Partition constrained covering of a symmetric crossing supermodular function by a graph

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Abstract

Given a symmetric crossing supermodular set function $p$ on $V$ and a partition $P$ of $V$, we solve the problem of finding a graph with vertex set $V$ having edges only between the classes of $P$ such that for every subset $X$ of $V$ the cut of the graph defined by $X$ contains at least $p(X)$ edges. The objective is to minimize the number of edges of the graph.

This problem is a common generalization of the global edge-connectivity augmentation of a graph with partition constraints, which was solved by Bang-Jensen, Gabow, Jordán and Szigeti [SIAM J. Discrete Math. Vol. 12, No. 2 (1999), pp. 160-207] and the problem of covering a symmetric crossing supermodular set function solved by Benczúr and Frank [Math. Program. Vol. 84, No. 3 (1999), pp. 483-503]. Our problem can be considered as an abstract form of the problem of global edge-connectivity augmentation of a hypergraph with partition constraints, which was earlier solved by the authors [Journal of Graph Theory, Vol. 72, No. 3 (2013), pp. 291-312].

Key words: Edge-connectivity augmentation, splitting off, connectivity function

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1 Introduction

This paper is concerned with edge-connectivity augmentation problems in graphs, hypergraphs and abstract forms of the problems for "connectivity" set functions. For a survey, we refer to [9].

Our starting point is the problem of global edge-connectivity augmentation of a graph, where we have to add a minimum number of new edges to a given graph $G = (V,E)$ in order to obtain a $k$-edge-connected graph, for a given $k \geq 2$. A natural lower bound can be obtained as follows: for a set $X$ of degree $d(X)$ less than $k$, the deficiency of $X$ is $k - d(X)$, that is, we must add at least $k - d(X)$ edges between $X$ and $V \setminus X$. The deficiency of a subpartition of $V$ is the sum of the deficiencies of its sets. By adding a new edge we may decrease the deficiency of at most two sets of this subpartition so we may decrease the deficiency of the subpartition by at most two, hence we obtain the so-called subpartition lower bound: $\alpha_G := \lceil$ half of the maximum deficiency of a subpartition of $V \rceil$. The minimax theorem due to Watanabe and Nakamura [10] says that this lower bound $\alpha_G$ can always be achieved.

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The next step is a generalization of the above problem, namely the problem of global edge-connectivity augmentation of a hypergraph, where we have to add a minimum number of new graph edges to a given hypergraph \( \mathcal{G} = (V, E) \) in order to obtain a \( k \)-edge-connected hypergraph, for a given \( k \). Of course the subpartition lower bound holds also for hypergraphs. However, a new lower bound arises: after deleting \( k - 1 \) hyperedges the connected components must be connected by the new graph edges, hence we obtain the components lower bound: \( \omega_\mathcal{G} - 1 \), where \( \omega_\mathcal{G} := \max \{ \alpha_\mathcal{G}, \omega_\mathcal{G} - 1 \} \) can always be achieved.

Benczúr and Frank \[4\] considered the abstract form of the previous problem, namely covering of a symmetric crossing supermodular function by a graph: given a symmetric, crossing supermodular set function \( p \) on \( V \), what is the minimum number of edges of a graph on vertex set \( V \) that covers \( p \), that is, for all subsets \( X \subseteq V \), the cut defined by \( X \) contains at least \( p(X) \) edges? The subpartition and the component lower bounds can be extended for this problem: \( \alpha_p := \lceil \text{half of the maximum of the sum of the values of the sets in a subpartition of } V \rceil \) and \( \dim(p) - 1 := \max \{ \text{size of a } p \text{-full partition } -1 \} \), where a partition is \( p \)-full if each union of some of its sets, has value at least one. The minimax theorem due to Benczúr and Frank \[4\] says that the lower bound \( \max \{ \alpha_p, \dim(p) - 1 \} \) can always be achieved.

Now we consider the partition constrained versions of the above problems.

Motivated by a problem from the theory of rigidity, Bang-Jensen et al. \[2\] introduced the problem of partition constrained global edge-connectivity augmentation of a graph: given a graph \( G = (V, E) \), an integer \( k \geq 2 \) and a partition \( \mathcal{P} = \{ P_1, \ldots, P_r \} \) of \( V \), what is the minimum number of new edges, between different members of \( \mathcal{P} \), whose addition results in a \( k \)-edge-connected graph? We have a new partition constrained lower bound because we can not add a new edge in \( P \in \mathcal{P} \): \( \beta_G := \max \{ \text{deficiency of a subpartition of } P \text{ over all } P \in \mathcal{P} \} \). The minimax theorem due to Bang-Jensen et al. \[2\] says that the lower bound \( \max \{ \alpha_p, \dim(p) - 1 \} \) can be achieved, except if the graph contains a \( C_4 \)- or a \( C_6 \)-configuration, in which case one more edge is needed.

Bernáth et al. \[6\] considered a generalization of the above problem, the problem of partition constrained global edge-connectivity augmentation of a hypergraph: given a hypergraph \( \mathcal{G} = (V, E) \), an integer \( k \) and a partition \( \mathcal{P} = \{ P_1, \ldots, P_r \} \) of \( V \), what is the minimum number of new graph edges, between different members of \( \mathcal{P} \), whose addition results in a \( k \)-edge-connected hypergraph? The minimax theorem due to Bernáth et al. \[6\] says that the lower bound \( \max \{ \alpha_\mathcal{G}, \beta_\mathcal{G}, \omega_\mathcal{G} - 1 \} \) can be achieved, except if the hypergraph contains a \( C_4 \)- or a \( C_6 \)-configuration, extension of the above configurations, in which case one more edge is needed.

We emphasize that the above mentioned papers contain polynomial algorithms solving the corresponding problems.

In this paper we solve the abstract version of the previous problem, a common generalization of all the above mentioned problems, namely the partition constrained covering of a symmetric crossing supermodular function by a graph: given a symmetric, crossing supermodular set function \( p \) on \( V \) and a partition \( \mathcal{P} \) of \( V \), what is the minimum number of edges, between different members of \( \mathcal{P} \), resulting in a graph that covers \( p \)? The partition constrained lower bound can be extended for this problem: \( \beta_p := \max \{ \text{sum of the values of the sets in a subpartition of } P \text{ over all } P \in \mathcal{P} \} \) can be achieved except if a \( C_4 \)-, \( C_6 \)- or a \( C_8 \)-configuration exists for \( (p, \mathcal{P}) \), in which case one more edge is needed. This result strictly generalizes the partition constrained problem for hypergraphs. Indeed, a new configuration arises, and it extends an application of Benczúr and Frank \[4\] that can not be treated in the framework of hypergraphs, see Section 6.

We will follow the classical approach of Frank \[7\]. First we treat the so called degree-specified
version of the above problem in Section 4 which is the following: given a symmetric, crossing supermodular set function \( p \) on \( V \), a partition \( \mathcal{P} \) of \( V \), and a function \( m : V \rightarrow \mathbb{Z}_+ \) (called degree-specification) and the task is to decide whether a graph \( G \) covering \( p \) exists that has only edges connecting different members of \( \mathcal{P} \) and satisfies \( d_G(v) = m(v) \) for every \( v \in V \). We show the natural necessary conditions of the existence of such a graph and we characterize the exceptional structures (called obstacles): these are the only cases that satisfy these conditions but still there does not exist a solution. Then in Section 5 we turn to the above given minimization version of our problem and we solve it the following way. Firstly, in Section 5.2 we try to find a degree-specification \( m \) satisfying the necessary conditions given earlier and with \( m(V) \) as small as possible, but avoiding the obstacles: these necessary conditions correspond to natural lower bounds for \( m(V) \). Secondly, in Section 5.3 we exhibit the structures (called configurations) where we can only avoid creating obstacles if we allow two more additional edges. Thirdly, we derive our main result in Section 5.4. Finally, we provide applications of our theorem. We show in Section 7 that this approach provides a polynomial algorithm.

2 Definitions

Graphs and set functions: Let us be given a finite ground set \( V \). By \( X \subseteq V \) we mean a proper subset \( X \) of \( V \), and \( \overline{X} = V \setminus X \). Two subsets \( X \) and \( Y \) of \( V \) are crossing if none of \( X \setminus Y, Y \setminus X, X \cap Y \) and \( V \setminus (X \cup Y) \) is empty. A family \( \mathcal{F} \) of subsets of \( V \) is laminar if for all \( X, Y \in \mathcal{F} \) either \( X \) and \( Y \) are disjoint or one of them contains the other one. For a family \( \mathcal{M} = \{M_1, \ldots, M_t\} \) of subsets of \( V \), let \( M^*_0 = \bigcap_{i=1}^t M_i \) and \( M^*_i = M_i \setminus \bigcup_{j \neq i} M_j \).

Let \( G = (V, E) \) be a graph. For \( X,Y \subseteq V \), \( d_G(X,Y) \) denotes the number of edges between \( X \setminus Y \) and \( Y \setminus X \), \( d_G(X,Y) = d_G(X,V \setminus Y) \), and \( d_G(X) = d_G(X,V \setminus X) \). Given a partition \( \mathcal{X} \) of \( V \), \( E_{\delta}(X) \) will denote the set of edges connecting two members of \( X \) and \( E_{\mathcal{X}} \) the set of edges contained in members of \( \mathcal{X} \). It is well-known that the following equalities hold for all \( X,Y \subseteq V \):

\[
\begin{align*}
    d_G(X) + d_G(Y) &= d_G(X \cap Y) + d_G(X \cup Y) + 2d_G(X,Y), \\
    d_G(X) + d_G(Y) &= d_G(X \setminus Y) + d_G(Y \setminus X) + 2d_G(X,Y).
\end{align*}
\]

All the functions in this paper are integer valued but not necessarily non-negative and they have value 0 on the empty set. A set function \( p : 2^V \rightarrow \mathbb{Z} \) is symmetric if \( p(X) = p(V \setminus X) \) for all \( X \subseteq V \), and is called crossing supermodular if it satisfies (3) for all crossing sets \( X,Y \subseteq V \) with \( p(X),p(Y) > 0 \). Note that if \( X \cap Y = \emptyset \) or \( Y \setminus X = \emptyset \) then (3) is trivially satisfied with equality. A set \( X \subseteq V \) with \( p(X) > 0 \) is called p-positive. A symmetric crossing supermodular set function \( p \) also satisfies (4) for crossing \( p \)-positive set pairs \( X,Y \). Note that if \( X \cap Y = \emptyset \) or \( X \cup Y = V \) then (4) is trivially satisfied with equality.

\[
\begin{align*}
    p(X) + p(Y) &\leq p(X \cap Y) + p(X \cup Y). \\
    p(X) + p(Y) &\leq p(X \setminus Y) + p(Y \setminus X).
\end{align*}
\]

A non-negative function \( m : V \rightarrow \mathbb{Z}_+ \) is called a degree specification. The graph \( G \) is said to cover the function \( p \) if (5) holds and it is said to satisfy the degree specification \( m \) if (6) holds. Moreover, \( m \) is called p-admissible if (7) holds, where \( m(X) = \sum_{x \in X} m(x) \).

\[
\begin{align*}
    d_G(X) &\geq p(X) \quad \text{for all } \emptyset \neq X \subseteq V, \\
    d_G(v) &\geq m(v) \quad \text{for all } v \in V, \\
    m(X) &\geq p(X) \quad \text{for all } \emptyset \neq X \subseteq V.
\end{align*}
\]

The following result due to Frank is fundamental in every result on edge-connectivity augmentation: it gives the connection between the minimization version and the degree-specified version of these problems. The theorem is true under more general circumstances, we only state what we need in this paper.
Theorem 1. [7], [1] If $p : 2^V \to \mathbb{Z}$ is crossing supermodular then

$$\min\{m(V) : m \text{ is a } p\text{-admissible degree specification}\} = \max\{\sum_{i=1}^{t} p(V_i) : \{V_1, \ldots, V_t\} \text{ is a subpartition of } V\}. \quad (8)$$

The maximum value in (8) is denoted by $\sigma_p$. A degree specification $m$ that achieves the minimum value in (8) will be called minimal. Note that for the parameter $\alpha_p$ defined in the Introduction, we have

$$\alpha_p = \left\lceil \frac{1}{2} \sigma_p \right\rceil. \quad (9)$$

Given a partition $\mathcal{X} = \{X_1, \ldots, X_t\}$ of $V$, the index $j$ of $X_j$ is considered modulo $t$, that is for example $X_{t+1} = X_1$. Let $J \subset \{1, \ldots, t\}$ be an index set. Let $\overline{J} := \{1, \ldots, t\} \setminus J$. We call $J$ consecutive if $J = \{j, j+1, \ldots, k\}$ modulo $t$ for some $j$ and $k$. Let us say that $J$ is $p$-positive if $J \neq \emptyset$ and $p(\bigcup_{j \in J} X_j) > 0$.

Following [4], the partition $\mathcal{X}$ is called $p$-full if $t \geq 4$, every nonempty index set $I \subset \{1, \ldots, t\}$ is $p$-positive and $p(X_j) = 1$ for some $j \in \{1, \ldots, t\}$. The maximum cardinality of a $p$-full partition is the dimension of $p$ and is denoted by $\dim(p)$. If no $p$-full partition exists, then $\dim(p)$ is defined to be 0. A degree specification $m$ is called $p$-legal if (10) is satisfied.

$$m(V) \geq 2(\dim(p) - 1). \quad (10)$$

The following lemma comes from Benczúr, Frank [4].

Lemma 2. [4] Let $p : 2^V \to \mathbb{Z}$ be a symmetric crossing supermodular set function.

1. A graph that covers $p$ has at least $\dim(p) - 1$ edges.
2. If a partition $\mathcal{X} = \{X_1, \ldots, X_t\}$ of $V$ satisfies $t \geq 4$, $p(X_1) = 1$ and $p(X_1 \cup X_i) > 0$ for $i = 2, \ldots, t$, then $\mathcal{X}$ is $p$-full.

Let $G = (V, E)$ be a graph, $p_0$ a symmetric crossing supermodular set function on $V$ and $m_0$ a degree specification on $V$. Let us introduce the functions $p_G$ and $m_G$, that will play an important role in this paper, as follows:

$$p_G(X) = p_0(X) - d_G(X) \quad \text{for all } X \subset V \quad (11)$$

$$m_G(v) = m_0(v) - d_G(v) \quad \text{for all } v \in V. \quad (12)$$

By (1), the function $-d_G$ is symmetric crossing supermodular, hence so is $p_G$. Moreover, by (3) and (1) and respectively by (4) and (2), for crossing $p_0$-positive subsets $X$ and $Y$ of $V$, the following hold.

$$p_G(X) + p_G(Y) \leq p_G(X \cap Y) + p_G(X \cup Y) - 2d_G(X, Y), \quad (13)$$

$$p_G(X) + p_G(Y) \leq p_G(X \setminus Y) + p_G(Y \setminus X) - 2d_G(X, Y). \quad (14)$$

It is useful to define the following surplus function $s_G$:

$$s_G = m_G - p_G. \quad (15)$$

**Observation 3.** Note that $m_G$ is $p_G$-admissible is equivalent to

$$s_G(X) \geq 0 \quad \text{for all } \emptyset \neq X \subset V. \quad (16)$$

By modularity of $m_G$ and (13) (respectively (14)), for crossing $p_0$-positive subsets $X$ and $Y$ of $V$, the following hold.

$$s_G(X) + s_G(Y) \geq s_G(X \cap Y) + s_G(X \cup Y) + 2d_G(X, Y), \quad (17)$$

$$s_G(X) + s_G(Y) \geq s_G(X \setminus Y) + s_G(Y \setminus X) + 2(d_G(X, Y) + m_G(X \cap Y)). \quad (18)$$
Operations: Let $m_G : V \to \mathbb{Z}_+$ be a $p_G$-admissible degree specification. An element $v$ of $V$ is called $m_G$-positive if $m_G(v) > 0$. The set of $m_G$-positive elements is denoted by $V_+(m_G)$. For an element $v \in V$, $\chi_v$ denotes the incidence vector of the set $\{v\}$. Let $x, y$ be two different $m_G$-positive elements and $uv \in E$ an edge of $G$ that is not incident to $x$ and $y$. We will need the following operations:

1. **Splitting off** at $x, y$ means replacing $m_G$ by $m_{G_{xy}}$ and $p_G$ by $p_{G_{xy}}$, where $G_{xy}= G + xy$.

2. **Unsplitting** $uv$ is the reverse of splitting off: replace $m$ by $m_{G''}$, and $p_G$ by $p_{G''}$, where $G''= G - uv$. Note that $m_{G''}$ is $p_{G''}$-admissible.

3. The $(uv, ux)$-flip is defined as unsplitting $uv$ and splitting off at $x, u$, that is replacing $m_G$ by $m_G'$ and $p_G$ by $p_{G'}$, where $G'= G - uv + xu$. We will also call it **flipping** $uv$ for $ux$.

4. **Improving $uv$ to $ux, vy$** is defined as unsplitting $uv$ and splitting off at $x, u$ and at $v, y$, that is replacing $m_G$ by $m_{G''}$ and $p_G$ by $p_{G''}$, where $G'' = G - uv + xu + vy$. **Improving** $uv$ **by** $x$ **and** $y$ means improving $uv$ to either $xu, vy$ or $xv, uy$. The corresponding operation is an improvement.

Any of the above operations is called **$p_G$-admissible** if the new degree specification is admissible with the new set function. If the splitting off at $x, y$ is $p_G$-admissible, then we say that the pair $x, y$ is $p_G$-admissible.

**Observation 4.** $s_G(X) - s_{G_{uv}}(X) = 2$ if both $u$ and $v$ belong to $X$, and $0$ otherwise.

Special sets: The following special sets will be used frequently in the paper. A set $X \subseteq V$ is called

1. **tight** if $m_G(X) = p_G(X)$, that is if $s_G(X) = 0$,

2. **dangerous** if $m_G(X) \leq p_G(X) + 1$, that is if $s_G(X) \leq 1$,

3. **(uv, ux)-perilous** if $p_G(X) = 0 = m_G(X) - 1$, $x, u \in X$ and $v \notin X$. A $(uv, ux)$- or $(vu, vx)$-perilous set is called **(uv, x)-perilous**.

A partition is called **tight** if its members are tight. For an $m_G$-positive element $u$, $X_u$ and $T_u$ denote the minimal and the maximal tight set containing $u$. If $u$ belongs to no tight set, then $X_u$ is defined to be equal to $V$.

