy \\
S_p(x, y) = x + y - xy \\
T_d(x, y) = \begin{cases} 
  x, & \text{if } y = 1 \\
  y, & \text{if } x = 1 \\
  0, & \text{otherwise}
\end{cases} \\
S_d(x, y) = \begin{cases} 
  x, & \text{if } y = 0 \\
  y, & \text{if } x = 0 \\
  1, & \text{otherwise}
\end{cases} \\
T_i(x, y) = \max\{0, (x + y - 1)\} \\
S_i(x, y) = \min\{1, (x + y)\}.
\]

These simplest functions will later be used for the construction of parametric conjunction and disjunction operations. Generally, for any \( T \)-norm and \( T \)-conorm it follows that

\[
T_d(x, y) \leq T(x, y) \leq T_c(x, y) \leq S_c(x, y) \leq S_d(x, y).
\]

Hence, \( T \)-norms \( T_d \) and \( T_c \) are the minimal and the maximal boundaries for all \( T \)-norms (see Fig. 1). Similarly \( T \)-conorms \( S_c \) and \( S_d \) are the minimal and the maximal boundaries for all \( T \)-conorms (see Fig. 2). These inequalities are very important from a practical point of view because they establish the boundaries of the possible range of operations \( T \) and \( S \).

\( T \)-norms and \( T \)-conorms, as functions satisfying the associativity property, can be generated by generators of several types [1], [2], [15], [19], [21], [30]. For example, \( T \)-norms may
be generated as follows: $T(x, y) = \varphi^{-1}(\varphi(x)\varphi(y))$, where $\varphi$ is any increasing bijection (automorphism) $\varphi: [0, 1] \to [0, 1]$ with $\varphi(0) = 0$ and $\varphi(1) = 1$ and $\varphi^{-1}$ is the inverse of $\varphi$. $T$-norm may be generated by means of another $T$-norm $T_0$ as follows: $T(x, y) = \varphi^{-1}(T_0(\varphi(x), \varphi(y)))$. Recent discussions of methods of generation of $T$-norms and $T$-conorms may be particularly found in [17] and [27]. Parametric classes of $T$-norms are generally complicated due to the necessity of using inverse functions for their construction. Below one can find examples of parametric classes of $T$-norms varying from $T_d$ to $T_c$.

Two simplest parametric classes of $T$-norms of Schweizer and Sklar have the following form:

$$T(x, y) = \max(0, x^{-p} + y^{-p} - 1)^{-1/p},$$

$$T(x, y) = 1 - [(1 - x)^p + (1 - y)^p - (1 - x)(1 - y)]^{1/p}.\right.$$  

The behavior of a class of two-rule systems where the implication operator is formed by the first parametric class of $T$-norms is studied in [35]. Application of the second class of $T$-norms to optimal fuzzy modeling is discussed in [32].

Yager has introduced the following popular class of $T$-norms:

$$T(x, y) = 1 - \min(1, ((1 - x)^p + (1 - y)^p)^{1/p}), \quad p > 0.$$  

These operations are used in [20] for the construction of additive hybrid operators considered as nodes in network structures for decision making and a training algorithm based on gradient descent method for this network is developed.

The class of the following $T$-norms is proposed by Dombi:

$$T(x, y) = 1/(1 + ((1/x - 1)^p + (1/y - 1)^p)^{1/p}), \quad p > 0.$$  

The tuning of these parametric operators in fuzzy modeling by means of gradient descent optimization method is considered in [7].

III. GENERALIZED CONJUNCTION AND DISJUNCTION OPERATIONS

Generalizations of the concepts of fuzzy conjunction and disjunction operators are considered in several papers. Here we consider a generalization of these operations with the aim of constructing parametric classes of these operations.

**Definition 1**: A conjunction operation and a disjunction operation are functions $T, S: [0, 1] \times [0, 1] \to [0, 1]$ satisfying the following properties:

$$T(x, 1) = T(1, x) = x,$$

$$S(x, 0) = S(0, x) = x,$$

$$T(x, y) \leq T(u, v)$$

and

$$S(x, y) \leq S(u, v) \quad \text{if} \quad x \leq u, \ y \leq v.$$  

Naturally any $T$-norm and $T$-conorm will be a conjunction and a disjunction with respect to this definition. Below $T_c$,}
\( S_T, T_b, S_d \) will denote the simplest \( T \)-norms and \( T \)-conorms considered above. It is easy to prove the following properties of fuzzy connectives.

**Proposition 1:** Conjunction and disjunction operations satisfy the following properties:

\[
\begin{align*}
T(0, y) &= T(x, 0) = 0 \\
S(1, y) &= S(x, 1) = 1 \\
T_2(x, y) &\leq T(x, y) \leq T_c(x, y) \leq S_2(x, y) \\
&\leq S(x, y) \leq S_d(x, y).
\end{align*}
\]

(3)

(4)

From (1)–(4) we obtain:

\[
\begin{align*}
T(0, 0) &= T(0, 1) = T(1, 0) = 0, \quad T(1, 1) = 1 \\
S(0, 0) = 0, \quad S(0, 1) = S(1, 0) = S(1, 1) = 1.
\end{align*}
\]

(5)

We note that our definition of conjunction and disjunction operation coincides with the definition of \( T \)-seminorm and \( T \)-semiconorm on partially ordered sets in [8]. Conjunction operation belongs to the class of weak \( T \)-norms [12]. The properties (2) and (5) are considered in [22] as axioms of axiomatic skeleton for fuzzy sets intersection and union. Most general definition of fuzzy conjunction defined only by the properties (5) for \( T \) is considered in [13]. Nonstandard conjunctions are discussed also in [14].

It is easy to prove the following statement establishing the ways for generation disjunctions from conjunctions and vice versa.

**Theorem 1:** Suppose \( N \) is an involutive negation on \([0, 1]\) and \( T, S \) are some conjunction and disjunction, then the following relations define correspondingly disjunction and conjunction functions:

\[
\begin{align*}
S_T(x, y) &= N(T(N(x), N(y))) \\
T_S(x, y) &= N(S(N(x), N(y))).
\end{align*}
\]

(6)

It follows from (6) and from involutivity of \( N \) that for any \( T \) and \( S = S_T \) and similarly for any \( S \) and \( T = T_S \) the following De Morgan laws are fulfilled:

\[
\begin{align*}
N(S(N(x, y))) &= T(N(x), N(y)) \\
N(T(N(x, y))) &= S(N(x), N(y)).
\end{align*}
\]

IV. **Generation of Conjunction and Disjunction Operations**

Below we introduce two functions that will be used for generating conjunction and disjunction operations.

**Definition 2:** The functions \( t, s : [0, 1] \times [0, 1] \rightarrow [0, 1] \) satisfying the following properties:

\[
\begin{align*}
t(0, y) &= t(x, 0) = 0 \\
s(1, y) &= s(x, 1) = 1 \\
t(x, y) &\leq t(u, v) \\
s(x, y) &\leq s(u, v) \quad \text{if } x \leq u, \ y \leq v
\end{align*}
\]

(7)

and

(8)

will be called a pseudoconjunction and a pseudodisjunction, respectively.

It is evident that any \( T \)-norm (\( T \)-conorm) will be a conjunction (disjunction) and any conjunction (disjunction) will be a pseudoconjunction (pseudodisjunction).

**Theorem 2:** Suppose \( T_1, T_2 \) are conjunctions, \( t \) is a pseudoconjunction, \( S_1 \) and \( S_2 \) are disjunctions and \( s \) is a pseudodisjunction, then the following functions:

\[
\begin{align*}
T_3(x, y) &= T_2(T_1(x, y), s(x, y)) \\
S_3(x, y) &= S_2(S_1(x, y), t(x, y)) \\
T_4(x, y) &= T_2(s(x, y), T_1(x, y)) \\
S_4(x, y) &= S_2(t(x, y), S_1(x, y))
\end{align*}
\]

(9)

(10)

will be conjunctions and disjunctions, respectively.

**Proof:**

\[
\begin{align*}
T_3(x, 1) &= T_2(T_1(x, 1), s(x, 1)) = T_2(x, 1) = x \\
T_3(1, y) &= T_2(T_1(1, y), s(1, y)) = T_2(y, 1) = y.
\end{align*}
\]

A monotonicity of \( T_3 \) follows from the monotonicity of \( T_1, T_2, \) and \( s. \) Similarly we can show that \( T_4 \) is also a conjunction and \( S_3, S_4 \) are disjunctions.

Conjunctions (9) and (10) have the following properties.