**Observation 5.** Tight sets containing an $m_G$-positive element, dangerous sets containing two $m_G$-positive elements and perilous sets are $p_0$-positive.

Partition constraint: Let $\mathcal{P} = \{P_1, \ldots, P_r\}$ be a partition of $V$ with $r \geq 2$. An element $v \in V$ that belongs to some $P_i$ is said to be of color $i$. The notation $c(v)=i$ will also be used for $v \in P_i$. We say that the graph $G = (V, E)$ is **$\mathcal{P}$-partite** if every edge of $G$ goes between two different classes of $\mathcal{P}$.

**Observation 6.** Note that there always exists a $\mathcal{P}$-partite spanning tree on $V$. Indeed, let $u \in P_1$ and $v \in P_2$. Then the edge set $\{ux : x \in V \setminus P_1\} \cup \{vy : y \in P_1 \setminus \{u\}\}$ form a spanning tree on $V$ that is clearly $\mathcal{P}$-partite.
A degree specification $m$ is called $\mathcal{P}$-feasible if (19) and (20) are satisfied. If $m$ is $p_G$-admissible and $\mathcal{P}$-feasible, then $m$ is called $(p_G, \mathcal{P})$-allowed.

\[
\begin{align*}
m(V) & \text{ is even,} \\
m(P_i) & \leq \frac{m(V)}{2} \text{ for all } P_i \in \mathcal{P}.
\end{align*}
\]

We call $P_i \in \mathcal{P}$ dominating if $m(P_i) = \frac{1}{2}m(V)$. A pair of $m$-positive elements is called rainbow if they are of different colors and any dominating color class contains one of them. A splitting off, a flip, or an improvement, is called if they are of different colors and any dominating color class contains one of them. A splitting off, a flip, or an improvement, is called if it is $p_G$-admissible and it uses only rainbow pairs. We will simply write allowed for $(p_G, \mathcal{P})$-allowed, and we will precise the function to be considered when it differs from $p_G$. A complete allowed splitting off is a sequence of allowed splitting off that decreases $m(V)$ to zero. If the splitting off at $x, y$ is allowed, then we say that the pair $x, y$ is allowed.

Let $m$ be a degree specification and $P \in \mathcal{P}$. A pair $(X_1, X_2)$ of disjoint sets of $V$ is called a $P$-pair if there exists a subpartition $X_0$ of $X_1 \cap P$ such that $\sum_{X \in X_0} p(X) = p(X_i)$ for $i = 1, 2$, while it is called an $(m, P)$-pair if the $m$-positive elements of $X_1 \cup X_2$ are the $m$-positive elements of $P$. A subpartition $X$ of $V$ is called a $P$-subpartition if there exist a set $X' \subseteq X \cap P$ for every $X \in \mathcal{X}$ such that $p(X') = 1$, while it is called an $(m, P)$-subpartition if each $X \in \mathcal{X}$ contains an $m$-positive element of $P$.

Constructions, obstructions, obstacles: Let $p : 2^V \to \mathbb{Z}$ be a symmetric crossing supermodular function, $m : V \to \mathbb{Z}_+$ a degree specification and $\mathcal{P} = \{P_1, \ldots, P_r\}$ a partition of $V$.

Definition 7. A partition $\mathcal{A} = \{A_1, \ldots, A_4\}$ of $V$ is called a $C_4^*$-obstacle for $(p, \mathcal{P}, m)$ if

1. (a) $p(A_i) + p(A_{i+1}) - p(A_i \cup A_{i+1})$ is odd for $i = 1, \ldots, 4$,
   (b) $p(A_{i-1} \cup A_i) + p(A_i \cup A_{i+1}) = p(A_{i-1}) + p(A_{i+1})$ for $i = 1, \ldots, 4$,
   (c) if $p(A_i) = 1$ for $i = 1, \ldots, 4$, then $p(A_1 \cup A_3) = p(A_2 \cup A_4) \leq 0$,
   (d) $p(A_1) + p(A_3) = p(A_2) + p(A_4) = \frac{1}{2} \sigma_p$.

2. $m$ is minimally $p$-admissible.

3. $m$ is $\mathcal{P}$-feasible and there exist $\ell \in \{1, 2\}$ and $P \in \mathcal{P}$ such that $(A_\ell, A_{\ell+2})$ is an $(m, P)$-pair.

A partition $\mathcal{A}$ is called a $C_4^*$-construction for $p$ (respectively $C_4^*$-obstruction for $(p, m)$) if it satisfies 1 (resp. 1 and 2). $C_4^*$-constructions, $C_4^*$-obstructions and $C_4^*$-obstacles satisfying $p(A_i) = 1$ for $i = 1, \ldots, 4$ are called simple.

Definition 8. A partition $\mathcal{A} = \{A_1, A_2, A_3, A_4, B_1, \ldots, B_t\}$ of $V$ ($t \geq 1$) is called a $C_5^*$-obstacle for $(p, \mathcal{P}, m)$ if

1. (a) $p(A_i) = 1$ for $i = 1, \ldots, 4$,
   (b) $p(B_j) = 2$ for $j = 1, \ldots, t$,
   (c) $p(A_i \cup B_j) = 1$ for $i = 1, \ldots, 4$ and $j = 1, \ldots, t$,
   (d) $p(A_i \cup A_{i+1}) = 1$ for $i = 1, \ldots, 4$,
   (e) $p(A_i \cup A_{i+2}) \leq 0$ for $i = 1, 2$,
   (f) $\sigma_p = \sum_{X \in \mathcal{A}} p(X) = 2t + 4$.

2. $m$ is minimally $p$-admissible.
3. $m$ is $\mathcal{P}$-feasible and

(a) either there exist $\ell \in \{1, 2\}$ and $P \in \mathcal{P}$ such that $\{A_\ell, A_{\ell+2}, B_1, \ldots, B_t\}$ is an $(m, P)$-subpartition,

(b) or there exist $j_0 \in \{1, \ldots, t\}$ and distinct $P_{k_1}, P_{k_2} \in \mathcal{P}$ such that for $i = 1, 2,$ 
\{A_i, A_{i+2}\} \cup \{B_j : j \neq j_0\}$ is an $(m, P_{k_i})$-subpartition.

A partition $A$ is called a $C_5^*$-construction for $p$ (respectively $C_5^*$-obstruction for $(p, m)$) if it satisfies 1 (resp. 1 and 2). A $C_5^*$-obstacle is of type 1 (respectively type 2) if $3a$ (resp. $3b$) is satisfied. For a $C_5^*$-construction $A$, consecutive elements of $A$ mean sets $A_i$, $A_{i+1}$ where the index $i$ is considered modulo 4.

**Definition 9.** A partition $A = \{A_1, \ldots, A_6\}$ of $V$ is called a $C_6^*$-obstacle for $(p, \mathcal{P}, m)$ if

1. $(a)$ $p(A_i) = 1$ for $i = 1, \ldots, 6$,

(b) $p(A_i \cup A_{i+1}) = 1$ for $i = 1, \ldots, 6$,

(c) $p(A_i \cup A_j) \leq 0$ for all non-consecutive sets $A_i$ and $A_j$,

(d) $\sigma_p = \sum_{i=1}^6 p(A_i) = 6$.

2. $m$ is minimally $p$-admissible.

3. $m$ is $\mathcal{P}$-feasible and there exist distinct $P_{k_i} \in \mathcal{P}$ such that $(A_i, A_{i+3})$ is a $(m, P_{k_i})$-pair for $i = 1, 2, 3$.

A partition $A$ is called a $C_6^*$-construction for $p$ (respectively $C_6^*$-obstruction for $(p, m)$) if it satisfies 1 (resp. 1 and 2).

A construction (resp. obstruction, obstacle) is a $C_4^*$- or a $C_5^*$- or a $C_6^*$-construction (resp. obstruction, obstacle). Note that an obstruction is a special type of construction, and an obstacle is a special type of obstruction.

**Observation 10.** If there exists a construction then $\sigma_p$ is even.

**Observation 11.** Two $m$-positive elements that belong to consecutive sets $A_i$ and $A_{i+1}$ of an obstacle have different colors.

### 3 Preliminaries

In this section, $p_0 : 2^V \to \mathbb{Z}$ is a symmetric crossing supermodular function, $G = (V, E)$ is a graph, and $m_G$ is a $p_G$-admissible degree specification with $m_G(V) \geq 4$.

#### 3.1 Positive sets

**Claim 12.** 1. If a family $\{X_1, \ldots, X_k\}$ of $p_0$-positive subsets of $V$ satisfies that $X_j$ crosses $\bigcup_{i=1}^{j-1} X_i$ and $p_G(X_j \cap (\bigcup_{i=1}^{j-1} X_i)) \leq p_G(X_j)$ for $j = 2, \ldots, k$, then $p_G(X_1) \leq p_G(\bigcup_{i=1}^{j} X_i) \leq p_G(\bigcup_{i=1}^{k} X_i)$ for $j = 1, \ldots, k$.

2. If a subpartition $\{W_1, \ldots, W_k\}$ of $V$ satisfies $\bigcup_{i=1}^{k} W_i \neq V$, and for every $j = 2, \ldots, k$ there exists an $i_j$ such that $1 \leq i_j < j$, $p_G(W_{i_j}) = 1$, and $p_G(W_{i_j} \cup W_{i_j+1}) \geq 1$, then $p_G(\bigcup_{i=1}^{k} W_i) \geq 1$.

**Proof.** 1. We prove it by induction on $k$. For $k = 1$ the inequalities hold (with equalities). Suppose that the inequalities hold for $k - 1$, that is $p_G(X_1) \leq p_G(\bigcup_{i=1}^{j} X_i) \leq p_G(\bigcup_{i=1}^{k-1} X_i)$ for $j = 1, \ldots, k - 1$. To finish the proof we have to show that $p_G(\bigcup_{i=1}^{k-1} X_i) \leq p_G(\bigcup_{i=1}^{k} X_i)$. Applying (13) to $X_k$ and $\bigcup_{i=1}^{k-1} X_i$, and $p_G(X_k \cap (\bigcup_{i=1}^{k-1} X_i)) \leq p_G(X_k)$, gives the result.
2. Apply 12.1 to \( \{W_i \cup W_j : j = 2, \ldots, k\} \).

\[\]

\textbf{Claim 13.} If a partition \( A = \{A_1, \ldots, A_t\} \) of \( V \) (\( t \geq 4 \)) satisfies \( p_G(A_i) = p_G(A_i \cup A_{i+1}) = 1 \) for \( i = 1, \ldots, t \), then

1. Every edge of \( G \) connects consecutive members of \( A \).
2. \( p_G(\bigcup_{j \in J} A_j) = 1 \) for all nonempty consecutive \( J \subset \{1, \ldots, t\} \),
3. if \( p_G(\bigcup_{j \in J} A_j) \geq 1 \) for some non-consecutive index set \( J \), then \( p_G(A_k \cup A_{\ell}) \geq 1 \) for all non-consecutive pair \( \{k, \ell\} \) such that with \( K = (1, 2, \ldots, t) \), we have \( p_G(A_i) = 1 \) for all \( i \),

4. there exists a \( p_G \)-positive non-consecutive index set if and only if \( A \) is a \( p_G \)-full partition.

\textbf{Proof.} Let \( A_J := \bigcup_{j \in J} A_j \) for any index set \( J \subset \{1, \ldots, t\} \), and let \( \overline{J} = \{1, \ldots, t\} \setminus J \).

1. Suppose that there exists an edge of \( G \) between \( A_i \) and \( A_j \) where \( j \notin \{i - 1, i + 1\} \). By (14) applied to \( A_{i-1} \cup A_i \) and \( A_i \cup A_{i+1} \), we have \( 1 + 1 = p_G(A_{i-1} \cup A_i) + p_G(A_i \cup A_{i+1}) \leq p_G(A_{i-1}) + p_G(A_{i+1}) - 2d(\bigcup_{i-1}^{i+1}) \), a contradiction.

2. Without loss of generality, we may assume that \( J = \{1, \ldots, j\} \), for some \( j \leq t - 1 \). Let \( X_i = A_j \cup A_{i+1} \) for \( i = 1, \ldots, t - 2 \). Since \( A \) is a partition of \( V \), we have \( X_j \cap (\bigcup_{i=1}^{j-1} X_i) = A_j \) for \( j = 2, \ldots, t - 1 \), hence Claim 12.1 applies to \( \{X_1, \ldots, X_{t-2}\} \). It gives, since \( p_G \) is symmetric, \( 1 = p_G(X_1) \leq p_G(\bigcup_{j \in J} A_j) \leq p_G(\bigcup_{j=1}^{j-1} A_j) = p_G(A_j) = 1 \), and we have equality.

3. By \( p_G(A_i) = p_G(A_i \cup A_{i+1}) = 1 \) for \( i = 1, \ldots, t \), there exist consecutive pairs \( J_2, \ldots, J_r \) such that with \( J_1 := J \) the corresponding sets satisfy the conditions of Claim 12.1 and \( \bigcup_{j=1}^{r} A_{j} = \overline{A_{k \cup A_{\ell}}} \). Then, by Claim 12.1, \( p_G(A_j) \geq 1 \) and by the symmetry of \( p_G \), the assertion follows.

4. We prove only the non-trivial direction. Suppose that \( p_G(A_K) \geq 1 \) for some non-consecutive index set \( K \). Then, there exists a non-consecutive pair \( \{q, r\} \subset K \) such that \( \{q + 1, \ldots, r - 1\} \) and \( \{r + 1, \ldots, q - 1\} \) both intersect \( K \). By Claim 13.3 applied to \( K \), we have \( p_G(A_q \cup A_r) \geq 1 \). Then by Claim 13.3 applied to \( \{q, r\} \), we have \( p_G(A_{q-1} \cup A_{q+1}) \geq 1 \), and then by Claim 13.3 applied to \( \{q - 1, q + 1\} \), which gives \( p_G(A_q \cup A_j) \geq 1 \) for all \( j \neq q - 1, q + 1 \). Moreover, by assumption \( p_G(A_q) = p_G(A_{q-1} \cup A_q) = p_G(A_q \cup A_{q+1}) = 1 \), thus by Claim 2.2, \( A \) is a \( p_G \)-full partition.

\[\]

\textbf{3.2 Tight sets}

The following properties of tight sets are well-known. In this section, we will often implicitly use Observation 5.

\textbf{Claim 14.} Let \( X \) and \( Y \) be \( p_0 \)-positive tight sets. Then

1. if \( X \cap Y \neq \emptyset \) and \( X \cup Y \neq V \), then \( X \cap Y \) is tight and \( X \cup Y \) is \( p_0 \)-positive tight,

2. if \( X \setminus Y \neq \emptyset \) and \( Y \setminus X \neq \emptyset \), then \( X \setminus Y \) and \( Y \setminus X \) are \( p_0 \)-positive tight and \( m_G(X \cap Y) = 0 \).

3. If an \( m_G \)-positive element \( v \) belongs to a tight set, then \( v \) belongs to a unique minimal and a unique maximal tight set.
we have equality everywhere, in particular minimality of $m$ decreases by $m$ of $X$.

Claim 17. Let $X$ be a tight set.

Proof. 1. Suppose that for some $u,v \in D$, none of $X_u - X_v, X_v - X_u$ and $X_u \cap X_v$ is empty. Then, by Claim 14.2, $X_u \setminus X_v \subset X_u$ is a tight set containing $u$, that contradicts the minimality of $X_u$.

2. Let $X$ be the maximal sets of $\{X_u : u \in D\}$. Then, by 15.1, $X$ is a partition of $\bigcup_{u \in D} X_u$ and, by $m_G$ is non-negative and each element of $X$ is tight, we have $m_G(D) \leq m_G(\bigcup_{u \in D} X_u) = \sum_{X \in \mathcal{X}} m_G(X) = \sum_{X \in \mathcal{X}} p_G(X)$. 

By Claim 14.3, for an $m_G$-positive element $u$, the definitions of $X_u$ and $T_u$, the minimal and the maximal tight sets containing $u$, are correct.

Claim 15. Let $D$ be a subset of the $m_G$-positive elements such that each element of $D$ belongs to a tight set.

1. The family $\{X_u : u \in D\}$ is laminar.

2. There exists a partition $\mathcal{X}$ of $\bigcup_{u \in D} X_u$ such that $\sum_{X \in \mathcal{X}} p_G(X) \geq m_G(D)$.

Proof. 1. Suppose that for some $u,v \in D$, none of $X_u - X_v, X_v - X_u$ and $X_u \cap X_v$ is empty. Then, by Claim 14.2, $X_u \setminus X_v \subset X_u$ is a tight set containing $u$, that contradicts the minimality of $X_u$.

2. Let $X$ be the maximal sets of $\{X_u : u \in D\}$. Then, by 15.1, $X$ is a partition of $\bigcup_{u \in D} X_u$ and, by $m_G$ is non-negative and each element of $X$ is tight, we have $m_G(D) \leq m_G(\bigcup_{u \in D} X_u) = \sum_{X \in \mathcal{X}} m_G(X) = \sum_{X \in \mathcal{X}} p_G(X)$.

It is important to mention that a degree specification $m$ can be modified without destroying $p$-admissibility as follows.

Claim 16. Let $u$ be an $m$-positive element and $u' \in X_u$. If $m$ is $p$-admissible, then so is $m' := m - \chi_u + \chi_{u'}$.

Proof. Suppose that $m'$ is not $p$-admissible, that is, there exists a set $Y \subset V$ such that $m'(Y) + 1 \leq p(Y)$. Then, by $m$ is $p$-admissible and the definition of $m'$, we have $p(Y) \leq m(Y) \leq m'(Y) + 1 \leq p(Y)$. Thus equality holds everywhere, that is $Y$ contains $u$ but not $u'$ and it was tight. Then $u \in X_u \cap Y$, so, by assumption, $m(X_u \cap Y) > 0$ and then, by Claim 14.2, $X_u \cup Y \neq V$ and hence, by Claim 14.1, $X_u \cap Y \subset X_u$ is tight, that contradicts the minimality of $X_u$.

Claim 17. Suppose that $m_G$ is minimally $p_G$-admissible. If the pair $u,v$ is $p_G$-admissible, then $m_{G_{uv}}$ is minimally $p_{G_{uv}}$-admissible.

Proof. The pair $u,v$ being $p_G$-admissible, $m_{G_{uv}}$ is $p_{G_{uv}}$-admissible. Moreover, the splitting off decreases by 1 the $p_G$-value of at most two sets of any partition achieving $\sigma_{p_G}$. Then, by the minimality of $m_G$, we have $m_{G_{uv}}(V) \geq \sigma_{p_{G_{uv}}} \geq \sigma_{p_G} - 2 = m_G(V) - 2 = m_{G_{uv}}(V)$. Hence we have equality everywhere, in particular $m_{G_{uv}}(V) = \sigma_{p_{G_{uv}}}$, that is $m_{G_{uv}}$ is minimally $p_{G_{uv}}$-admissible.

Claim 18. If $p_G \leq 1$ and $X \subset V$ crosses a $p_G$-positive tight set, then $p_G(X) \leq 0$. 

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Proof. Assume that $X$ and a tight set $Y$ are $p_G$-positive and crossing. Then, by (13) for $X$ and $Y$ and $p_G \leq 1$, we have $1 + 1 \leq p_G(X) + p_G(Y) \leq p_G(X \cap Y) + p_G(X \cup Y) \leq 1 + 1$, so, $p_G(Y) = 1 = p_G(X \cap Y)$. By $Y$ is tight, $m_G(Y) = p_G(Y) = 1$. Hence, by possibly complementing $X$, we may assume that $m_G(X \cap Y) = 0$. Then, by $p_G$ is symmetric and $m_G$ is $p_G$-admissible, we get $1 = p_G(X \cap Y) \leq m_G(X \cap Y) = 0$, a contradiction. \hfill \qed

3.3 Dangerous sets

We start this subsection by the characterization of admissible pairs, see [4]. In the light of Lemma 19 it is natural to study the properties of dangerous sets.

Lemma 19. [4] A pair of $m_G$-positive elements $u, v$ is $p_G$-admissible if and only if no dangerous set contains both $u$ and $v$.

The following technical claims will be applied throughout the paper. From now on we suppose that $m_G(V)$ is even.

Claim 20. For a dangerous set $Y$,

1. $m_G(Y) \leq \frac{1}{2} m_G(V)$,
2. if $Y$ contains an $m_G$-positive element of a dangerous set $X$, then $m_G(V \setminus Y \setminus X) \geq 1$,
3. if $Y$ intersects a tight set $X$, $X$ and $Y$ are $p_0$-positive, and $X \cup Y \neq V$, then $Y \cup X$ is dangerous.

Proof. 1. By $Y$ is dangerous, $p_G$ is symmetric, $m_G$ is $p_G$-admissible and modular, we have $m_G(Y) \leq p_G(Y) + 1 = p_G(V \setminus Y) + 1 \leq m_G(V \setminus Y) + 1 = m_G(V) - m_G(Y) + 1$, and then, by $m_G(V)$ is even, 20.1 is satisfied.

2. By $m_G$ is modular, 20.1 and $m_G(Y \cap X) \geq 1$, we have $m_G(V \setminus Y \setminus X) = m_G(V) - m_G(Y) - m_G(Y \cap X) \geq m_G(V) - \frac{1}{2} m_G(V) - \frac{1}{2} m_G(V) + 1 = 1$, so 20.2 is satisfied.

3. By $X$ is tight, $Y$ is dangerous, (17) and (16), we have $0 + 1 \geq s_G(X) + s_G(Y) \geq s_G(X \cap Y) + s_G(Y \cup X) \geq 0 + s_G(Y \cup X)$, and 20.3 is satisfied. \hfill \qed

Claim 21. Let $M = \max \{ p_G(X) : X \subseteq V \}$. If $W$ is an inclusionwise minimal set satisfying $p_G(W) = M$, $X$ is a dangerous set, $w, x$ is a pair of $m_G$-positive elements, $w \in W \cap X$ and $x \in X \setminus W$, then

1. $W \subseteq X$,
2. $p_G(X) = M$, and $m_G(X \setminus W) = 1$.

Proof. 1. Suppose that $W \setminus X \neq \emptyset$. Then, by $X$ is dangerous, $p_G(W) = M$, (14), (7), the minimality of $W$ and $w \in W \cap X$, we have $m_G(X) - 1 + M \leq p_G(X) + p_G(W) \leq p_G(X \setminus W) + p_G(W \setminus X) < m_G(X \setminus W) + M \leq m_G(X) - 1 + M$, a contradiction and 21.1 follows.

2. By the definition of $M$, $X$ is dangerous, 21.1, the modularity of $m_G$, (7), $x \in (X \setminus W) \cap V_4(m_G)$ and $p_G(W) = M$, we have $M + 1 \geq p_G(X) + 1 \geq m_G(X) = m_G(W) + m_G(X \setminus W) \geq p_G(W) + 1 = M + 1$, so equality holds everywhere and 21.2 follows. \hfill \qed

Lemma 22. If $m_G$ is $P$-feasible and no $(p_G, P)$-allowed splitting off exists, then $p_G(X) \leq 1$ for all $X \subseteq V$. 

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Proof. Suppose that $M = \max\{p_G(X) : X \subseteq V\} \geq 2$. Let $Y$ be an inclusionwise minimal set satisfying $p_G(Y) = M$. By the symmetry of $p_G$, we have $p_G(V \setminus Y) = M$, and let $Z \subseteq V \setminus Y$ be a minimal set satisfying $p_G(Z) = M$. By $m_G$ is $p_G$-admissible, $Y$ and $Z$ contain $m_G$-positive elements.

Let $y \in Y, z \in Z$ be $m_G$-positive elements. No dangerous set $X$ contains $y$ and $z$, since otherwise Claim 21 would imply $Y \cup Z \subseteq X$ and $1 = m_G(X \setminus Y)$, and then, by $m_G$ is modular and non-negative and (7), we would have $1 = m_G(X \setminus Y) \geq m_G(Z) \geq p_G(Z) = M \geq 2$, a contradiction. It follows, by Lemma 19, that the pair $y, z$ is $p_G$-admissible. Since there exists no allowed splitting off, the pair $y, z$ is not rainbow.

Then, the set of the $m_G$-positive elements of $Y \cup Z$ is either a subset of some $P \in \mathcal{P}$ or is disjoint from a dominating color class $P' \in \mathcal{P}$. Since $m_G$ is $\mathcal{P}$-feasible, there exists an $m_G$-positive element $x$ of $V \setminus (Y \cup Z)$ that belongs to a dominating color class (if there exists one). Then, in the second case, $x \in P'$. For $y \in V_+(m_G) \cap Y$ and $z \in V_+(m_G) \cap Z$, $x, y$ and $x, z$ are rainbow pairs, hence there exist a dangerous set $X$ containing $x, y$ and a dangerous set $X'$ containing $x, z$. Claim 21 applies to $X$ and $Y$ and also to $X'$ and $Z$, hence $p_G(X) = p_G(X') = M$ and $m_G(X \cap X') = 1$. By Claim 20.2, $X$ and $X'$ are crossing, and now (13) implies that $p_G(X \cap X') = M \geq 2 > m_G(X \cap X')$, which contradicts the $p_G$-admissibility of $m_G$.

Claim 23. Let $\mathcal{M} := \{M_1, M_2\}$ be a family of maximal dangerous sets. If $m_G(M_i^*) \geq 1$ for $0 \leq i \leq 2$, then

1. $M_1 \cap M_2$ is tight and $s_G(M_1 \cup M_2) = 2$,
2. $M_1 \setminus M_2$ and $M_2 \setminus M_1$ are tight and $m_G(M_0^*) = 1$,
3. $M_i^*$ is a maximal tight set for $0 \leq i \leq 2$.

Proof. By Claim 20.2, $m_G(M_i^*) \geq 1$ for $0 \leq i \leq 2$, and Observation 5, $M_1$ and $M_2$ are crossing $p_0$-positive sets. Thus (17) and (18) apply to $M_1$ and $M_2$.

1. By $M_1$ and $M_2$ are dangerous, (17), (16) and $M_1 \cup M_2$ is not dangerous, we have $1 + 1 \geq s_G(X) + s_G(Y) \geq s_G(X \cap Y) + s_G(X \cup Y) \geq 0 + 2$ so equality holds everywhere and 23.1 follows.

2. By $M_1$ and $M_2$ are dangerous, $M_i^* \neq \emptyset$, (18), (16) and $m_G(M_0^*) \geq 1$, we get $1 + 1 \geq s_G(X) + s_G(Y) \geq s_G(X \setminus Y) + s_G(Y \setminus X) + 2m_G(X \cap Y) \geq 0 + 0 + 2$ so equality holds everywhere and 23.2 follows.

3. By 23.1-2, $M_1$ and $M_2$ are dangerous and Claim 20.3, we have 23.3.

Claim 24. Let $\mathcal{M} = \{M_1, \ldots, M_\ell\}$ be a family of maximal dangerous sets with $\ell \geq 3$. If $m_G(M_i^*) \geq 1$ for $0 \leq i \leq \ell$, then

1. $M_i^*$ is maximal tight and $M_i = M_i^* \cup M_0^*$,
2. $m_G(M_i^*) = 1$,
3. $s_G(M_j^* \cup M_k^*) = 1$ for $1 \leq j < k \leq \ell$.

Proof. 1. By $m_G(M_i^*) \geq 1$, there exists an $m_G$-positive element $u_i$ in $M_i^*$ for $0 \leq i \leq \ell$. By applying Claim 23.3 to $M_j, M_k$ ($1 \leq j < k \leq \ell$), we get that $M_j \cap M_k = T_{u_j}$, $M_j \setminus M_k = T_{u_j}$ and $M_k \setminus M_j = T_{u_k}$. Then it follows that $M_i^* = T_{u_i}$ and $M_i = M_i^* \cup M_0^*$ for $0 \leq i \leq \ell$ so 24.1 is satisfied.
2. Let $i, j$ and $k$ be three different indices between 1 and $\ell$. By Claim 23.1, (18) applied to $M_i \cup M_j$ and $M_i \cup M_k$, 24.1, modularity of $m_G$, (16), Claim 23.2 and $m_G(M^*_i) \geq 1$, we have $2 + 2 = s_G(M_i \cup M_j) + s_G(M_i \cup M_k) \geq s_G(M^*_i) + 2m_G(M^*_i) \geq 0 + 0 + 2 + 2$, so equality holds everywhere and 24.2 is satisfied.

3. By $M_i$ is dangerous, Claim 23.1, (18) applied to $M_i$ and $M_j \cup M_k$, 24.1, (16) and Claim 23.2, we have $1 + 2 \geq s_G(M_i) + s_G(M_j \cup M_k) \geq s_G(M^*_i) + s_G(M^*_j \cup M^*_k) + 2m_G(M^*_i) \geq 0 + s_G(M^*_j \cup M^*_k) + 2$, so equality holds everywhere and 24.3 is satisfied.

Using the above claims, we generalize a theorem of [6] on admissible edges. It will help us to find an allowed pair when no simple $C^1_4$-obstacle exists but an admissible pair exists.

For an $m$-positive element $t$, let $S_t$ be the set of $m$-positive elements admissible with $t$.

**Lemma 25.** Let $p : 2^V \rightarrow \mathbb{Z}$ be a symmetric crossing supermodular set function, $m : V \rightarrow \mathbb{Z}_+$ a $p$-admissible degree specification with $m(V) \geq 4$ even. Suppose that an admissible pair exists. Then

(i) either there is an $m$-positive element $t$ such that $m(S_t) \geq \frac{1}{2}m(V)$,

(ii) or there is a simple $C^1_4$-obstruction.

**Proof.** By Lemma 19, let $M_t = \{M_1, \ldots, M_t\}$ be a minimal family of maximal dangerous sets such that $t \in M^*_t$ and $V_+(m) \setminus S_t = V_+(m) \cap \bigcup_{i=1}^t M_i$. Suppose that (i) is violated that is (⋆) $m(S_t) \leq \frac{1}{2}m(V) - 1$ for all $t \in V_+(m)$.

**Claim 26.** For all $t \in V_+(m)$, $|M_t| = 2$, $m(M^*_0) = 1$, $M_t^*$ is maximal tight and $M_t = M_t^* \cup M^*_0$ for all $M_t \in M_t$.

**Proof.** If for some $t \in V_+(m)$, $|M_t| \leq 1$, then, by Claim 20.1, $m(S_t) \geq m(V) - m(M_t) \geq m(V) - \frac{1}{2}m(V) = \frac{1}{2}m(V)$ which contradicts (⋆). Thus $|M_t| \geq 2$ for all $t \in V_+(m)$. Suppose that for some $t_0 \in V_+(m)$, $\ell = |M_{t_0}| \geq 3$. By Claim 24 and Lemma 19, $S_{t_0} = S_{t_0}$ for all $t_i \in V_+(m) \setminus S_{t_0}$. The existence of an admissible pair implies that there exists $u \in S_{t_0}$. Since $S_0 \subseteq S_u$, $\{t_0, t_1, \ldots, t_\ell\} \subseteq S_u$. Then (⋆) applied to $u$, Claim 24.2 and (⋆) applied to $t_0$, imply that $\frac{1}{2}m(V) - 1 \geq m(S_u) \geq \ell - 1 = m(V \setminus S_u) = m(V) - m(S_u) \geq 4m(V) + 1$, contradiction. Thus $|M_t| = 2$ for all $t \in V_+(m)$ and, by Claim 23 applied to $M_i$ and $M_{t_0}$, 26 follows. □

By Claim 26, for all $t \in V_+(m)$, there exist $t_1, t_2 \in V_+(m)$ such that $M_t = \{T_{t_1} \cup T_{t_1}, T_{t_1} \cup T_{t_2}\}$ and $m(T_t) = 1$. Then, by (⋆) and $m(V) \geq 4$, for all $t \in V_+(m)$, $m(T_t) = 4m(T_t) + m(T_{t_2}) = m(T_{t_1} \cup T_{t_2}) \geq \frac{1}{2}m(V) + 1 \geq 3$ and hence $m(V) = |V_+(m)| = 4$. Let $V_+(m) = \{a_1, a_2, a_3, a_4\}$ so that $M_{a_1} = \{T_{a_1} \cup T_{a_2}, T_{a_1} \cup T_{a_4}\}$.

**Claim 27.** $A = \{T_{a_1}, T_{a_2}, T_{a_3}, T_{a_4}\}$ is a simple $C^1_4$-obstruction.

**Proof.** By Claim 26 and $M_{a_1} = \{T_{a_1} \cup T_{a_2}, T_{a_1} \cup T_{a_4}\}$, $M_{a_3} = \{T_{a_2} \cup T_{a_3}, T_{a_3} \cup T_{a_4}\}$. Since $T_{a_1} \cup T_{a_2}$ is dangerous, so is $V \setminus (T_{a_1} \cup T_{a_2})$ by $m(V) = 4$. By maximality, $V \setminus (T_{a_1} \cup T_{a_2}) = T_{a_3} \cup T_{a_4}$ so $\{T_{a_1}, T_{a_2}, T_{a_3}, T_{a_4}\}$ is a partition of $V$. By $m(V) = 4$, $m$ is $p$-admissible, the definition of $\sigma_p$ and Claim 26, we have $4 = m(V) \geq \sigma_p = \sum_{t=1}^4 p(T_{a_t}) = 4$, that implies Definitions 7.2 and 7.1d for $A$. Claim 26 also implies that $p(T_{a_6}) = p(T_{a_1} \cup T_{a_6+1})$, so Definitions 7.1a and 7.1b hold for $A$. Since there exists an admissible pair, Definition 7.1c also holds for $A$. □

By Claim 27, (ii) is satisfied and the theorem is proved. □

**Corollary 28.** There exists no $p_G$-admissible pair if and only if there exists a partition $\{V_1, \ldots, V_\ell\}$ of $V$ such that
1. \( \ell \geq 4 \),

2. for \( 1 \leq i < j \leq \ell \), \( m_G(V_i) = p_G(V_i) = p_G(V_i \cup V_j) = 1 \) and \( V_i \) is a maximal tight set, that is \( \{V_1, \ldots, V_\ell\} = \{T_w : w \in V_+(m_G)\} \).

3. for all \( e = uv \in E(G) \), there exists \( 1 \leq i_e \leq \ell \) such that \( u, v \subseteq V_{i_e} \).

4. \( p(\bigcup_{j \in J} V_j) = 1 \) for all nonempty \( J \subset \{1, \ldots, \ell\} \).

**Proof.** The sufficiency follows from 28.2 and Lemma 19. Let us see the necessity. For an \( m_G \)-positive element \( t \), by Lemma 19, let \( M_t = \{M_1, \ldots, M_{\ell-1}\} \) be a minimal family of maximal dangerous sets containing \( t \) and covering all the \( m_G \)-positive elements.

1. By Claim 20.2, \( \ell - 1 \geq 3 \).

2. By Claim 24 applied to \( M_t, \{V_{i+1} := M_i^* : i = 0, \ldots, \ell - 1\} \) is a subpartition of \( V \) satisfying 28.2. It is in fact a partition of \( V \) because if \( Z := V \setminus \bigcup_i V_i \neq \emptyset \), then by \( M_t \) covers all the \( m_G \)-positive elements, \( m_G \) is \( p_G \)-admissible, \( p_G \) is symmetric and \( M_t \) is \( p_G \)-positive for \( i = 1, \ldots, \ell - 1 \), so by Claim 12.1 applied to \( M_t \), we have \( 0 = m_G(Z) \geq p_G(Z) = p_G(\bigcup_{i=1}^{\ell-1} V_i) = p_G(\bigcup_{i=1}^{\ell-1} M_i) \geq p_G(M_t) \geq 1 \), a contradiction.

3.-4. Note that, by 28.2, Claim 13 can be used for any order of the sets in \( V_1, \ldots, V_\ell \) and then, by Claim 13.1-2, 28.3-4 follow.

**Corollary 29.** If there exists no \( p_G \)-admissible pair, \( m_G(V) \geq 6 \) and \( G' \) is obtained from \( G \) by a \( p_G \)-admissible improvement, then no \( p_{G'} \)-admissible pair exists.

**Proof.** Suppose that \( G' \) is obtained from \( G \) by the \( p_G \)-admissible improvement of \( uv \) to \( ux, vy \). Let \( V_1, \ldots, V_\ell \) be the partition of \( V \) provided by Corollary 28 applied for \( p_G \). Then there exist \( 1 \leq i, j, k \leq \ell \) such that \( x \in V_i, y \in V_j, u, v \subseteq V_k \). Let \( X = V_i \cup V_j \cup V_k \). By Corollary 28.2, since the improvement is \( p_G \)-admissible, and by Corollary 28.4, we have \( 1 = 1 + 1 + 1 - 2 = m_G(V_i) + m_G(V_j) + m_G(V_k) - 2 \geq m_G(X) - 2 = m_G' \geq p_G(X) = p_G(X) = 1 \), and then, by Corollary 28 for \( p_G \) and \( m_G(V) \geq 6 \), it follows that \( \{V_1, \ldots, V_\ell\} \setminus \{V_i, V_j, V_k\} \cup \{X\} \) satisfies 28.1-4 for \( p_{G'} \) and hence, by Corollary 28, no \( p_{G'} \)-admissible pair exists.