**Proposition 2:**

\[
\begin{align*}
T(T_b, s) &= T_d \quad \text{for all conjunctions } T \text{ and pseudodisjunctions } s. \\
T_d(T, s) &= T_d \quad \text{for all conjunctions } T \text{ and pseudodisjunctions } s \text{ such that } s(x, y) < 1 \text{ if } x, y < 1 \text{; } \quad T(T_c, S_c) = T \text{ for all conjunctions } T \text{ and disjunctions } S; \\
T(T_r, S_r) &= T \text{ for all commutative conjunctions } T; \\
T_b(T, S) &= T_b \quad \text{for all pairs } (T, S) \text{ of operators } (T_c, S_c), \quad (T_p, S_p), \text{ and } (T_b, S_b).
\end{align*}
\]

**Proof:** From (7) and Theorem 2 it follows that it is sufficient to consider the cases when \( x, y < 1. \) Then we have \( T(T_d(x, y), s(x, y)) = T(0, s(x, y)) = 0 = T_d(x, y). \)

\[
T_d(T(x, y), s(x, y)) = 0 = T_d(x, y) \quad \text{since } T(x, y) \leq \min(x, y) < 1 \text{ and } s(x, y) < 1 \text{ for } x, y < 1. \quad \text{From (4) we have } T_c(T(x, y), S(x, y)) = \min(T(x, y), S(x, y)) = T(x, y). \text{ From commutativity of } T \text{ it follows that } T(T_c(x, y), S(x, y)) = T(x, y). \text{ Let us show that } T_b(T(x, y), S(x, y)) = \max(0, T(x, y) + S(x, y) - 1) = T_b(x, y). \text{ It is sufficient to show that } T(x, y) + S(x, y) = x + y \text{ for all considered pairs of } T, S \text{ operators.}
\]

\[
\begin{align*}
T_c(x, y) + S_c(x, y) &= \max(0, x + y) \\
T_p(x, y) + S_p(x, y) &= xy + x + y - xy = x + y \\
T_b(x, y) + S_b(x, y) &= \max(0, x + y - 1) + \min(1, x + y) = x + y \\
\end{align*}
\]

for both possibilities \( x + y < 1 \) and \( x + y \geq 1. \)

Similarly, it can be shown that disjunctions (9) and (10) have the following properties.

**Proposition 3:**

\[
\begin{align*}
S(t, S_d) &= S_d \quad \text{for all pseudoconjunctions } t \text{ and disjunctions } S;
\end{align*}
\]
for all disjunctions \( S \) and all pseudoconjunctions \( t \) such that \( t(x, y) > 0 \) if \( x, y > 0 \);
\[
S_c(T, S) = S \quad \text{for all conjunctions } T \text{ and disjunctions } S;
\]
\[
S(T_c, S_c) = S \quad \text{for all commutative disjunctions } S;
\]
\[
S_b(T, S) = S_b \quad \text{for all pairs } (T, S) \text{ of } T \text{-operators } (T_c, S_c),
\]
\[
(T_p, S_p), \text{ and } (T_b, S_b).
\]
As follows from Theorem 2, we can build conjunction and disjunction operations from well-known \( T \)-norms and \( T \)-conorms using them as (pseudo) conjunction and (pseudo) disjunction operations. But for obtaining new operators we must take into account Propositions 2 and 3. For example, starting from \( T_c, T_p, S_p \) we can get the following commutative conjunction and disjunction operations related to each other by De Morgan laws [with \( N(x) = 1 - x \)]:
\[
T(x, y) = (x + y - xy) \min(x, y)
\]
\[
S(x, y) = \max(x, y) + xy - \max(x, y)xy
\]
\[
T(x, y) = \max(x, y)xy
\]
\[
S(x, y) = \min(x, y) + x + y - xy
\]
\[
T(x, y) = xy(x + y - xy)
\]
\[
S(x, y) = x + y - xy(x + y - xy).
\]
For obtaining more interesting parametric classes of conjunction and disjunction operations we may use in (9) and (10) pseudoconnectives different from \( T \)-norms and \( T \)-conorms.

Proposition 4: Suppose \( N \) is any negation on \([0, 1]\) and \( t, s \) are some pseudoconjunctive and pseudodisjunctive then the following relations define correspondingly pseudodisjunction and pseudoconjunction functions:
\[
s_t(x, y) = N(t(N(x), N(y)))
\]
\[
t_s(x, y) = N(s(N(x), N(y))).
\]
Proof: \( s_t(1, x) = N(N(t(N(x), N(y)))) = N(t(N(x), 0)) = N(0) = 1 \). Similarly, we obtain \( s_t(1, x) = 1 \). Since \( t \) is monotonically increasing on both arguments and \( N \) is monotonically decreasing function we get the monotonicity property of \( s_t \). Proof for \( t_s \) is similar.