**3.4 Perilous sets**

In the previous section, the study of admissible pairs lead to dangerous sets. Here, we are interested in admissible flips and improvements. This is where perilous sets come into play, see Lemma 31. We will often implicitly use the fact that a perilous set is \( p_0 \)-positive, see Observation 5.

**Lemma 30.** Let \( x \) and \( y \) be two distinct \( m_G \)-positive elements and \( uv \) an edge of \( G \).

(i) Flipping \( uv \) for \( ux \) is \( p_G \)-admissible if and only if no dangerous set contains \( x \) and \( u \) but not \( v \).

(ii) Improving \( uv \) to \( ux, vy \) is \( p_G \)-admissible if and only if both flipping \( uv \) for \( ux \) and flipping \( vu \) for \( vy \) are \( p_G \)-admissible, and no dangerous set contains \( x, u, v \) and \( y \).

**Proof.** (i) Recall that flipping \( uv \) for \( ux \) consists of first unsplitting \( uv \) and afterwards splitting off at \( x, u \). Since unsplitting \( uv \) is \( p_G \)-admissible, \( m_G^{uv} \) is \( p_G^{uv} \)-admissible. Then, flipping \( uv \) for \( ux \) is not \( p_G \)-admissible if and only if splitting off at \( u, x \) is not \( p_G^{uv} \)-admissible which is equivalent, by Lemma 19, to the fact that there exists a dangerous set \( X \) with
With respect to $p_{G_{uv}}$ and $m_{G_{uv}}$ (that is $1 \geq s_{G_{uv}}(X)$) containing $x$ and $u$. Then, Observation 4 and $s_G(X) \geq 0$ imply $v \notin X$ and $s_G(X) = s_{G_{uv}}(X) \leq 1$, that is $X$ is dangerous with respect to $p_G$ and $m_G$ containing $x$ and $u$ but not $v$, proving the first statement of the lemma.

(ii) Note that improving $uv$ to $ux$, $vy$ can be considered as flipping $uv$ for $ux$ and then splitting off at $v, y$. Let $H$ (resp. $K$) be the graph obtained from $G$ after flipping $uv$ for $ux$ (resp. improving $uv$ to $ux, vy$).

(a) To prove the necessity, suppose that improving $uv$ to $ux, vy$ is $p_G$-admissible, that is $m_K$ is $p_K$-admissible. Since unsplitting $vy$ is $p_K$-admissible, $m_H$ is $p_H$-admissible, that is flipping $uv$ for $ux$ is $p_G$-admissible. Similarly, flipping $vu$ for $vy$ is $p_G$-admissible. If a dangerous set $X$ of $G$ contained $x, u, v$ and $y$, then, by Observation 4 and since $m_K$ is $p_K$-admissible, we have $1 \geq s_G(X) = s_K(X) + 2 \geq 2$, a contradiction.

(b) To prove the sufficiency, suppose that improving $uv$ to $ux, vy$ is not $p_G$-admissible. If the $(uw, ux)$-flip or the $(vu, vy)$-flip is not $p_G$-admissible, then we are done, hence suppose they are both $p_G$-admissible. Since the splitting off at $v, y$ is not $p_H$-admissible, there exists, by Lemma 19, a set $X$ containing $y$ and $v$ which is dangerous with respect to $p_H$ and $m_H$. Note that, since flipping $vu$ for $vy$ is $p_G$-admissible, $X$ is not dangerous with respect to $p_{G_{uv}}$ and $m_{G_{uv}}$. Hence we have $s_{G_{uv}}(X) \geq 2 > 1 \geq s_H(X)$. Since $G_{uv} = H[ux]$, Observation 4 implies that $X$ contains $u$ and $x$. Therefore $s_G(X) = s_H(X)$, thus $X$ is also dangerous with respect to $p_G$ and $m_G$, and contains $x, u, v$ and $y$.

As suggests Lemma 22, it is reasonable to study the admissibility of flips and improvements when $p_G \leq 1$. The following lemma reveals how perilous sets arise in the process. In fact, perilous sets will always be studied when $p_G \leq 1$ and $\{T_w, w \in V_+(m_G)\}$ is a partition of $V$. We derive some of their properties in this situation. Note that, then, $T_x \cap T_y = \emptyset$ whenever $x$ and $y$ are distinct $m_G$-positive elements.

**Lemma 31.** Suppose that $p_G \leq 1$, $\{T_w, w \in V_+(m_G)\}$ is a partition of $V$, $x, y, z \in V_+(m_G)$, $uv \in E$ and $u, v \in T_z$.

(i) Flipping $uv$ for $ux$ is $p_G$-admissible if and only if there exists no $(uw, ux)$-perilous set.

(ii) Improving $uv$ to $ux, vy$ is $p_G$-admissible if and only if neither a $(uw, ux)$-nor a $(vu, vy)$-perilous set exists.

**Proof.** (i) The necessity comes from Lemma 30(i) and the definition of perilous sets. To see the sufficiency, by Lemma 30(i), we just have to show that a dangerous set $X$ containing $u$ and $x$ but not $v$ is a perilous set. Indeed, by $x \in V_+(m_G) \cap X$, $m_G$ is modular and non-negative, $X$ is dangerous, $p_G \leq 1$, we have $0 \leq m_G(x) - 1 \leq m_G(X) - 1 \leq p_G(X) \leq 1$. Then, by $m_G(V) \geq 4$, $X$ crosses $T_z$, therefore, by Claim 18, we have $p_G(X) = 0$, and 31(i) is proved.

(ii) We apply Lemma 30(ii). First, no dangerous set contains $x, u, v, y$: if $X$ was such a set, since $p_G \leq 1$, we would have $2 \leq m_G(X) = p_G(X) + 1 \leq 2$. It would imply $z \notin X$, hence $X$ and $T_z$ would be crossing because $m_G(V) \geq 4$, contradicting Claim 18. Then, applying 31(i) to the $(uw, ux)$-or the $(vu, vy)$-flip gives 31(ii).

The following results will be applied when either no admissible splitting exists or a simple $C_4^*$-obstacle exists.
Claim 32. Suppose that $p_G \leq 1$, \( \{T_w, w \in V_+(m_G)\} \) is a partition of \( V \), \( x, z \in V_+(m_G) \), \( x \neq z \), \( uv \in E \) and \( u \in T_z \). If \( X \) is a \((uv, ux)\)-perilous set, then

1. \( m_G(T_w) = 1 \) for all \( w \in \{T_w, w \in V_+(m_G)\} \),
2. \( m_G(X \cap T_z) = 0 \), \( p_G(X \cap T_z) = 0 \), \( p_G(X \cup T_z) = 1 \) and \( d(X, T_z) = 0 \),
3. \( p_G(X \setminus T_z) \geq 0 \), \( p_G(T_z \setminus X) \geq 0 \) and \( \bar{d}(X, T_z) = 0 \),
4. \( X \cup T_z = T_x \cup T_z \), that is \( X \setminus T_z = T_z \).

**Proof.** By \( u \in X \cap T_z \), \( x \in V_+(m_G) \cap (X \setminus T_z) \), \( m_G(X) = 1 \) and \( m_G(V) \geq 4 \), (13) and (14) apply to \( X \) and \( T_z \).

1. By \( w \in V_+(m_G) \cap T_w \), \( T_w \) is tight and \( p_G \leq 1 \), we have \( 1 \leq m_G(T_w) = p_G(T_w) \leq 1 \) and 32.1 follows.
2. By (7), \( m_G(X) = 1 \) and \( x \in V_+(m_G) \cap (X \setminus T_z) \), we have \( p_G(X \cap T_z) \leq m_G(X \cap T_z) = 0 \).
   Then, by \( X \) is perilous, Observation 5, (13) applied to \( X \) and \( T_z \) and \( p_G \leq 1 \), we get \( 0 + 1 \leq p_G(X) + p_G(T_z) \leq p_G(X \cap T_z) + p_G(X \cup T_z) - 2d(X, T_z) \leq 0 + 1 - 0 \) and 32.2 follows.
3. By \( X \) is perilous, Observation 5, (14) and \( p_G \leq 1 \), we get \( 0 + 1 \leq p_G(X) + p_G(T_z) \leq p_G(X \cap T_z) + p_G(T_z \setminus X) - 2d(X, T_z) \leq 0 + 1 - 2d(X, T_z) \) and 32.3 follows.
4. Since \( \{T_w, w \in V_+(m_G)\} \) partitions \( V \), \( m_G(V) \geq 4 \) and, by \( m_G \) is modular, \( X \) is perilous and 32.1-2, we have \( m_G(X \cup T_z) = m_G(X) + m_G(T_z) - m_G(X \cap T_z) = 1 + 1 - 0 = 2 \), Claim 18 implies \( X \cup T_z = T_x \cup T_z \). Then, \( T_x \cap T_z = \emptyset \) implies \( X \setminus T_z = T_z \). \( \square \)

Corollary 33. Suppose that \( p_G \leq 1 \), \( \{T_w, w \in V_+(m_G)\} \) is a partition of \( V \), \( z \in V_+(m_G) \), \( uv \in E \) and \( u \in T_z \). If \( x \in V_+(m_G) \setminus z \) and \( v \notin T_z \), then no \((uv, ux)\)-perilous set exists.

**Proof.** Indeed, if \( X \) is \((uv, ux)\)-perilous for some \( x \in V_+(m_G) \setminus z \), then, by Claim 32.4, we have \( X \setminus T_z = T_z \). Moreover, by Claim 32.3, we have \( \bar{d}(X, T_z) = 0 \), hence \( v \in X \cup T_z \). Since, by definition, \( v \notin X \), we get \( v \in T_z \). However, by assumption, \( v \notin T_z \). This contradiction proves the corollary. \( \square \)

For \( e = uv \in E, z \in V_+(m_G) \) such that \( u, v \in T_z \), let \( R_e = \{ x \in V_+(m_G) \setminus \{z\} : \) there exists an \((uv, x)\)-perilous set, \} and \( R_e \) the family of \((uv, ux)\)-perilous sets, for \( x \in R_e \).

Claim 34. Suppose that \( p_G \leq 1 \) and \( \{T_w, w \in V_+(m_G)\} \) is a partition of \( V \), \( e = uv \in E, z \in V_+(m_G) \), \( u, v \in T_z \).

1. If \( X \) and \( X' \) are \((uv, ux)\)- and \((uv, ux')\)-perilous sets for distinct \( x, x' \in R_e \), then \( \bar{d}(X, X') = 1 \), \( X \cap X' = X \cap T_z = X' \cap T_z \), \( p_G(X \cap X') = 0 \) and \( p_G(X \cup X') = 0 \).
2. If \( X \) and \( X' \) are \((uv, ux)\)- and \((uv, ux')\)-perilous sets for \( x, x' \in R_e \), then \( X \cap X' = \emptyset \), \( x \neq x' \), \( p_G(T_z \setminus X) = p_G(T_z \setminus X') = 0 \) and \( p_G(T_z \setminus (X \cup X')) = 1 \).
3. If \( |R_e| \geq 2 \), then there exists a unique \( X_e \subset T_z \) such that \( R_e = \{ X_e \cup T_z, x \in R_e \} \).
4. If all the edges of \( G \) are contained in members of the partition \( \{T_w, w \in V_+(m_G)\} \) of \( V \), then \( d_G(X_e) = 1 \) and \( p_0(X_e) = 1 \).
5. The family \( \{T_w, w \in V_+(m_G)\} \cup \{X_e, e \in E : X_e \text{ exists}\} \) is laminar.
Proof. 1. Since e connects $X \cap X'$ and $V \setminus (X \cup X')$ and $p_G \leq 1$, applying (14) to the perilous sets $X$ and $X'$ gives $p_G(X \setminus X') = p_G(X' \setminus X) = 1$ and $d(X, X') = 1$.

By Claim 32.4, we have $T_x = X \setminus T_z$ and $T_z \cap X' = \emptyset$, hence $T_x \subseteq X \setminus X'$. Since $m_G(X \setminus X') = 1$, the maximality of $T_x$ implies $X \setminus X' = T_x$. Similarly, we have $X' \setminus X = T_{x'}$. Thus $X \cap X' = X \cap T_z = X' \cap T_{x'}$.

Then, by Claim 32.2, $p_G(X \cap X') = 0$. By (13) applied to $X$ and $X'$ and Claim 18, we get $0 + 0 = p_G(X) + p_G(X') \leq p_G(X \cap X') + p_G(X \cup X') \leq 0 + 0$ and 34.1 follows.

2. By Claim 32.1 and $x \in R_e$, we have $x \in X \setminus T_z$. Note that $u \in (X \setminus X') \cap T_z$, $v \in (X' \setminus X) \cap T_z$, and, by Claim 32.2, $z \in T_z \setminus (X \cup X')$. If $X \cap X' \neq \emptyset$, then, $X$ and $X'$ are crossing, and, by $m_G(V) \geq 4$ and Claim 32.1-4, $X \cup X'$ and $T_z$ are also crossing. Then, by $X$ and $X'$ are perilous, (13) applied to $X$ and $X'$, $e$ connects $X \setminus X'$ and $X' \setminus X$, $p_G \leq 1$ and Claim 18 applied for $X \cup X'$ and $T_z$, we have $0 + 0 = p_G(X) + p_G(X') \leq p_G(X \cap X') + p_G(X \cup X') - 2d(X, X') \leq 1 + 0 - 2$, a contradiction.

By $X \cap X' = \emptyset$, $x \in X$ and $x' \in X'$, we have $x \neq x'$.

Let $Y = T_z \setminus X$ and $Y' = T_z \setminus X'$. Claim 32.3 applied to $T_z$ and $X$ and $X'$ gives that $p_G(Y), p_G(Y') \geq 0$. By (13) applied to $Y$ and $Y'$, $p_G \leq 1$ and $e$ connects $Y \setminus Y'$ and $Y' \setminus Y$, we have $0 = p_G(Y) = p_G(Y')$ and $1 = p_G(Y \setminus Y')$, and 34.2 follows.

3. Defining $X_e = X \cap X'$, 34.1 gives 34.3.

4. By 34.1 and all the edges of $G$ are contained in members of the partition $\{T_w, w \in V_+(m_G)\}$ of $V$, $d_G(X_e) = 1$ and $p_G(X_e) = 0$. Then, by $p_0 = p_G + d_G$, 34.4 follows.

5. Since $\{T_w, w \in V_+(m_G)\}$ is a partition of $V$ and every $X_e$ is contained in some $T_w$, we just have to check that $\{X_e, e \in E : X_e$ exists$\}$ is laminar. Let $X_e$ and $X_f$ be two sets in this family for $e \neq f$. Note that it is possible that $e = uv$ and $f = vu$. If $e \subseteq T_s$ and $f \subseteq T_t$ for distinct $s, t \in V_+(m_G)$, then $X_e \subseteq T_s$ and $X_f \subseteq T_t$ do not intersect. Suppose that $e$ and $f$ are contained in $T_z$. Then, by 34.3, there exist two distinct $x, y \in V_+(m_G) \setminus \{z\}$ such that $X = X_e \cup T_z$ is a $(e, x)$-perilous set and $Y = X_f \cup T_y$ is a $(f, y)$-perilous set. If $X_e$ and $X_f$ are crossing, then so are $X$ and $Y$. Since $p_G \leq 1$, applying (14) to $X$ and $Y$, and then Claim 18, gives that $p_G(X \setminus Y) = p_G(Y \setminus X) = 0$. Then $e$ enters either the $(e, x)$-perilous set $X \setminus Y$ or the $(e, y)$-perilous set $Y$, both cases contradict the definition of $X_e$.

We reformulate an important part of Claim 34.2 as follows.

Lemma 35. Suppose that $p_G \leq 1$, $\{T_w, w \in V_+(m_G)\}$ is a partition of $V$, $x, z \in V_+(m_G)$, $x \neq z$, $uv \in E$ and $u, v \in T_z$. If there exists a $(uv, ux)$-perilous set, then no $(vu, vx)$-perilous set exists.

The following claim will be applied when no admissible splitting exists.

Claim 36. If $m_G$ is $P$-feasible, there exists no $p_G$-admissible pair, $z \in V_+(m_G)$, $e = uv \in E(G)$, $u, v \in T_z$ and no allowed improvement exists for $e$, then

1. there exists a $(uv, w)$-perilous set for some $w \in V_+(m_G) \setminus \{z\}$,

2. if $X$ is a $(uv, ux)$-perilous set for some $x \in V_+(m_G)$, then $T_w \cup (X \cap T_z)$ is a $(uv, uw)$-perilous set for all $w \in V_+(m_G) \setminus \{z\}$,

3. $X_e$ is well defined and $p_0(X_e \cup T_w) = 1$ for all $w \in V_+(m_G) \setminus \{z\}$. 

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Proof. By Lemma 22, we have $p_G \leq 1$. Moreover, by Corollary 28, we have $m_G(V) \geq 4$, \{T_w, w \in V_+(m_G)\} is a partition of $V$, and $p(T_z \cup T_y) = 1$ for all $x, z \in V_+(m_G)$. Suppose that $m_G(P_1) = \max\{m_G(P) : P \in P\}$.

1. Without loss of generality, there exist $x \in V_+(m_G) \cap P_1$ and $y \in V_+(m_G) \setminus P_1$ such that $x \neq z \neq y$. By possibly exchanging $u$ and $v$, we can assume that $c(u) \neq c(x)$ and $c(v) \neq c(y)$. Then, since improving $uv$ to $xu, vy$ is not allowed, it is not $p_G$-admissible, and, by Lemma 31(ii), 36.1 follows for $w = x$ or $w = y$.

2. Let $w \in V_+(m_G) \setminus \{x, z\}$. Since $m_G(V) \geq 4$, there exists $y \in V_+(m_G) \setminus \{x, w, z\}$. Apply (3) to $X$ and $W = T_x \cup T_y$ to get that $p_G(X \cup W) \geq 0$. Now apply (4) to $X \cup W$ and $T_x \cup T_y$, and then Claim 18, to obtain that $p_G(T_w \cup (X \cap T_2)) = 0$. Therefore, by Claim 32.1-2, $T_w \cup (X \cap T_2)$ is a $(w, uw)$-perilous set and 36.2 follows.

3. By $m_G(V) \geq 4$ and Claim 34.3, $X_e$ exists. By 36.2, $X_e \cup T_w$ is perilous, so $p_G(X_e \cup T_w) = 0$. Then, by Corollary 28.3, we have $d_G(T_w) = 0$, by Claim 34.4, we have $d_G(X_e) = 1$ and hence, by $p_0(X_e \cup T_w) = p_G(X_e \cup T_w) + d_G(X_e \cup T_w) = 0 + 1 = 1$, 36.3 follows.

\[\square\]

### 3.5 A special full partition

Let us introduce the following families:

\[
\begin{align*}
U &= \{T_w : w \in V_+(m_G)\} \cup \{X_e : e \in E(G)\}, \\
U^* &= U \setminus \bigcup\{U' : U' \in \mathcal{U}, U' \subseteq U\} \text{ for all } U \in \mathcal{U}, \\
U^* &= \{U^* : U \in \mathcal{U}\}.
\end{align*}
\]

**Lemma 37.** Suppose that $m_G$ is allowed, $m_G(V) \geq 4$, neither a $p_G$-admissible splitting off nor an allowed improvement exists. Then the following properties hold.

1. $\{T_w, w \in V_+(m_G)\}$ is a tight partition of $V$ and, for all $e = uv \in E(G)$, there exists $1 \leq i_e \leq \ell$ such that $u, v \subseteq V_{i_e}$ and $X_e$ exists.

2. $U^*$ is a partition of $V$.

3. $U^* \neq \emptyset$ for all $U \in \mathcal{U}$.

4. $p_0(T_w \cup U) = 1$ for all $w \in V_+(m_G)$ and $U \in \mathcal{U}$.

5. $p_0(T_w \cup U^*) \geq 1$ for all $w \in V_+(m_G)$ and $U \in \mathcal{U}$.

6. $p_0(U^* \cup W^*) \geq 1$ for all $U, W \in \mathcal{U}$.

7. $U^*$ is a $p_0$-full partition of $V$.

8. $|U^*| = m_G(V) + |E(G)|$.

9. $m_0$ is not $p_0$-legal.

10. There exists a $\mathcal{P}$-partite graph $F$ on $V$ that covers $p_0$ with $|E(F)| \leq \dim(p_0) - 1$.

**Proof.** By Lemma 22, we have $p_G \leq 1$. 

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1. By Corollary 28.2-3, \( \{ T_w, w \in V_+(m_G) \} \) is a tight partition of \( V \) and, for all \( e \in E(G) \), there exists \( z_e \in V_+(m_G) \) such that \( e \subseteq T_{z_e} \). Let \( e = uw \in E(G) \). By Claim 36, we may assume that there exists a \((uw, uw)\)-perilous set for all \( w \in V_+(m_G) \setminus \{ z_e \} \). Now, by \( m_G(V) \geq 4 \) and Claim 34.3, there exists a unique \( X_e \subseteq T_{z_e} \setminus \{ z_e \} \) such that \( X_e \cap T_w \) is the \((uw, uw)\)-perilous set, for all \( e \in E(G) \) and \( w \in V_+(m_G) \setminus \{ z_e \} \).

2. By 37.1 and Claim 34.5, the family \( U \) is laminar and hence 37.2 follows.

3. If \( U \subseteq U \), then either \( U = T_w \) for \( w \in V_+(m_G) \) or \( U = X_e \) for \( e \in E(G) \). In the former case, by Claim 32.2, \( w \in U^* \), and in the latter case, by Claim 34.4, \( f \) is the only edge entering \( X_f \) for all \( f \in E(G) \), hence one end-vertex of \( e \) is in \( U^* \).

4. By Corollary 28.2-3, 37.4 is satisfied for all \( U \in \{ T_w, w \in V_+(m_G) \} \). By Claim 36.3, 37.4 is satisfied for all \( U \in \{ X_e : e \in E(G) \} \).

5. We may assume, by 37.4, that \( U \neq U^* \) and \( U \subseteq T_x \) where \( x \neq w \). By \( m_G(V) \geq 4 \), there exists \( y \in V_+(m_G) \setminus \{ w, x \} \). Apply Claim 12.1 for \( \{ T_y \cup W : W \text{ maximal set of } U \text{ strictly contained in } U \} \) and 37.4 to deduce that \( p_0(T_y \cup X) \geq 1 \) where \( X = \bigcup_{W \in U \cup W \subseteq U} W = U \setminus U^* \neq \emptyset \). Now, (4) applied to \( T_w \cup U \) and \( T_y \cup X \) gives, by 37.4, that \( p_0(T_w \cup U^*) + p_0(T_y \cup X) \geq p_0(T_w \cup U) + p_0(T_y) \geq 1 + 1 = 2 \), as claimed.

6. By \( m_G(V) \geq 4 \), let \( x \in V_+(m_G) \) be such that \( U \cap T_x = \emptyset \) and \( W \cap T_x = \emptyset \). Claim 12.1 applies to the family \( \{ T_x \cup X^* : X^* \in U^* \setminus \{ U^*, W^* \} \} \) and gives, by 37.5 and 37.2, that \( p_0(U^* \cup W^*) = p_0(V \setminus (U^* \cup W^*)) \geq 1 \), as claimed.

7. If there are no edges, then 37.7 comes from Corollary 28.4. Otherwise, a minimal \( X_e \in U \) belongs to \( U^* \) and satisfies \( p_0(X_e) = 1 \) by 37.1 and Claim 34.4. Then, by 37.2 and 37.6, Lemma 2.2 implies that \( U^* \) is a \( p_0 \)-full partition.

8. By 37.3, definition of \( U \), Corollary 28.2-3, \( m \) is modular and 37.1, we have \( |U^*| = |U| = |\{ T_w : w \in V_+(m_G) \}| + |\{ X_e : e \in E(G) \}| = \sum_{w \in V_+(m_G)} m_G(T_w) + |E(G)| = m_G(V) + |E(G)| \).

9. By 37.7, \( U^* \) is a \( p_0 \)-full partition of \( V \) and, hence, by 37.8, \( \dim(p_0) \geq |U^*| = m_G(V) + |E(G)| \). Then, by \( m_G(V) \geq 4 \), we have \( \frac{1}{2} m_0(V) = \frac{1}{2} m_G(V) + |E(G)| \geq m_G(V) + |E(G)| - 2 \leq \dim(p_0) - 2 \), that is \( m_0 \) is not \( p_0 \)-legal.

10. By 37.1, \( |V_+(m_G)| = m_G(V) \). By Observation 6, there exists a \( P \)-partite spanning tree \( T \) on \( V_+(m_G) \). Since \( T \) is a tree, we have \( |E(T)| = m_G(V) - 1 \). We claim that \( T \) covers \( p_G \). Otherwise, by \( p_G \leq 1 \), there exists a set \( X \) such that \( 0 \leq d_T(X) < p_G(X) \leq 1 \), that is \( 0 = d_T(X) \) and \( p_G(X) = 1 \). Since \( T \) is connected and \( p_G \) is symmetric, we may suppose that \( X \) does not contain any vertex of \( V_+(m_G) \). Then, by \( m_G \) is \( p_G \)-admissible, we have \( 0 = m_G(X) \geq p_G(X) = 1 \), a contradiction. Let \( F := (V, E(T) \cup E(G)) \). Since \( T \) and \( G \) are both \( P \)-partite, so is \( F \). Since \( T \) covers \( p_G \), we have \( d_T(X) \geq p_G(X) = p_0(X) - d_G(X) \) for all \( X \subseteq V \), so \( d_T(X) \geq p_0(X) \) for all \( X \subseteq V \), that is \( F \) covers \( p_0 \). Finally, by 37.8 and 37.7, we have \( |E(F)| = |E(T)| + |E(G)| = m_G(V) - 1 + |E(G)| = |U^*| - 1 \leq \dim(p_0) - 1 \).

\[ \square \]

4 Degree specified version

In this section we solve the degree specified version of our problem: given a symmetric crossing supermodular function \( p_0 : 2^V \to \mathbb{Z} \), a degree specification \( m_0 : V \to \mathbb{Z}_+ \) and a partition \( P \) of \( V \), find a \( P \)-partite graph \( G \) on \( V \) covering \( p_0 \) and satisfying \( m_0 \). Note that finding such a graph is equivalent to finding a complete allowed splitting off.
4.1 Necessary conditions

Here, we provide necessary conditions for the existence of a complete allowed splitting off.

**Lemma 38.** Let \( p_0 : 2^V \to \mathbb{Z} \) be a symmetric crossing supermodular function, \( m_0 : V \to \mathbb{Z}_+ \) a degree specification and \( \mathcal{P} \) a partition of \( V \). If there exists a complete allowed splitting off, then \( m_0 \) is \( \mathcal{P} \)-feasible, \( p_0 \)-admissible and \( p_0 \)-legal.

**Proof.** Let \( G = (V, E) \) be a graph obtained by a complete allowed splitting off. Then \( G \) covers \( p_0 \) and satisfies \( m_0 \). Note that \( m_0(V) = \sum_{v \in V} m_0(v) = \sum_{v \in V} d_G(v) = 2|E| \), hence (19) holds. Since every splitting off is rainbow, we get (20). Hence \( m_0 \) is \( \mathcal{P} \)-feasible. By (6) and (5), we have \( m_0(X) = \sum_{x \in X} m_0(x) = \sum_{x \in X} d_G(x) \geq d_G(X) \geq p_0(X) \), hence we obtain (7), that is \( m_0 \) is \( p_0 \)-admissible. Since \( G \) covers \( p_0 \), \( G \) has, by Lemma 2.1, at least \( \dim(p_0) - 1 \) edges. By \( m_0(V) = 2|E| \), we get (10), that is \( m_0 \) is \( p_0 \)-legal. \( \square \)

4.2 Obstacles

It turns out that the conditions that appeared in Section 4.1 are not sufficient to have a complete allowed splitting off. Exceptional structures must be forbidden in order to get the sufficiency. The description of these structures are given in Section 2. An obstacle is a partition \( A \) of \( V \) satisfying two types of conditions. On the one hand, the partition \( A \) fulfill rigorous conditions. On the other hand, the partition \( A \) is closely related to the partition \( \mathcal{P} \). We mention that the \( C_5^* \)-obstacle arises only for our abstract form of the problem, it does not exist in the framework of graphs or hypergraphs.

4.3 Properties of constructions

In this section we provide some basic properties of constructions.

**Claim 39.** Let \( A = \{A_1, \ldots, A_4\} \) be a \( C_4^* \)-construction for \( p \). Then, for \( i = 1, \ldots, 4 \),

1. \( p(A_i) + p(A_{i+1}) - 1 \geq p(A_i \cup A_{i+1}) \),
2. \( p(A_i) \geq 1 \),
3. \( p(A_i \cup A_{i+1}) \geq 1 \),
4. \( A \) is a simple \( C_4^* \)-construction if and only if \( p(A_i \cup A_{i+1}) = 1 \) for \( i = 1, \ldots, 4 \).

**Proof.**

1. By Definition 7.1d and the definition of \( \sigma_p \), we have \( p(A_i) + p(A_{i+1}) \geq p(A_i \cup A_{i+1}) \) and then, by Definition 7.1a, 39.1 follows.

2. By Definition 7.1b and 39.1, we have \( p(A_{i-1}) + p(A_{i+1}) = p(A_{i-1} \cup A_i) + p(A_i \cup A_{i+1}) \leq p(A_{i-1}) + p(A_{i+1}) + 2p(A_i) - 2 \), and 39.2 follows.

3. By Definitions 7.1b and 7.1d, 39.1, the symmetry of \( p \) and Definition 7.1d, we have \( p(A_i \cup A_{i+1}) = \frac{1}{2} \sigma_p - p(A_{i-1} \cup A_i) \geq \frac{1}{2} \sigma_p - \frac{1}{2}((p(A_{i-1}) + p(A_i) - 1) + (p(A_{i+1}) + p(A_{i+2}) - 1)) = 1 \) and 39.3 follows.

4. By Definition 7.1b and 39.2 and 39.3, \( p(A_i) = 1 \) for \( i = 1, \ldots, 4 \) and only if \( p(A_i \cup A_{i+1}) = 1 \) for \( i = 1, \ldots, 4 \) and 39.4 follows. \( \square \)

**Claim 40.** If \( A = \{A_1, A_2, A_3, A_4, B_1, \ldots, B_t\} \) is a \( C_5^* \)-construction for \( p_G \), then for \( J \subseteq \{1, \ldots, t\} \) and \( i = 1, \ldots, 4 \),

1. \( p_G(A_i \cup A_{i+1} \cup \bigcup_{j \in J} B_j) = 1 \),
2. \( p_G(A_i \cup \bigcup_{j \in I} B_j) = 1 \),
3. \( p_G(\bigcup_{j \in I} B_j) = 2 \) if \( J \neq \emptyset \),
4. \( p_G(A_i \cup A_{i+2} \cup \bigcup_{j \in I} B_j) \leq 0 \).

**Proof.** We denote for \( I \subseteq \{1, \ldots, 4\} \) and \( J \subseteq \{1, \ldots, t\} \), \( \bar{I} := \{1, \ldots, 4\} \setminus I \) and \( \bar{J} := \{1, \ldots, t\} \setminus J \), 
\( A_I := \bigcup_{j \in I} A_j, B_j := \bigcup_{j \in J} B_j, A := \bigcup_{i=1}^{4} A_i \) and \( B := \bigcup_{i=1}^{4} B_i \). Let \( i \in \{1, \ldots, 4\} \) and \( J \subseteq \{1, \ldots, t\} \).

1. By Definitions 8.1d, 8.1a and 8.1c, Claim 12.1 applies to \( \{A_i \cup A_{i+1}\} \cup \{A_i \cup B_j : j \in J\} \cup \{A_i \cup B_j : j \in J\} \). Let \( I = \{i, i+1\} \). Then, using the symmetry of \( p_G \) and Definition 8.1d, we get \( 1 = p_G(A_I) \leq p_G(A_I \cup B_J) \leq p_G(A_I \cup B) = p_G(A_I) = p_G(A_{i+2} \cup A_{i+3}) = 1 \), and 40.1 follows.

2. By Definitions 8.1d, 8.1a, (13) applied to \( A_{i+1} \cup A_{i+2} \) and \( A_{i+2} \cup A_{i+3} \), we have \( p_G(A_{i+1 \cup i+2}) \geq 1 \). Hence, by Definition 8.1c, Claim 12.1 applies to \( \{A_{i+1 \cup i+2}\} \cup \{A_{i+1 \cup B_j} : j \in J\} \cup \{A_{i+1 \cup B_j} : j \in J\} \). Then, by the symmetry of \( p_G \), we get \( 1 \leq p_G(A_{i+1 \cup i+2}) \leq p_G(A_{i+1 \cup B_J}) \leq p_G(A_{i+1 \cup B_J} \cup B) = p_G(A_i) = 1 \), and we have equality everywhere. In particular, by the symmetry of \( p_G \), we have \( p_G(A_i \cup B_J) = p_G(A_{i+1 \cup B_J}) = 1 \).

3. 40.2 and the symmetry of \( p_G \) imply \( p_G(A_{i+1 \cup i+2}) = p_G(A_{i+2 \cup i+3}) = 1 \). Therefore, (13) applied to these two sets and Definition 8.1e give \( p_G(A) \geq p_G(A_{i+1 \cup i+2}) + p_G(A_{i+2 \cup i+3}) - p_G(A_{i+1 \cup B_J}) \geq 1 + 1 - 2 = 0 \). Since \( J \neq \emptyset \), there exists \( k \in J \). By Definitions 8.1a and 8.1c, Claim 12.1 applies to \( \{A\} \cup \{A_{i+1 \cup B_j} : j \in J\} \cup \{A_{i+1 \cup B_j} : j \in J\} \). Then, by the symmetry of \( p_G \), we have \( 2 \leq p_G(A) \leq p_G(A_{i+1 \cup B_J}) \leq p_G(A_{i+1 \cup B_J} \cup B) = p_G(A_{i+1}) = 2 \), and we have equality everywhere. In particular, by the symmetry of \( p_G \), we have \( p_G(B_J) = p_G(A_{i+1 \cup B_J}) = 2 \).

4. Let \( I = \{i, i+2\} \). If \( p_G(A_i \cup B_J) > 0 \), then, by Definitions 8.1a and 1c, Claim 12.1 applies to \( \{A_i \cup B_J\} \cup \{A_i \cup B_j : j \in J\} \). Then, by the symmetry of \( p_G \) and Definition 8.1e, we have \( 0 < p_G(A_i \cup B_J) \leq p_G(A_I \cup B) = p_G(A_{i+1 \cup B_J}) = p_G(A_i \cup A_{i+1} \cup A_{i+3}) \leq 0 \), a contradiction.

\[ \square \]

### 4.4 Properties of obstructions

We show now that obstructions are unique up to cyclically reordering their elements.

**Claim 41.** Every set of an obstruction is tight.

**Proof.** Let \( \mathcal{A} \) be an obstruction. Using that \( \mathcal{A} \) is a partition of \( V \), \( m \) is \( p \)-admissible by Definitions 7.2, 8.2, 9.2, then Definitions 7.1d, 8.1f, 9.1d, and finally Theorem 1, we have \( m(V) = \sum_{X \in \mathcal{A}} m(X) \geq \sum_{X \in \mathcal{A}} p(X) = \sigma_p = m(V) \). Thus, there is equality everywhere and every set of \( \mathcal{A} \) is tight. \[ \square \]

**Lemma 42.** If \( \mathcal{A} \) is an obstruction for \( (p, m) \) and the \( m \)-positive elements \( x \) and \( y \) belong to distinct non-consecutive elements of \( \mathcal{A} \), then splitting off at \( x, y \) is \( p \)-admissible.