We note that if in (11) the negation \( N \) is an involution then pseudoconnectives \( t \) and \( s = s_t \), \( s \) and \( t = t_s \) will be mutually related by De Morgan laws. We will consider the following pairs of pseudoconnectives mutually related with negation \( N(x) = 1 - x \):
\[
t_p(x, y) = 0, \quad \text{for all } x, y \in [0, 1]
\]
\[
s_p(x, y) = 1, \quad \text{for all } x, y \in [0, 1]
\]
\[
t_p(x, y) = 0, \quad \text{if } x = 0
\]
\[
0, \quad \text{if } y = 0
\]
\[
1, \quad \text{otherwise}
\]
\[
s_p(x, y) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } y = 1 \\ 0, & \text{otherwise} \end{cases}
\]
\[
t_p(x, y) = \begin{cases} 0, & \text{if } y = 0 \\ 1, & \text{otherwise} \end{cases}
\]
\[
s_p(x, y) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } y = 1 \\ 0, & \text{otherwise} \end{cases}
\]
\[
t_p(x, y) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } y = 1 \\ 0, & \text{otherwise} \end{cases}
\]
\[
s_p(x, y) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } y = 1 \\ 0, & \text{otherwise} \end{cases}
\]
It is easy to see that for all pseudconjunctions \( t \) and pseudodisjunctions \( s \) the following inequalities are fulfilled:
\[
t_p(x, y) \leq t(x, y) \leq t_d(x, y)
\]
\[
s_p(x, y) \leq s(x, y) \leq s_b(x, y).
\]
Any pseudconjunctive \( t \) differs from any pseudodisjunction \( s \) at least in two points \((0, 1)\) and \((1, 0)\) because of the following:
\[
t(0, 1) = t(1, 0) = 0, \quad s(0, 1) = s(1, 0) = 1.
\]

Theorem 3: Suppose \( s \) is some parametric pseudodisjunction varying from \( s_B \) to \( s_B \) and \( T_1 \) is an arbitrary conjunction, then by means of any conjunction \( T_2 \) applying (9) and (10) we can generate conjunctions varying from \( T_2 \) to \( T_1 \).

Proof: From (9) we have \( T_2(T_1(x, y), s_B(x, y)) = T_2(T_1(x, y), 1) = T_1(x, y) \). Denote \( T(x, y) = T_2(T_1(x, y), s_B(x, y)) \). If \( x = 1 \) then we have \( T(1, y) = T_2(T_1(1, y), s_B(1, y)) \). Hence, \( T = T_d \), and we obtain by (9) \( T_2 \) and \( T_1 \) when \( s = s_B \) and \( s = s_p \), respectively. Suppose by varying parameter in \( s \) one can build pseudonections \( s_a \) and \( s_b \) such that \( s_a \leq s_b \). Denote conjunctions obtained by (9) on the base of \( s_a \) and \( s_b \) as \( T_a \) and \( T_b \), respectively. Then we have \( s_D \leq s_a \leq s_b \) and from monotonicity of all functions in (9) we obtain \( T_d \leq T_a \leq T_b \leq T_1 \).

The proof of the theorem for (10) is similar.

As follows from this theorem, if we can build parametric class of pseudodisjunctions \( s \), varying from \( s_B \) to \( s_B \), then applying \( s \) and \( T_1 = T_2 \) in (9) or (10) we can vary conjunctions in all possible ranges between \( T_2 \) and \( T_1 \). Of course the types of conjunctions that will be generated between \( T_2 \) and \( T_1 \) depend on the form of \( s \) and \( T_2 \).

As for obtaining more interesting parametric classes of conjunction and disjunction operations we may use in (9) and (10) pseudoconnectives different from \( T \)-norms and \( T \)-conorms.

Proposition 4: Suppose \( N \) is any negation on \([0, 1]\) and \( t, s \) are some pseudoconjunctive and pseudodisjunctive then the following relations define correspondingly pseudodisjunction and pseudoconjunction functions:
\[
s_t(x, y) = N(t(N(x), N(y)))
\]
\[
t_s(x, y) = N(s(N(x), N(y))).
\]
Proof: \( s_t(1, x) = N(N(t(N(x), N(y)))) = N(t(N(x), 0)) = N(0) = 1 \). Similarly, we obtain \( s_t(1, x) = 1 \). Since \( t \) is monotonically increasing on both arguments and \( N \) is monotonically decreasing function we get the monotonicity property of \( s_t \). Proof for \( t_s \) is similar.