**Proof.** By Claim 41, every set of \( \mathcal{A} \) is tight. Suppose that the splitting off at \( x, y \) is not \( p \)-admissible. Then, by Lemma 19, there exists a maximal dangerous set \( Y \) containing \( x \) and \( y \). By Observation 5, \( p(Y) \geq 1 \). By Claim 20.1, \( Y \cup A \neq V \) for all \( A \in \mathcal{A} \). Therefore, by Claim 20.3, \( Y \) is the union of elements of \( \mathcal{A} \).
1. If \( \mathcal{A} \) is a \( C_4^* \)-obstruction. Then, by non-consecutiveness, \( x \in A_i \) and \( y \in A_{i+2} \) for some \( i \in \{1, \ldots, 4\} \) and hence \( A_i \cup A_{i+2} \subseteq Y \). By Claim 20.1, modularity and non-negativity of \( m \), Claim 41, and Definitions 7.1d, 7.2, we have \( \frac{1}{2}m(V) \geq m(Y) \geq m(A_i) + m(A_{i+2}) = p(A_i) + p(A_{i+2}) = \frac{1}{2}\sigma_p = \frac{1}{2}m(V) \), thus \( m(Y) = \frac{1}{2}\sigma_p \) and, by Claims 41 and 39.2, \( A_i \cup A_{i+2} = Y \). By Claim 39.3, \( p(A_i \cup A_{i+1}) \geq 1 \), therefore, (3) and (4) apply to \( Y \) and \( A_i \cup A_{i+1} \). By Claim 39.3, (3) and (4), the symmetry of \( p \), Definitions 7.1d and since \( Y \) is dangerous, we get \( p(Y) + 1 \leq p(Y) + p(A_i \cup A_{i+1}) \leq \frac{1}{4}((p(A_i) + p(A_{i+3})) + (p(A_{i+1}) + p(A_{i+2}))) = \frac{1}{2}\sigma_p = m(Y) \leq p(Y) + 1 \). Thus \( p(A_i \cup A_{i+1}) = 1 \). Similarly, \( p(A_{i+1} \cup A_{i+2}) = 1 \). Then, by Claim 39.4, \( \mathcal{A} \) is a simple \( C_4^* \)-obstruction. Then, by Definition 7.1c, we have \( 0 \geq p(Y) \geq 1 \), a contradiction.

2. If \( \mathcal{A} \) is a \( C_5^* \)-obstruction. Let \( a \) and \( b \) be the number of \( A_i \) and \( B_j \) belonging to \( Y \). Note that \( a + b \geq 2 \). By \( Y \) is dangerous, \( m \) is modular, Claim 41 and Definitions 8.1a, 8.1b, we have \( p(Y) \geq m(Y) - 1 = a + 2b - 1 \). By Claim 40 and the symmetry of \( p \), we get \( 2 \geq p(Y) \). Thus \( b \leq 1 \) and \( a \leq 3 \). Then, by \( a + b \geq 2 \), we have \( 1 \leq a \leq 3 \). In this case, Claim 40 gives \( 1 \geq p(Y) \). It follows that \( b = 0 \) and \( a = 2 \). Then, by Definition 8.1e and the non-consecutiveness of the two \( A_i \)'s contained in \( Y \), and since \( Y \) is dangerous, we get \( 0 \geq p(Y) \geq 1 \), a contradiction.

3. If \( \mathcal{A} \) is a \( C_6^* \)-obstruction. By Definition 9.1c, \( \mathcal{A} \) is not \( p \)-full so, by Claim 13.4 and \( p(Y) \geq 1 \), \( Y \) is consecutive. Then, Claim 13.2 gives \( p(Y) = 1 \). Since \( Y \) is dangerous, \( 1 = p(Y) \geq m(Y) - 1 \geq 1 \), and hence \( m(Y) = 2 \) that is \( x \) and \( y \) belong to consecutive sets, a contradiction.

\[ \square \]

Lemma 43. If \( \mathcal{A} \) is an obstruction for \( (p_G, m_G) \), then \( \mathcal{A} \) is the unique partition of \( V \) into maximal \( p_0 \)-positive tight sets.

Proof. Let \( X \) be an element of \( \mathcal{A} \). By Claim 39.2, Definitions 8.1a-1b, 9.1a and Claim 41, \( X \) is \( p_0 \)-positive and tight. Let \( Y \) be a maximal tight set containing \( X \) and suppose that \( X \neq Y \), that is, since \( \mathcal{A} \) is a partition, \( Y \) intersects some other \( X' \in \mathcal{A} \). Since \( X \) is \( p_0 \)-positive, so is \( Y \). By Claims 14.1, we have \( X' \subset Y \). By \( Y \) is tight and Lemmas 42 and 19, the exists an index \( i \) such that \( Y = A_i \cup A_{i+1} \). Then, by \( m_G \) is modular and Claim 41, we have \( 0 = m_G(A_i) + m_G(A_{i+1}) - m_G(A_i \cup A_{i+1}) = p_G(A_i) + p_G(A_{i+1}) - p_G(A_i \cup A_{i+1}) \) is even, that contradicts Definitions 7.1a or 8.1a-1d or 9.1a-1b. Suppose that \( A' \) is a partition of \( V \) into maximal \( p_0 \)-positive tight sets. Since the elements 37.7 of \( \mathcal{A} \) and \( A' \) are maximal \( p_0 \)-positive tight sets, by Claim 14.1, the two partitions coincide.

\[ \square \]

Corollary 44. If there exists a simple \( C_4^* \)-, a \( C_5^* \)- or a \( C_6^* \)-obstruction for \( (p_G, m_G) \), then it is the unique obstruction for \( (p_G, m_G) \), up to cyclically reordering its members.

Proof. By Lemma 43, an obstruction is the unique partition of \( V \) into maximal \( p_0 \)-positive tight sets. Since, by Definition 7.1 or 8.1 or 9.1, in a simple \( C_4^* \)-, a \( C_5^* \)- or a \( C_6^* \)-obstruction for \( (p_G, m_G) \), the union of any two sets of the obstruction has \( p_G \)-value 1 if they are consecutive, and non-positive \( p_G \)-value otherwise, we get the desired result.

\[ \square \]

4.5 Inherited obstructions

In the next three claims we prove that splitting off a special admissible pair in an obstruction gives rise to another obstruction. These results will be used in three different ways. First, they will help us to show that the existence of an obstruction implies the existence of a complete \( p \)-admissible splitting off. Second, we use that one about \( C_5^* \)-obstructions to show that if a \( C_5^* \)-obstruction exists that is not a \( C_4^* \)-obstacle, then a complete allowed splitting off exists. Finally, they will be applied in the next section about inherited obstacles.
We consider the three different obstructions separately.

**Claim 45.** A $C_4^*$-obstruction $\mathcal{A} = \{A_1, \ldots, A_4\}$ for $(p_G, m_G)$ is a $C_4^*$-obstruction for $(p_\bar{G}, m_\bar{G})$, where $\bar{G}$ is obtained from $G$ by a $p_\bar{G}$-admissible splitting off at $u \in A_j$ and $v \in A_{j+1}$ for some $j \in \{1, \ldots, 4\}$.

**Proof.** By definition, $m_\bar{G}(X) = m_G(X) - \chi_X(\chi_a + \chi_v)$ and $p_\bar{G}(X) = p_G(X) - d_{uv}(X)$. We verify the conditions of Definition 7 one by one.

7.1a Since, by (1) applied to $A_i$ and $A_{i+1}$, $d_{uv}(A_i) + d_{uv}(A_{i+1}) - d_{uv}(A_i \cup A_{i+1}) = 2d_{uv}(A_i, A_{i+1})$ is even, Definition 7.1a for $p_G$ implies that Definition 7.1a holds for $p_\bar{G}$.

7.1b Since $d_{uv}(A_{i-1} \cup A_i) + d_{uv}(A_i \cup A_{i+1}) = 1 = d_{uv}(A_{i-1}) + d_{uv}(A_{i+1})$, Definition 7.1b for $p_G$ implies that Definition 7.1b holds for $p_\bar{G}$.

7.1c If Definition 7.1c did not hold for $p_\bar{G}$ and $m_\bar{G}$, then $p_\bar{G}(A_i) = 1$ for $i = 1, \ldots, 4$ and $p_G(A_j \cup A_{j+2}) \geq 1$. Then, $p_G(A_j \cup A_{j+2}) \geq 2$, $p_G(A_{j+2}) = p_G(A_{j+3}) = 1$, and $p_G(A_j) = p_G(A_{j+1}) = 2$. Since the splitting is allowed, by Claim 19, $A_j \cup A_{j+1}$ is not dangerous for $m_G$ and $p_G$. Then, by Claim 41 and modularity of $m_G$, $4 = m_G(A_j \cup A_{j+1}) - p_G(A_j \cup A_{j+1}) + 2$ and hence, by Definition 7.1b, $p_G(A_j \cup A_{j+3}) \geq 1$. Then, by (14), $1 + 2 \leq p_G(A_j \cup A_{j+3}) + p_G(A_j \cup A_{j+2}) \leq p_G(A_{j+3}) + p_G(A_{j+2}) = 1 + 1$, a contradiction.

7.1d By $m_\bar{G}$ is $p_\bar{G}$-admissible and the definition of $\sigma_p$, we have $\sigma_{p_\bar{G}} \leq m_\bar{G}(V) = m_G(V) - 2 = \sum 4 p_G(A_i) \leq \sigma_{p_G}$ and 7.1d follows.

7.2 By Claim 17, $m_\bar{G}$ is minimally $p_\bar{G}$-admissible.

\[ \square \]

**Claim 46.** Let $\mathcal{A} = \{A_1, A_2, A_3, A_4, B_1, \ldots, B_i\}$ be a $C_5^*$-obstruction for $(p_G, m_G)$, $G$ obtained from $G$ by a $p_G$-admissible splitting off at a vertex of $A_i$ (resp. $B_i$) and a vertex of $B_j$ (resp. $B_i \neq B_j$) and $\mathcal{A}'$ obtained from $\mathcal{A}$ by deleting $B_j$ and replacing $A_i$ (resp. $B_i$) by $A_i \cup B_j$ (resp. $B_i \cup B_j$).

1. If $t = 1$, then $\mathcal{A}'$ is a simple $C_4^*$-obstruction for $(p_G, m_G)$.

2. If $t \geq 2$, then $\mathcal{A}'$ is a $C_5^*$-obstruction for $(p_G, m_G)$.

**Proof.** Since the splitting is $p_G$-admissible $m_G$ is $p_\bar{G}$-admissible. The minimality of $m_G$ follows from Claim 17. Hence Definition 7.2 if $t = 1$ (resp. Definition 8.2 if $t \geq 2$) holds for $m_G$ and $p_G$.

1. Definition 7.1a-1d for $\mathcal{A}'$ and $p_G$ follows from Definition 8.1a-1f for $\mathcal{A}$ and $p_G$, and Claim 40. Note that $p(A_i') = 1$ for all $A_i' \in \mathcal{A}'$ also follows.

2. Definition 8.1a-1e for $\mathcal{A}'$ and $p_G$ follows from Definition 8.1a-1e and Claim 40 for $\mathcal{A}$ and $p_G$. By Claim 17, Theorem 1 and Definition 8.1f for $\mathcal{A}$ and $p_G$, we have $\sigma_{p_G} = m_G(V) - 2 = \sigma_{p_G} - 2 = 2t + 4 - 2 = 2(t - 1) + 4$, hence Definition 8.1f holds for $\mathcal{A}'$ and $p_G$.

\[ \square \]

**Claim 47.** If $\mathcal{A}$ is a $C_6^*$-obstruction for $(p_G, m_G)$ and $\bar{G}$ is obtained from $G$ by splitting off at $u \in A_i \cup A_j$, $v \in A_i \cup A_j$ for some $i \in \{1, \ldots, 4\}$, then $\mathcal{A}' = \{A_1', A_2', A_3', A_4'\} = \{A_{i-1} \cup A_i \cup A_{i+1}, A_{i+2}, A_{i+3}, A_{i+4}\}$ is a simple $C_4^*$-obstruction for $(p_G, m_G)$.

\[ 22 \]
Proof. By Definition 9.1a and Claim 13.2 for $A$, we have $p_G(G_j' \cup G_{j+1}') = p_G(G_j' \cup G_{j+1}') = 1 = p_G(G_j') = p_G(G_{j+1}')$ for $j = 1, \ldots, 4$. Thus $A'$ satisfies 7.1a and 7.1b. By Definition 9.1c applied to $A_{j+2} \cup A_{j+4}$ and the symmetry of $p_G$, $A'$ satisfies 7.1c. Since, by Lemma 42, the splitting off is $p_G$-admissible, $m_G$ is $p_G'$-admissible. Then $m_G \leq m_G(V) = m_G(V) - 2 = 6 - 2 = 4 = \sum_1^4 p_G(G_j') \leq \sigma_{p_G}$ and 7.1d and 7.2 follows. 

\begin{lemma}
If $A$ is an obstruction for $(p, m)$, then there exists a complete $p$-admissible spilling off.
\end{lemma}

Proof. Let us consider the following cases.

(a) If $A$ is a simple $C_4^*$-obstruction, then, by Lemma 42, we are done.

(b) If $A$ is a $C_4^*$-obstruction that is not simple, then, by Corollary 44, no simple $C_4^*$-obstruction exists. Moreover, by Claim 41 and Definitions 7.1d and 7.2, we have $m(A_i \cup A_{i+1}) = \frac{1}{2}m(V)$. Therefore, by Lemma 25, there exists an admissible pair $u, v$ such that $u \in A_j$ and $v \in A_{j+1}$ for some $j \in \{1, \ldots, 4\}$. By Claim 45, after splitting off this pair, $A$ remains a $C_4^*$-obstruction. By repeating this, we arrive in the first case and then we are done.

(c) If $A$ is a $C_5^*$-obstruction, then, by the repeated application of Lemma 42 and Claim 46, we arrive in the first case and then we are done.

(d) If $A$ is a simple $C_6^*$-obstruction, then, by Lemma 42 and Claim 47, we arrive in the first case and then we are done.

\begin{lemma}
Suppose that $m_G$ is $\mathcal{P}$-feasible. If there exists a simple $C_4^*$-obstruction (resp. $C_5^*$-obstruction) for $(p_G, m_G)$ but no $C_4^*$-obstacle (resp. $C_5^*$-obstacle) for $(p_G, \mathcal{P}, m_G)$, then there exists a complete $(p_G, \mathcal{P})$-allowed splitting-off.
\end{lemma}

Proof. 1. First, suppose that $A = \{A_1, \ldots, A_4\}$ is a simple $C_4^*$-obstruction for $(p_G, m_G)$ but not an obstacle for $(p_G, \mathcal{P}, m_G)$. Let $a_i = A_i \cap V^+(m_G)$ for $i = 1, \ldots, 4$. Then, by Claim 42, splitting off at $a_1, a_3$ is $p_G$-admissible. Since Definition 7.3 does not hold, it is $\mathcal{P}$-feasible, hence $(p_G, \mathcal{P})$-allowed, and we are done.

2. Now, suppose that $A = \{A_1, A_2, A_3, A_4, B_1, \ldots, B_t\}$ is a $C_5^*$-obstruction for $(p_G, m_G)$ but not an obstacle for $(p_G, \mathcal{P}, m_G)$. We proceed by induction on $t$. Let $a_i = A_i \cap V^+$ for $i = 1, \ldots, 4$ and $b_j, b_j' = B_j \cap V^+$ for $j = 1, \ldots, t$. Without loss of generality, we may assume that $a_1, b_1$ is rainbow, thus splitting off at $a_1, b_1$ is $(p_G, \mathcal{P})$-allowed by Claim 42.

Let $p' := p_{G_{a_1}}$ and $m' := m_{G_{a_1}}$.

Our approach is the following. First, split off at $a_1, b_1$, and let $A' = \{A_1 \cup B_1, A_2, A_3, A_4, B_2, \ldots, B_t\}$ be the obstruction for $(p', m')$ given by Claim 46. By Corollary 44, if an obstacle exists for $(p', \mathcal{P}, m')$, then it is $A'$. If there exists none, then we are done by the previous case or by induction. Otherwise, we will provide a $(p', \mathcal{P})$-allowed flip to get rid of the obstacle. We will perform either a $(b_1a_1, b_1a_2)$- or a $(a_1b_1, a_1b_1')$-flip, note that both are $p'$-admissible because they consist in an unsplitting and a splitting off which is $p_G$-admissible by Claim 42.

Distinguish the following cases, where we assume that $m'(P_1) \geq m'(P_2) \geq \cdots \geq m'(P_t)$, and say that the color of $P_1$ and $P_2$ are red and blue.

(a) $A'$ is a simple $C_4^*$-obstacle or a $C_5^*$-obstacle of type 1 for $(p', \mathcal{P}, m')$. By Definition 8.1a, we may assume that the color of $b_j$ is red for $j = 2, \ldots, t$ and either $a_2, a_4$ or $b_1', a_3$ are both red.
i. Since $\mathcal{A}$ is neither a $C_5^-$-obstacle of type 1 nor a $C_5^+$-obstacle of type 2 for $(p_G, \mathcal{P}, m_G)$, none of $b_1, b_1'$ is red, and $a_1, a_3, b_2', \ldots, b_i'$ are not all of the same color. Therefore, $b_1, a_2$ is a rainbow pair. As noted before, the $(b_1 a_1, b_1 a_2)$-flip is $p'$-admissible, thus it is $(p', \mathcal{P})$-allowed. Let $m'' = m_{G_{a_2 b_1}}$ and $p'' = p_{G_{a_2 b_1}}$. By Claim 46, $\mathcal{A}'' = \{A_1, A_2 \cup B_1, A_3, A_4, B_2, \ldots, B_i\}$ is an obstruction for $(p'', m'')$, and it is not an obstacle of type 2 for $(p'', \mathcal{P}, m'')$ because $c(b_1') \neq c(a_4)$, and neither of type 1 because $a_1, a_3, b_2', \ldots, b_i'$ are not all of the same color.

ii. Then, the red $m'$-positive elements are $a_3, b_1', b_2', \ldots, b_i$. Moreover, since $\mathcal{A}$ is not a $C_5^-$-obstacle of type 1 for $(p_G, \mathcal{P}, m_G)$, $a_1$ is not red. We may assume that $m'(P_2) < m'(V)/2$, that is $a_2, a_4, b_2', \ldots, b_i'$ are not all of the same color, as we dealt with this case above.

A. If $b_1$ is red, then $b_1, a_2$ is a rainbow pair, hence the $(b_1 a_1, b_1 a_2)$-flip is $(p', \mathcal{P})$-allowed. Let $m'' = m_{G_{a_2 b_1}}$ and $p'' = p_{G_{a_2 b_1}}$. Then, by Claim 46, $\mathcal{A}'' = \{A_1, A_2 \cup B_1, A_3, A_4, B_2, \ldots, B_i\}$ is an obstruction for $(p'', m'')$, and it is not an obstacle for $(p'', \mathcal{P}, m'')$ because $b_1'$ and $a_3$ are $m''$-positive elements with the same color that belong to distinct consecutive sets.