We note that if in (11) the negation \( N \) is an involution then pseudoconnectives \( t \) and \( s = s_t \), \( s \) and \( t = t_s \) will be mutually related by De Morgan laws. We will consider the following pairs of pseudoconnectives mutually related with negation \( N(x) = 1 - x \):
\[
t_p(x, y) = 0, \quad \text{for all } x, y \in [0, 1]
\]
\[
s_p(x, y) = 1, \quad \text{for all } x, y \in [0, 1]
\]
\[
t_p(x, y) = 0, \quad \text{if } x = 0
\]
\[
0, \quad \text{if } y = 0
\]
\[
1, \quad \text{otherwise}
\]
\[
s_p(x, y) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } y = 1 \\ 0, & \text{otherwise} \end{cases}
\]
\[
t_p(x, y) = \begin{cases} 0, & \text{if } y = 0 \\ 1, & \text{otherwise} \end{cases}
\]
\[
s_p(x, y) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } y = 1 \\ 0, & \text{otherwise} \end{cases}
\]
\[
t_p(x, y) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } y = 1 \\ 0, & \text{otherwise} \end{cases}
\]
\[
s_p(x, y) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } y = 1 \\ 0, & \text{otherwise} \end{cases}
\]
It is easy to see that for all pseudconjunctions \( t \) and pseudodisjunctions \( s \) the following inequalities are fulfilled:
\[
t_p(x, y) \leq t(x, y) \leq t_d(x, y)
\]
\[
s_p(x, y) \leq s(x, y) \leq s_b(x, y).
\]
Any pseudconjunctive \( t \) differs from any pseudodisjunction \( s \) at least in two points \((0, 1)\) and \((1, 0)\) because of the following:
\[
t(0, 1) = t(1, 0) = 0, \quad s(0, 1) = s(1, 0) = 1.
\]
will be pseudoconjunctions and pseudodisjunctions, respectively.

Proof: From \( f_1(0) = f_2(0) = 0 \) and from fulfillment of (7) for \( t_1 \) and \( t_2 \), we obtain fulfillment of (7) for functions \( t \) in (12)–(15). The monotonicity of functions \( t \) follows from the monotonicity of \( t_1, t_2, f_1, f_2, \) and \( h \). The proof for pseudodisjunctions is similar.

We note that due to possible noncommutativity of functions \( f_1, f_2, g_1, \) and \( g_2 \), the functions (14) and (15) may be different.

Multiple recursive application of (12)–(15) gives possibility to build various pseudo-conjunctions and pseudodisjunctions and then, by means of Theorem 2 and Proposition 4, various conjunctions and disjunctions may be constructed.

Functions \( f \) and \( g \) defined in Proposition 5 will be called \( f \)-generators and \( g \)-generators, respectively. It is easy to see that by means of any negation \( N \) we can obtain from \( f \)-generator some \( g \)-generator and vice versa

\[
g(x) = N(f(N(x)), \quad f(x) = N(g(N(x)).
\]

For example, by using (9) and (12) we may obtain the following conjunction:

\[
T(x, y) = T_2(T_1(x, y), s(g_1(x, p_1), g_2(y, p_2)))
\]

where \( T_2, T_1 \) some \( T \)-norms, \( s \) is a \( T \)-conorm and \( g_1(x, p_1), g_2(y, p_2) \) some generators dependent on parameters \( p_1, p_2 \). For obtaining more or less simple parametric classes of conjunctions we may choose \( T_2, T_1 \) between \( T_c, T_p, T_d, T_i \), choose \( s \) between \( S_c, S_p, S_d, S_i \), and use simple functions \( g_1 \) and \( g_2 \).

In what follows, mainly conjunction operations are considered. Corresponding disjunction operations can be obtained dually or from conjunction operations by means of negation operations.

Let us consider the following generators:

\[
\begin{align*}
 f_B(x) &= 0, \quad \text{for all } x \in [0, 1] \\
 g_B(x) &= 1, \quad \text{for all } x \in [0, 1] \\
 f_D(x) &= \begin{cases} 
 0, & \text{if } x = 0 \\
 1, & \text{otherwise}
\end{cases} \\
 g_D(x) &= \begin{cases} 
 1, & \text{if } x = 1 \\
 0, & \text{otherwise}.
\end{cases}
\end{align*}
\]

For any \( f \)- and \( g \)-generators we have

\[
f_B(x) \leq f(x) \leq f_D(x), \quad \text{and} \quad g_B(x) \leq g(x) \leq g_D(x).
\]

Taking into account that the following functions

\[
f_x(x) = x, \quad g_x(x) = x
\]

are also generators, we can change in (12)–(15) generators and functions \( h \) by their arguments.