B. If $b_1$ is not red, then, since $m'(P_2) < m'(V)/2$ and $b_1'$ is red, $a_1, b_1'$ is a rainbow pair, hence the $(a_1 b_1, a_1 b_1')$-flip is $(p', \mathcal{P})$-allowed. Since $b_1$ is not red but $a_3$ is red, and $a_2, a_4, b_2', \ldots, b_i'$ are not all of the same color, the obstruction $\mathcal{A}'$ for $(p'', m'')$ is not an obstacle for $(p'', \mathcal{P}, m'')$.

(b) $\mathcal{A}'$ is a $C_5^+$-obstacle of type 2 (but not of type 1) for $(p', \mathcal{P}, m')$. Note that $m_G(P_i) \leq m'(P_i) + 1 = m'(V)/2 = m_G(V)/2 - 1$, for $i = 1, 2$ so we may assume that $b_1' \geq 1$ and $a_2' = 1$ are red and $a_2$ and $a_4$ are blue.

i. If $b_1$ is not blue, then $b_1, a_2$ is a rainbow pair, hence the $(b_1 a_1, b_1 a_2)$-flip is $(p', \mathcal{P})$-allowed. $\mathcal{A}'' = \{A_1, A_2 \cup B_1, A_3, A_4, B_2, \ldots, B_i\}$ is an obstruction for $(p'', m'')$, yet not an obstacle for $(p'', \mathcal{P}, m'')$ since $a_3$ and $b_1'$ are red $m''$-positive elements and belong to distinct consecutive sets of $\mathcal{A}''$.

ii. If $b_1$ is blue, then, since $\mathcal{A}$ is not a $C_5^+$-obstacle of type 2 for $(p_G, \mathcal{P}, m_G)$, $a_1$ is not red. Then $a_1, b_1'$ is a rainbow pair, hence the $(a_1 b_1, a_1 b_1')$-flip is $(p', \mathcal{P})$-allowed. Let $m'' = m_{G_{a_2 b_1}}$ and $p'' = p_{G_{a_2 b_1}}$, and note that $\mathcal{A}'$ is also an obstruction for $(p'', m'')$, but not an obstacle for $(p'', \mathcal{P}, m'')$ because $b_1$ and $a_2$ are blue $m''$-positive elements that belong to distinct consecutive sets of $\mathcal{A}'$.

4.5.1 Inherited obstacles

In this section, we will see that splitting off an allowed pair in an obstacle gives rise to another obstacle. This implies a link between obstacles and complete allowed splitting off, see Lemma 54. From now on, in this section, $p_0 : 2^V \rightarrow \mathbb{Z}$ is a symmetric crossing supermodular function, $G = (V, E)$ is a graph, and $m_G$ is a $p_G$-admissible degree specification with $m_G(V) \geq 4$.

First let us see a result about an inherited simple $C_4^+$-obstacle after an allowed flipping.

Claim 50. Let $\mathcal{A}$ be a simple $C_4^+$-obstacle for $(p_G, \mathcal{P}, m_G)$, $\{a_j\} = A_j \cap V^+$ for $j = 1, \ldots, 4$, $uv \in E(G)$, $u \in A_i$, $v \in A_{i+1}$, $c(a_{i+1}) = c(a_{i-1})$ and $G'$ obtained from $G$ by the allowed $(vu, va_i)$-flip. Then $\mathcal{A}$ is a simple $C_4^+$-obstacle for $(p_G, \mathcal{P}, m_G')$.

Proof. Note that $p_G(A_i \cup A_j) = p_G(A_i \cup A_j)$ for $1 \leq i \leq j \leq 4$. By Definition 7.2 for $p_G$, the fact that the flipping is allowed, definition of $\sigma_{p_G'}$ and Definition 7.1d for $p_G$, we have $\sigma_{p_G} = m_G(V) = m_{G'}(V) \geq \sigma_{p_G'} \geq \sum_{i=1}^4 p_G(A_i) = p_G(A_i) = \sigma_{p_G}$. Then Definitions 7.1-2 for $p_G$ imply that Definitions 7.1-2 are satisfied for $p_G'. \sigma_{c(a_{i+1})} = c(a_{i-1})$, Definition 7.3 is also satisfied for $p_G'$. \qed
Now we show that an obstacle is inherited after an allowed splitting off. Let us see the three different obstacles separately.

**Claim 51.** A $C^*_l$-obstacle $A$ for $(p_G, \mathcal{P}, m_G)$ is a $C^*_l$-obstacle for $(p_{\overline{G}}, \mathcal{P}, m_{\overline{G}})$, where $G$ is obtained from $\overline{G}$ by an allowed splitting off.

**Proof.** By Definition 7.3, there exist $\ell \in \{1, 2\}$ and $P \in \mathcal{P}$ such that the $m_G$-positive elements of $A_t \cup A_{t+2}$ are the $m_G$-positive elements of $P$. Therefore, by Definitions 7.1d and 7.2, $P$ is dominating, hence we may assume that the two $m_G$-positive elements $u$ and $v$ of $V$ involved in the allowed splitting off satisfy $u \in A_j$ and $v \in A_{j+1}$ for some $j$. By Claim 45, $A$ is a $C^*_l$-obstruction for $(p_G, m_G)$. Definition 7.3 for $m_G$ and the fact that splitting off is allowed immediately imply that Definition 7.3 holds for $m_{\overline{G}}$. $\square$

**Claim 52.** Let $A = \{A_1, A_2, A_3, A_4, B_1, \ldots, B_t\}$ be a $C^*_l$-obstacle for $(p_G, \mathcal{P}, m_G)$ and let $G$ be obtained from $\overline{G}$ by an allowed splitting off. Then there exists $(p_{\overline{G}}, \mathcal{P}, m_{\overline{G}})$ a $C^*_l$-obstacle if $t \geq 2$ and a simple $C^*_l$-obstacle if $t = 1$.

**Proof.** By Lemmas 19 and 42 and Definition 8.1d, the pair $\{u, v\}$ of elements that we split off is either (a) $\{a_i, b_j\}$ for some $a_i \in A_i$ ($1 \leq i \leq 4$) and $b_j \in B_j$ ($1 \leq j \leq t$), or (b) $\{b_j, b_k\}$ for some $b_j \in B_j, b_k \in B_k$ ($1 \leq j < k \leq t$). In case (a), let $A'$ be obtained from $A$ by replacing $A_i$ by $A_i \cup B_j$ and deleting $B_j$, and in case (b), let $A'$ be obtained from $A$ by replacing $B_j$ and $B_k$ by $B_j \cup B_k$. By Claim 46, $A'$ is a simple $C^*_l$-obstruction or a $C^*_l$-obstruction for $(p_{\overline{G}}, \mathcal{P}, m_{\overline{G}})$. We show below that Definition 8.3 also holds for $A'$ and $(p_G, \mathcal{P}, m_G)$. There are two possible cases.

1. If $A$ is a $C^*_l$-obstacle of type 1 for $(p_G, \mathcal{P}, m_G)$, then there exists $\ell \in \{1, 2\}$ and a color $P$, say red, such that $A_t, A_{t+2}$ and every $B_j$ contains a red $m_l$-positive element. By Definitions 8.1f, 8.2, (20) and then 8.3a, we have $t + 2 = \frac{\nu_t}{2} = \frac{m_G(V)}{2} \geq m_G(P) \geq t + 2$, thus $m_G(P) = \frac{m_G(V)}{2}$, that is red is dominating. Therefore, the splitting off being allowed, exactly one of $u$ and $v$ is red.

   (a) If $i \in \{\ell, \ell+2\}$, then $A_i \cup B_j$ contains one red $m_G$-positive element, namely the red $m_G$-positive element contained in $B_j$. Otherwise, $A_t, A_{t+2}$ and every $B_j$ ($j' \neq j$) contain a red $m_G$-positive element, that is Definition 8.3a holds for $A'$ and $(p_{\overline{G}}, \mathcal{P}, m_{\overline{G}})$.

   (b) We may assume without loss of generality that $b_j$ is red. Then $B_j \cup B_k$ contains one red $m_G$-positive element, namely the red $m_G$-positive element contained in $B_k$, hence $A_t, A_{t+2}$ and every $B_j$ ($j' \neq j, k$) contain a red $m_G$-positive element, that is Definition 8.3a holds for $A'$ and $(p_{\overline{G}}, \mathcal{P}, m_{\overline{G}})$.

2. If $A$ is a $C^*_l$-obstacle of type 2 but not of type 1 for $(p_G, \mathcal{P}, m_G)$, then there exist $i' \in \{1, 2\}, j_0 \in \{1, \ldots, t\}$ and two colors, say red and blue, such that $A_{i'}, A_{i'+2}$ and every $B_{j'}$ ($j' \neq j_0$) contain a red $m_G$-positive element, $A_{i'+1}, A_{i'+3}$ and every $B_{j'}$ ($j' \neq j_0$) contain a blue $m_G$-positive element, and $B_{j_0}$ contains no red and no blue $m_G$-positive element. Note that the situation is symmetric for red and blue.

   (a) Without loss of generality we may assume that $a_i$ is red. If $b_j$ is blue (that is $j \neq j_0$), then $A_i \cup B_j$ contains one red $m_G$-positive element, namely the red $m_G$-positive element contained in $B_j$. Then Definition 8.3b holds for $A'$ and $(p_{\overline{G}}, \mathcal{P}, m_{\overline{G}})$. If $b_j$ is not blue (that is $j = j_0$), then $A_{i'+1}, A_{i'+3}$ and every $B_{j'}$ ($j' \neq j$) contain a blue $m_G$-positive element, that is Definition 8.3a also for $A'$ and $(p_{\overline{G}}, \mathcal{P}, m_{\overline{G}})$.

   (b) Without loss of generality we may assume that $b_j$ is red. If $b_k$ is blue (that is $k \neq j_0$), then $B_j \cup B_k$ contains one red and one blue $m_G$-positive element, namely the red (resp. blue) $m_G$-positive element contained in $B_k$ (resp. in $B_j$). Then, Definition 8.3b holds for $A'$ and $(p_{\overline{G}}, \mathcal{P}, m_{\overline{G}})$. If $b_k$ is not blue (that is $k = j_0$), then $A_{i'+1}, A_{i'+3}$ and every $B_{j'}$ ($j' \neq j, k$) and $B_j \cup B_k$ contain a blue $m_G$-positive element, that is Definition 8.3a holds for $A'$ and $(p_{\overline{G}}, \mathcal{P}, m_{\overline{G}})$.
In the cases 1(a) and 2(a), if \( t \geq 2 \), then \( A' \) is a \( C_5^* \)-obstacle and if \( t = 1 \), then, by Definition 8.1e, Definition 7.1c holds for \( A' \) and \((p_G, \mathcal{P}, m_G)\), so \( A' \) is a simple \( C_5^* \)-obstacle for \((p_G, \mathcal{P}, m_G)\).

**Claim 53.** If \( A \) is a \( C_6^* \)-obstacle for \((p_G, \mathcal{P}, m_G)\) and \( \bar{G} \) is obtained from \( G \) by an allowed splitting off, then there exist \( i \in \{1, \ldots, 4\} \) and \( a_{i-1} \in A_{i-1}, a_{i+1} \in A_{i+1} \) such that \( \bar{G} = G_{a_{i-1}a_{i+1}} \) and \( \{A_{i-1} \cup A_i \cup A_{i+1}, A_{i+2}, A_{i+3}, A_{i+4}\} \) is a simple \( C_4^* \)-obstacle for \((p_G, \mathcal{P}, m_G)\).

**Proof.** By Lemmas 19 and 42, and by Definitions 9.1b and 9.3, the only allowed pairs are \( a_{i-1}, a_{i+1} \) for all \( i \). By Claim 47, \( A' = \{A'_1, A'_2, A'_3, A'_4\} = \{A_{i-1} \cup A_i \cup A_{i+1}, A_{i+2}, A_{i+3}, A_{i+4}\} \) is a simple \( C_4^* \)-obstruction for \((p_G, \mathcal{P}, m_G)\). By \( m_G(V) = 4 \) and Definition 9.3 for \( A, A' \) satisfies 7.3.

The main result of this section, Lemma 54, motivates the definitions of obstacles by revealing the close link between them and the existence of a complete allowed splitting off.

**Lemma 54.** If there is an obstacle for \((p, \mathcal{P}, m)\), then there exists no complete \((p, \mathcal{P})\)-allowed splitting off.

**Proof.** Suppose that there exists an obstacle for \((p, \mathcal{P}, m)\). By Claims 51, 52 and 53, after any sequence of allowed splitting off, there exists an obstacle and then, by Claim 39.2, Definitions 8.1a and 9.1a and Claim 41, the new degree specification satisfies \( m'(V) > 0 \), that is no complete allowed splitting off exists.

### 4.5.2 Obstacles and split edges

An important subcase will occur when there is a simple \( C_4^* \)-obstacle \( A = \{A_1, \ldots, A_t\} \) for \((p_G, \mathcal{P}, m_G)\). Let us show some properties of such obstacles when \( G \) is not the edgeless graph. Let \( a_i := V_+(m_G) \cap A_i \) for \( i = 1, \ldots, 4 \). For an edge \( e = uv \) where \( u, v \) are contained in a member \( A_i \) of the \( C_4^* \)-obstacle, a **consecutive improvement for** \( e \) is an admissible improvement of \( e \) by \( a_{i-1} \) and \( a_{i+1} \).

We start with some technical properties.

**Corollary 55.** If \( A \) is a simple \( C_4^* \)-obstacle for \((p_G, \mathcal{P}, m_G)\), then \( p_G \leq 1 \).

**Proof.** Since \( A \) is simple, Lemma 54 implies that there exists no allowed splitting off, and then, by Lemma 22, we have \( p_G \leq 1 \).

**Claim 56.** Let \( A = \{A_1, \ldots, A_t\} \) be a simple \( C_4^* \)-obstacle for \((p_G, \mathcal{P}, m_G)\), \( e = uv \in E, i \in \{1, \ldots, 4\}, u, v \in A_i \). Suppose that no allowed improvement exists.

1. No \((uv, a_{i+2})\)-perilous set exists.
2. There exists a \((uv, a_j)\)-perilous set \( X_j \) for \( j \in \{i - 1, i + 1\} \).
3. If \( X_{i-1} \cap \{u, v\} = X_{i+1} \cup \{u, v\} \), then no edge connects distinct consecutive members of \( A \).
4. If \( X_{i-1} \cap \{u, v\} = \{u\} \) and \( X_{i+1} \cap \{u, v\} = \{v\} \), then \( c(u) = c(a_{i+1}), c(v) = c(a_{i+1}) \) and \( c(a_j) = c(a_{i+2}) \).

**Proof.** By Corollary 55 and Lemma 43, we have \( p_G \leq 1 \) and \( T_{a_i} = A_\ell \) for all \( \ell \in \{1, \ldots, 4\} \). We may assume that \( i = 1 \).

1. By Claim 32.2-4 and Definition 7.1c, there is no \((uv, a_3)\)-perilous set.
2. Suppose that, for some \( j \in \{2, 4\} \), there is no \((uv, a_j)\)-perilous set. Then, by 56.1 and Lemma 31(ii), the improvement of \( uv \) to \( ua_j, va_3 \) and that of \( uv \) to \( ua_3, va_j \) are \( p_G \)-admissible. Since \( c(u) \neq c(v) \) and, by Observation 11, \( c(a_j) \neq c(a_3) \), one of these improvements is allowed, contradicting our assumption.
3 By Claim 34.3, \( X_j = X_e \cup A_j \) for \( j = 2, 4 \). Applying for \( A_1, A_2 \) and \( A_4 \), Claims 34.1, 32.2-3, we get that no edge connects \( A_1 \) and \( A_2 \cup A_4 \). By Claim 39.4, (13) applied for \( X_2 \cup X_4 \) and \( A_3 \cup A_4 \) (and \( A_3 \cup A_2 \)), \( p_G \leq 1 \), and since \( p_G(X_2 \cup X_4) = 0 \) by Claim 34.1, we get that no edge connects \( A_3 \) and \( A_2 \cup A_4 \), and 56.3 follows.

4 By 56.1 and Lemma 35, no \((uv, a_3)\)-, \((uv, a_2)\)- and \((vu, v_4)\)-perilous sets exist. Then, by Lemma 31(ii), improving \( uv \) to \( ux, vy \) is \( p_G \)-admissible for \((x, y) = (a_2, a_3), (a_2, a_4), (a_3, a_4)\). If \( c(a_2) \neq c(u) \), then, by the improvements of \( uv \) to \( u_2, v_3 \) and that of \( uv \) to \( u_2, v_4 \) are not \((p_G, P)\)-allowed and Observation 11, we have \( c(a_3) = c(v) = c(a_4) \neq c(a_3) \), a contradiction. Hence \( c(a_2) = c(u) \), and similarly \( c(v) = c(a_4) \). Therefore, \( c(a_2) \neq c(a_4) \), and then, by Definition 7.3 for \( A \), we have \( c(a_1) = c(a_3) \).

The following lemma will handle the case of \( C_4^* \)-obstacles.

**Lemma 57.** Let \( \mathcal{A} = \{A_1, \ldots, A_4\} \) be a simple \( C_4^* \)-obstacle for \((p_G, P, m_G)\), \( e = uv \in E \), \( i \in \{1, \ldots, 4\} \), \( u \in A_i \) and \( v \in A_{i+1} \). Suppose that no allowed improvement exists.

1. Then either \( c(u) = c(a_i) = c(a_{i+2}) \) or \( c(v) = c(a_{i-1}) = c(a_{i+1}) \).

2. If all the edges of \( G \) connect distinct members of \( \mathcal{A} \) and there exists no \( C_4^* \)-obstacle for \((p_0, P, m_0)\), then there exist two edges \( f \) and \( g \) such that there exists a complete \((p_G, P, m_G)\)-allowed splitting off.

**Proof.** By Corollary 55 and Lemma 43, we have \( p_G \leq 1 \) and \( T_{0h} = A_h \) for all \( h \in \{1, \ldots, 4\} \). We may assume that \( i = 1 \).

1. By Definition 7.3 for \( A \), we may assume \( c(a_1) = c(a_3) \). Assume that \( c(u) \neq c(a_1) \). Then, \( u \neq a_1 \). By Corollary 33, no \((uv, u_3)\)-, \((vu, v_4)\)- or \((vu, v_4)\)-perilous sets exist. Therefore, by Lemma 31(ii), the improvements of \( uv \) to \( u_3, v_4 \) and that of \( uv \) to \( u_3, v_4 \) are \( p_G \)-admissible. Since these improvements are not allowed, we have, by Observation 11, \( c(v) = c(a_4) \neq c(a_4) = c(v) \), a contradiction.