It is easy to see that by substituting \( t_1 = T \) and \( s_1 = S \) in (12) we obtain for arbitrary conjunction \( T \) and disjunction \( S \) and for any generators \( f \) and \( g \) the following relations:

\[
\begin{align*}
 T(f_B(x), f(y)) &= T(f(x), f_B(y)) = t_B(x, y) \\
 S(g_B(x), g(y)) &= S(g(x), g_B(y)) = s_B(x, y) \\
 T(f_D(x), f(y)) &= t_D(x, y) \\
 S(g_D(x), g(y)) &= s_D(x, y) \\
 T(f_D(x), y) &= t_D(x, y) \\
 S(g_D(x), y) &= s_D(x, y) \\
 T(f_2(x), y) &= t_2(x, y) \\
 S(g_2(x), y) &= s_2(x, y) \\
 T(x, y) &= t_2(x, y) \\
 S(x, y) &= s_2(x, y).
\end{align*}
\]

Taking these relations into account we obtain the following result from Theorem 3.

Theorem 5: Suppose \( T_1 \) and \( T_2 \) are arbitrary conjunctions, \( S \) is an arbitrary disjunction, \( g_1 \) and \( g_2 \) are some parametric classes of \( g \)-generators such that one of them varies from \( g_D \) to \( g_B \) and another from \( g_D \) to some \( g^* \), then by means of relation

\[
T(x, y) = T_2(T_1(x, y), S(g_1(x), g_2(y)))
\]

we can generate conjunctions varying from \( T_d \) to \( T_1 \).

VI. EXAMPLES OF PARAMETRIC CLASSES OF CONJUNCTIONS

Example 1: We can propose the following parametric classes of threshold dependent generators:

\[
\begin{align*}
 f(x, p) &= \begin{cases} 
 0, & \text{if } x \leq p \\
 1, & \text{otherwise}
\end{cases} \\
 g(x, p) &= \begin{cases} 
 1, & \text{if } x \geq p \\
 0, & \text{otherwise}, \quad p \in [0, 1]
\end{cases}
\end{align*}
\]

with the following properties:

\[
\begin{align*}
 f_B(x) &= f(x, 1) \leq f(x, p) \leq f(x, 0) = f_D(x) \\
 g_D(x) &= g(x, 1) \leq g(x, p) \leq g(x, 0) = g_B(x).
\end{align*}
\]

For any \( T \) and \( S \) we have

\[
\begin{align*}
 T(f(x, p), f(y, q)) &= \begin{cases} 
 0, & \text{if } x \leq p \text{ or } y \leq q \\
 1, & \text{otherwise}
\end{cases} \\
 S(g(x, p), g(y, q)) &= \begin{cases} 
 1, & \text{if } p \leq x \text{ or } q \leq y \\
 0, & \text{otherwise}.
\end{cases}
\end{align*}
\]

Applying in (16) \( T_1 = T_c \) and generators \( g(x, p) \) and \( g(y, q) \) for arbitrary \( T_2 \) and \( S \), we obtain the following conjunction operation:

\[
T(x, y) = \begin{cases} 
 \min(x, y), & \text{if } p \leq x \text{ or } q \leq y \\
 0, & \text{otherwise}
\end{cases}
\]

and particularly \( T = T_d \) when \( p = 1, q = 1 \), and \( T = T_c \) when \( p = 0 \) or \( q = 0 \). The graph of this conjunction for \( p = 0.4, q = 0.8 \) is shown on Fig. 3.
Example 2: Let us consider the parametric classes of linear generators:

\[ f(x, p) = \min(px, 1) \]
\[ g(x, p) = \max(1 - p(1 - x), 0), \quad p \geq 0. \]

We have for these generators

\[ f_{B}(x) = f(x, 0) \leq f(x, p) \leq f(x, \infty) = f_{D}(x), \]
\[ g_{D}(x) = g(x, \infty) \leq g(x, p) \leq g(x, 0) = g_{B}(x) \]

where \( f(x, \infty) = \lim f(x, p), \ p \to \infty, \) and \( g(x, \infty) = \lim g(x, p), \ p \to \infty. \)

Applying in (16) \( T_{1} = T_{c}, \ T_{2} = T_{p}, \) and \( S = S_{c} \) we obtain the conjunction

\[ T(x, y) = \min(x, y) \max\{1 - f(1 - x), 1 - g(1 - y), 0\} \]

with \( T = T_{c} \) when \( p = 0 \) and \( T = T_{d} \) when \( p, q \to \infty. \) The graphs of this conjunction for \( p = 1.2, \ q = 4 \) and for \( p = 2 \) and \( q = 4 \) are shown on Fig. 4.

Example 3: Let us consider the parametric classes of power generators

\[ f(x, p) = x^{p} \]
\[ g(x, p) = x^{p}, \quad p \geq 0 \]

where we suppose that \( 0^{p} = 0 \) for all \( p > 0, \ f(0, 0) = 0. \ f(1, \infty) = 0, \) but \( g(0, 0) = 1, \ g(1, \infty) = 1. \) Then we have

\[ f_{B}(x) = f(x, \infty) \leq f(x, p) \leq f(x, 0) = f_{D}(x) \]
\[ g_{D}(x) = g(x, \infty) \leq g(x, p) \leq g(x, 0) = g_{B}(x). \]

By means of these generators we can build many conjunctions with attractive features. For example from Theorem 5 it follows that applying these generators in (16) with \( T_{1} = T_{c} \) we will obtain parametric classes of conjunctions varying from \( T_{c} \) (when \( p, q \to 0 \)) to \( T_{d} \) (when \( p, q \to \infty). \)

Example 3.1: For \( T_{2} = T_{c} \) and \( S = S_{c} \) we will obtain the following conjunction:

\[ T(x, y) = \min\{\min(x, y), \max(x^{p}, y^{q})\}. \]

Example 3.2: For \( T_{2} = T_{p} \) and \( S = S_{c} \) we will obtain another conjunction

\[ T(x, y) = \min(x, y) \max(x^{p}, y^{q}). \]

The graphs of this conjunction for \( p = 1.2, \ q = 4, \) and
Fig. 6. Conjunction in Example 3.2. (a) For $p = 1.2, q = 4$. (b) For $p = 2, q = 4$.

Fig. 7. Conjunction for $p = 1.2, q = 4$ in Example 3.3.

Fig. 8. Conjunction for $p = 0.8, q = 4$ in Example 3.4.

Fig. 9. Conjunction for $q = 2$ in Example 3.5.

for $p = 2, q = 4$ are shown on Fig. 6. When $p = q$ this
conjunction has the following form:

$$T(x, y) = \begin{cases} xy^p, & \text{if } x > y \\ x^q y^q, & \text{if } x \leq y. \end{cases}$$

When $p = q = 1$ we have $T = T_p$.

Example 3.3: For $T_2 = T_p$ and $S = S_p$ we will obtain a
new conjunction

$$T(x, y) = \min(x, y)(x^p + y^q - x^q y^q).$$

The graph of this conjunction for $p = 1.2, q = 4$ is shown
on Fig. 7.

Example 3.4: For $T_2 = T_p$ and $S = S_p$ we will obtain the
following conjunction:

$$T(x, y) = \min(x, y) \min(1, x^p + y^q).$$

The graph of this conjunction for $p = 0.8, q = 4$ is shown
on Fig. 8.

Example 3.5: For $T_2 = T_c, S = S_c$ and two generators
$g_D(x)$ and $g^q$ we will obtain a following conjunction:

$$T(x, y) = \min\{\min(x, y), \max(g_D(x), g^q)\}.$$
generator \( g(x, p) = x^p \) and \( T \)-norms \( T_2 = T_p \) and \( T_1 = T_c \)
we will obtain the following conjunction:

\[
T(x, y) = \min(x, y)(x + y - xy)^p
\]

varying from \( T_c \) (when \( p = 0 \)) to \( T_d \) (when \( p \to \infty \)).

**Example 5**: Let us propose another parametric class of conjunctions based on representations (9) and (14) with \( T_1 = T_c, T_2 = T_p, s_1 = s_D, h(y) = y^p, s_2 = S_c \):

\[
T(x, y) = \begin{cases} \min(x, y) \max(s_D(x, y), y^p) & \text{if } \max(x, y) = 1 \\ y^p \min(x, y), & \text{otherwise.} \end{cases}
\]

We have \( T = T_c \) when \( p = 0 \) and \( T = T_d \) when \( p \to \infty \). The graph of this conjunction for \( q = 2 \) is shown on Fig. 10.

Below one can find the conjunctions based on \( T_1 = T_p \):

\[
\begin{align*}
T(x, y) &= (xy) \max(x^p, y^p) \\
T(x, y) &= xy(x^p + y^p - x^p y^p) \\
T(x, y) &= (xy) \min(1, x^p + y^p).
\end{align*}
\]

These conjunctions vary from \( T_p \) to \( T_b \).

The following conjunction is based on representations (9) and (14) with \( T_1 = T_c, T_2 = T_p, s_1 = s_2 = S_c, h(y) = p^x \):

\[
T(x, y) = \begin{cases} p \min(x, y) \max(x, y, p) & \text{if } x, y \leq p \\ xy^p, & \text{otherwise, } \end{cases} \quad p \in [0, 1]
\]

and varies from \( T = T_c \) when \( p = 1 \) to \( T = T_p \) when \( p = 0 \).

**VII. Example of Modeling by New Operations**

In this section, the approximation of a given function \( f(x, y) \) by a fuzzy inference system (FIS) is considered. Let the function \( f(x, y) \) be the surface described by a first order Sugeno FIS with two inputs and one output. Each input variable has two terms: \( S \) (SMALL) and \( L \) (LARGE) defined by trapezoidal membership functions on \([0, 1]\) and FIS consists

![Fig. 10. Conjunction for \( q = 2 \) in Example 5.](image)

![Fig. 11. Surface of initial model.](image)

![Fig. 12. Surface of final model.](image)

of the following four rules:

- IF \( x \) is \( S \) AND \( y \) is \( S \) THEN \( z = x + 2y + 3 \)
- IF \( x \) is \( S \) AND \( y \) is \( L \) THEN \( z = 4x + 10y + 20 \)
- IF \( x \) is \( L \) AND \( y \) is \( S \) THEN \( z = 3x + 5y + 15 \)
- IF \( x \) is \( L \) AND \( y \) is \( L \) THEN \( z = 4x + 8y + 6 \).

In this FIS, let us first use the \( \min \) operation as the conjunction operation representing the connective \( \text{AND} \). The surface of the function defined by this FIS is shown on Fig. 11.

Let us now approximate this initial FIS by the same FIS where trapezoidal membership functions \( S \) and \( L \) are replaced by triangular membership functions and \( \min \) operation is replaced by the parametric operation \( T(x, y) = \min(x, y)(x^p + y^p - x^p y^p) \). Four parameters \( p_S, p_L, q_S, \) and \( q_L \) are used in this operation for processing the membership value of \( x \) in \( S \) and \( L \), \( y \) in \( S \) and \( L \), respectively. For example, the firing value of the second rule is calculated as

\[
\alpha = T(\mu_S(x^p), \mu_L(y^p)) = \min(\mu_S(x^p), \mu_L(y^p))(\mu_L(x^p)\mu_S(y^p) + \mu_L(y^p)\mu_S(x^p)) - \mu_L(x^p)\mu_S(y^p).
\]

The values of the parameters \( p \) and \( q \) are obtained as a result of the minimization of the mean-squared error between the surface of the initial FIS and the surface of the approximating FIS. Fifty points grid are used in each scale and, as result, 2500 points from initial surface is used for approximation. The optimal values of parameters are the following: \( p_S = 6.45, p_L = 6.45, q_S = 5.89, q_L = 5.89 \). The surface of the optimal
approximating FIS is shown on Fig. 12. The membership functions of fuzzy sets used in these models are presented on Fig. 13.

VIII. CONCLUSIONS

In the construction of optimal fuzzy models, the optimization of the parameters of the fuzzy operations may be used instead of or in addition to the optimization of the parameters of the fuzzy sets. However, the known parametric classes of $T$-norms and $T$-conorms are, in general, far too complicated for optimization procedures and hardware realization. It is therefore desirable to have simpler parametric classes of conjunction and disjunction operations. This is the motivation behind the work reported in this paper. A novel approach to the construction of parametric classes of conjunction and disjunction operations is proposed, based on the consideration that the inference procedures of fuzzy models do not always require commutativity and associativity of the operations used. Several such parametric classes of conjunction operations are derived that are simpler than the known parametric classes of $T$-norms. The efficacy of the proposed operation is tested on the approximation of a given function by a fuzzy inference system in which the proposed parametric class of conjunctions is used. It is seen that the parameters of the conjunction operation can easily be tuned and the resulting FIS is a good approximation of the given function.

REFERENCES


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