2. We will unsplit two edges and find a complete allowed splitting off by performing first an allowed flip, and then an improvement that is allowed for the resulting functions.

Since no allowed improvement exists, 57.1 implies that for every edge \( u'v' \) either \( c(u') = c(a_1) = c(a_3) \) or \( c(v') = c(a_2) = c(a_4) \). Then, since no \( C_4^* \)-obstacle exists for \((p_0, P, m_0)\), there exist edges \( f = u_1v_1 \) and \( g = u_2v_2 \) such that \( u_1, u_2 \in A_1 \cup A_3 \), \( c(u_1) \neq c(a_1) = c(a_3) = c(u_2) \) and \( c(v_2) \neq c(a_2) = c(a_4) = c(v_1) \).

Assume without loss of generality that \( u_1 \in A_1 \) and \( v_1 \in A_2 \). By Lemma 31(i) and Corollary 33, flipping \( v_1u_1 \) for \( v_1a_1 \) is \( p_G \)-admissible, and, by Observation 11, \( c(a_1) \neq c(a_2) = c(v_1) \), thus it is allowed. Let \( G' \) be the resulting graph. By Claim 50, \( \mathcal{A} \) is a simple \( C_4^* \)-obstacle for \((p_G, P, m_G)\), and \( u_1 = V_4(m_G) \cap A_1 \) is the new \( m_G \)-positive element of \( A_1 \). Since \( c(a_2) \neq c(v_2) \) and \( c(u_1) \neq c(a_3) \), 57.1 implies that there exists a \((p_G, P)\)-allowed improvement using the edge \( uv' \) which means that there exists a complete \((p_G, P, m_G)\)-allowed splitting off.

The following lemma considers the case of \( C_5^* \)-obstacles.

**Lemma 58.** Let \( \mathcal{A} = \{A_1, \ldots, A_4\} \) be a simple \( C_4^* \)-obstacle for \((p_G, P, m_G)\), \( e = uv \in E \), \( i \in \{1, \ldots, 4\} \), \( u, v \in A_i \). Suppose that no allowed improvement exists.
1. If $f = xy \in E(G)$, $x, y \in A_i$ and no consecutive improvement exists for $e$ and for $f$, then $X_e$ and $X_f$ exist and one of them contains the other one.

2. If all the edges of $G$ are contained in members of $A$ and no edge belongs to a consecutive improvement, then there exists a $C_{5}^{*}$-obstruction for $(p_0, \mathcal{P}, m_0)$.

Proof. By Corollary 55 and Lemma 43, we have $p_G \leq 1$ and $T_{ah} = A_h$ for all $h \in \{1, \ldots, 4\}$. We may assume that $i = 1$.

1 By Claim 56.2, let $X_j^f$ be a $(g, a_j)$-perilous set for $g = e, f$ and $j = 2, 4$. Since no consecutive improvement exists for $g$, by Lemmas 35 and 31(ii), we have $X^f_j \cap g = X^f_j \cap g$. Therefore, Claim 34.3 applies and defines a unique $X_g \subset A_1$ such that $X_g \cup A_j$ is the $(g, a_j)$-perilous set for $j = 2, 4$. By Claim 34.1, $p_G(Y_g) = 0$, where $Y_g = X_g \cup A_2 \cup A_4$. Note that, since $g$ enters $Y_g$, the latter implies that $Y_g$ is $p_0$-positive, for $g = e, f$.

Suppose that $X_e \cap X_f = \emptyset$. Then applying (13) to $Y_e$ and $Y_f$ and Definition 7.1c implies $0 = p_G(Y_e) + p_G(Y_f) \leq p_G(Y_e \cap Y_f) = p_G(A_2 \cup A_4) + p_G(Y_e \cap Y_f) \leq p_G(Y_e \cup Y_f)$. In particular, $Y_e \cup Y_f$ is $p_0$-positive. By $A$ is a simple $C_{5}^{*}$-obstacle, Claim 39.4, (14) applied to $Y_e \cup Y_f$ and $A_3 \cup A_4$, and, by $e$ enters $X_e \cup X_f \cup A_2$ and Claim 34.1, it is not $(e, a_2)$-perilous, we have $0 + 1 \leq p_G(Y_e \cup Y_f) + p_G(A_3 \cup A_4) \leq p_G(X_e \cup X_f \cup A_2) + p_G(A_3) \leq -1 + 1 = 0$, a contradiction.

Then, since, by Claim 34.5, $\mathcal{A} \cup \{X_h, h \in E(G)\}$ is laminar, 58.1 follows.

2 Let $j \in \{1, \ldots, 4\}$. It follows, by Claim 58.1, that the edges contained in $A_j$ can be ordered as $e_1, \ldots, e_\ell$ such that $1 \leq k < k' \leq \ell$ implies $X_{e_k} \subset X_{e_{k'}}$. Let $X_{e_{k+1}} = A_j$, $B_{e_k} = X_{e_{k+1}} \setminus X_{e_k}$, for $k = 1, \ldots, \ell$ and $A'_j = A_j \setminus \bigcup_{k=1}^\ell B_{e_k}$. Note that $A'_j = A_j$ if $A_j$ contains no edge, and $A'_j = X_{e_1}$ otherwise. Thus, $\{A'_j, B_f : f \in A_j\}$ partitions $A_j$, and, since $A$ partitions $V$, we get that $A' = \{A'_1, A'_2, A'_3, A'_4\} \cup \{B_f : f \in E(G)\}$ is a partition of $V$.

We show that $A'$ is a $C_{5}^{*}$-obstruction for $(p_0, \mathcal{P}, m_0)$. First, let us make a few observations.

Recall that $p_0 = p_G + d_G$, $E_\delta(A)$ is empty and, by Claim 34.4, $f$ is the only edge entering $X_f$, for all $f \in E(G)$. In particular, no member of $A'$ contains an edge, and $m_0(B_f) = m_G(B_f) + d_G(B_f) = 2$ and $d_G(A_j) = 0, d_G(A'_j) = 1$.

Let $f \in A_j$, say $f = e_k$ and $h \in \{j - 1, j + 1\}$. By Claims 34.4, 39.4, 56.2 and 34.3, we get $p_0(X_f) = p_0(A_j) = p_0(A_j \cup A_h) = p_0(X_f \cup A_h) = 1$. It follows that

$$p_0(A'_j) = p_0(A'_j \cup A_h) = 1. \quad (21)$$

Then, by applying (4) to $A_h \cup X_{e_{k+1}}$ and $A_{h+2} \cup X_{e_k}$, we get

$$p_0(A_h \cup B_f) \geq 1. \quad (22)$$

8.1a By (21), we have $p_0(A'_j) = 1$ for $j \in \{1, \ldots, 4\}$, hence Definition 8.1a follows.

8.1d By (21), we may suppose that $A'_j = X_f$ and $A'_{j+1} = X_g$ for $f, g \in E(G)$. By (22), we may apply Claim 12.2 to $\{A_{j+2} \cup A_{j+3}\} \cup \{A_{j+3} \cup B_f' : f' \in A_j, f' \neq f\} \cup \{A_{j+2} \cup B_{g'} : g' \in A_{j+1}, g' \neq g\}$ and then, by Claim 39.4, the symmetry of $p_0$ and (21), we have $1 = p_0(A_{j+2} \cup A_{j+3}) \leq p_0(V \setminus (A'_j \cup A'_{j+1})) \leq p_0(V \setminus A'_j) = p_0(A'_j) = 1$, that is equality holds everywhere and, by the symmetry of $p_0$, 8.1d follows.

8.1c Note that the previous proof works for $A'_j$ and $B_g$ if $g \in A_{j-1} \cup A_{j+1}$. Suppose now that $g \in A_j$. By (22), we may apply Claim 12.2 to $\{V \setminus A_j\} \cup \{A_{j+1} \cup B_{g'} : g' \in A_j, g' \neq g\}$ and then, by the symmetry of $p_0$ and (21), we have $1 = p_0(A_j) = p_0(V \setminus A_j) \leq p_0(V \setminus (A'_j \cup B_g)) \leq p_0(A'_j) = 1$, that is equality holds everywhere and, by the symmetry of $p_0$, 8.1c follows.
Finally, suppose that \( g \in A_{j+2} \). By (22), we may apply Claim 12.2 to \( \{A_{j+3} \cup B_f : f' \in A_j \} \cup \{A_{j+3} \cup B_f' : g' \in A_{j+1}, g' \neq g \} \cup \{A_{j+1} \cup A_{j+2} \} \cup \{A_{j+3} \cup B_g \} \) and then, by (22) and (21), we have \( 1 = p_0(A_{j+3} \cup B_f) \leq p_0(V \setminus (A_j' \cup B_j)) \leq p_0(A_j') = 1 \), that is equality holds everywhere and, by the symmetry of \( p_0 \), 8.1c follows.

8.1b Let \( f \in A_j \). By (21) and (3) applied to \( A_j' \cup A_{j-1} \) and \( A'_j \cup A_{j+1} \), we have \( p_0(Y) \geq 1 \), where \( Y = A_j' \cup A_{j-1} \cup A_{j+1} \). Then, by (3) applied to \( Y \) and \( V \setminus A_j \), the symmetry of \( p_0 \) and Definition 7.1c for \( A \), we get \( p_0(Z) \geq 2 \), where \( Z = A'_j \cup (V \setminus A_j) \). By (22), we may apply Claim 12.2 to \( \{Z \} \cup \{A_{j+1} \cup B_f' : f' \in A_j, f' \neq f \} \) and then, by the symmetry of \( p_0 \), \( m_0 \) is \( p_0 \)-admissible, we have \( 2 \leq p_0(Z) \leq p_0(V \setminus B_f) = p_0(B_f) \leq m_0(B_f) = 2 \), that is equality holds everywhere and 8.1b follows.

8.1e If \( p_0(A'_j \cup A'_{j+2}) \geq 1 \), then Claim 12.1 applies to \( \{A_j \} \cup \{A'_j \cup A'_{j+2} \} \cup \{A_{j+2} \} \), and we get, by Definition 7.1c, \( 1 \leq p_0(A_j \cup A_{j+2}) = p_G(A_j \cup A_{j+2}) \leq 0 \), a contradiction.

8.2 Recall that \( m_0 \) is \( (p_0, P) \)-allowed. Moreover, since no \( A \in \mathcal{A}' \) contains an edge, thus the above results on \( d_G \) and \( p_0 \) imply \( p_0(A) = m_0(A) \) for all \( A \in \mathcal{A}' \). Therefore, we have \( m_0(V) \geq \sigma_{p_0} \geq \sum_{A \in \mathcal{A}'} p_0(A) = \sum_{A \in \mathcal{A}'} m_0(A) = m_0(V) \).

Finally, \( C^*_e \)-obstacles are treated by the following lemmas.

**Lemma 59.** Let \( A = \{A_1, \ldots, A_4 \} \) be a simple \( C^*_e \)-obstacle for \( (p_G, \mathcal{P}, m_G) \), \( e = uv \in E \), \( i \in \{1, \ldots, 4 \} \), \( u, v \in A_i \). Suppose that no allowed improvement exists.

1. If an edge connects distinct consecutive members of \( A \), then a consecutive improvement exists for \( e \).

2. If a consecutive improvement exists for \( e \), then there exists a \( C^*_e \)-obstacle for \( (p_G, \mathcal{P}, m_G) \).

**Proof.** By Corollary 55 and Lemma 43, we have \( p_G \leq 1 \) and \( T_{ah} = A_h \) for all \( h \in \{1, \ldots, 4 \} \). We may assume that \( i = 1 \). By Claim 56.2, there exists a \((u, e_j)\)-perilous set \( X_j \), for \( j \in \{2, 4 \} \). Without loss of generality, \( X_2 \) is \((u, u)\)-perilous. Note that, for both parts of the lemma, we have \( X_2 \cap \{u, v\} \neq X_4 \cap \{u, v\} \), thus \( X_4 \) is \((uv, va_4)\)-perilous. Indeed, for 59.1, it comes from Claim 56.3, and for 59.2, it follows from Lemmas 35 and 31(ii), and the fact that \( e \) belongs to a consecutive improvement.

1 By Lemma 35, no \((vu, va_2)\)- and no \((uv, va_4)\)-perilous sets exist. Thus, by Lemma 31(ii), improving \( uv \) to \( va_4 \), \( va_2 \) is \( p_G \)-admissible, and 59.1 follows.

2 By Claim 32.4, \( X_j \setminus A_1 = A_j \), for \( j \in \{2, 4 \} \). By \( A \) is a partition of \( V \) and Claim 34.2, \( \mathcal{A}' = \{A'_1, \ldots, A'_6\} = \{X_2 \cap A_1, A_2, A_3, A_4, X_4 \cap A_1 \setminus (X_1 \cup X_2)\} \) is a partition of \( V \). Let us show that \( \mathcal{A}' \) is a \( C^*_e \)-obstacle for \( (p_{G^e}, \mathcal{P}, m_{G^e}) \).

9.1a -1b By Claim 34.2, we have \( 0 = p_G(A'_6 \cup A'_j) = p_G(A'_6 \cup A'_1) \) \( = 1 = p_G(A'_6) \). By Claim 32.2, \( 0 = p_G(A'_1) = p_G(A'_3) \). By \( A \) is a simple \( C^*_e \)-obstacle, Claim 39.4 and \( X_2 \) and \( X_4 \) are perilous, \( 1 = p_G(A'_j) = p_G(A'_6) = p_G(A'_1) = p_G(A'_3) = p_G(A'_2) = p_G( A'_i \cup A'_j) \), \( 0 = p_G(A'_6 \cup A'_2) = p_G(A'_2 \cup A'_3) \). Then, by \( p_G = p_{G^e} - d_e \), 9.1a-1b follows.

9.1c By \( u \in A'_1, v \in A'_3 \), \( m_0 \) is \( p_G \)-admissible and modular and Claim 32.2, we have \( p_{G^e}(A'_1 \cup A'_3) = p_G(A'_1 \cup A'_3) \leq m_G(A'_1 \cup A'_3) = m_G(X_2 \cap A_1) + m_G(X_4 \cap A_1) = 0 \); by Claim 18 for \( i = 2, 3, 4 \), we have \( p_{G^e}(A'_i \cup A'_j) = p_G(A'_i \cup A'_j) \leq 0 \); by Definition 7.1c, we have \( p_{G^e}(A'_2 \cup A'_3) = p_G(A_2 \cup A_4) \leq 0 \); by Claim 18, and by 56.1, no \((uv, a_3)\)-perilous set exists, we have \( p_{G^e}(A'_1 \cup A'_3) = p_G(A'_1 \cup A_3) + 1 \leq -1 + 1 \) for \( i = 1, 5 \); by Claim 18, and since no \((uv, ua_4)\)- and no \((vu, va_2)\)-perilous sets exist by Lemma 35, we have \( p_{G^e}(A'_1 \cup A'_3) \leq 0 \) for \( i = 1, 2 \).
9.1d and 9.2 Since \( e \) was obtained by an allowed splitting off, \( m_{G^e} \) is \( p_{G^e} \)-admissible and \( \mathcal{P} \)-feasible. By Definition 7.1d for \( \mathcal{A} \), the fact that \( m_{G^e} \) is \( p_{G^e} \)-admissible, definition of \( \sigma_{p_{G^e}} \) and Definition 9.1a for \( \mathcal{A}' \), we have \( 6 + 2 = m_G(V) + 2 = m_{G^e}(V) \geq \sigma_{p_{G^e}} \geq \sum_{X \in \mathcal{A}'} p_{G^e}(X) = 6 \) and 9.1d and 9.2 follows.

9.3 By Claim 56.4.

\[\square\]

**Lemma 60.** If \( \mathcal{A} \) is a \( C_4^e \)-obstacle for \( (p_G, \mathcal{P}, m_G) \) and \( e \) is an edge of \( G \), then there exists a complete \( (p_G, \mathcal{P}) \)-allowed splitting-off.

**Proof.** In each of the following cases, we will first perform a \( (p_G, \mathcal{P}) \)-allowed splitting off, hence find, by Claim 53, a simple \( C_4^e \)-obstacle. If \( p' \) denotes the resulting function, then, by Corollary 55, we have \( p' \leq 1 \). We will then find, by Lemma 31(ii), a \( (p', \mathcal{P}) \)-allowed improvement involving \( e \). It is equivalent to unsplit \( e \) and then find a complete \( (p_G, \mathcal{P}) \)-allowed splitting off.

Let \( \mathcal{A} = \{A_1, \ldots, A_6\} \) and \( a_i = A_i \cap V_+(m_G) \) for every \( i = 1, \ldots, 6 \). Denote \( e = uv \). We may assume, by Definitions 9.1a-1b and Claim 13.1, that \( u \in A_1 \) and \( v \in A_1 \) or \( v \in A_2 \).

Let \( G_i = G_{a_i} \) and \( A_i = \{A_i \cup A_{i+1} \cup A_{i+2}, A_{i+3}, A_{i+4}, A_{i+5}\} \) for \( i = 2, 3, 5 \). By Definition 9.3 for \( \mathcal{A} \), \( c(a_1) \neq c(a_5) \neq c(a_3) \) and \( c(a_6) \neq c(a_2) \neq c(a_4) \). Then, by Lemma 42, the pair \( a_i, a_{i+2} \) is \( (p_G, \mathcal{P}) \)-allowed, so \( G_{a_i} \) is \( (p_G, \mathcal{P}) \)-allowed, and, by Claim 53, \( A_i \) is a simple \( C_4^e \)-obstacle for \( (p_G, \mathcal{P}, m_{G_i}) \) for \( i = 2, 3, 5 \). By Corollary 55, we have \( p_{G_i} \leq 1 \) for \( i = 2, 3, 5 \).

1. \( v \in A_1 \). By Claim 56.1 for \( \mathcal{A}_2 \) (resp. \( \mathcal{A}_3 \)),

\[
\text{no } (uv, a_3) \text{-perilous set exists with respect to } p_{G_2} \text{ and } m_{G_2} \text{ (resp. } p_{G_3} \text{ and } m_{G_3}).
\]

By Claim 56.2 for \( \mathcal{A}_3 \), there exists, for \( j = 2, 6 \), a \( (uv, a_j) \)-perilous set \( X_j \) with respect to \( p_{G_3} \) and \( m_{G_3} \) and, by Claim 32.4, with respect to \( p_G \) and \( m_G \). We may assume that \( X_2 \) is \( (uv, a_2) \)-perilous.

We show that \( X' = X_2 \cup A_3 \cup A_4 \) is a \( (uv, ua_3) \)-perilous set with respect to \( p_{G_2} \) and \( m_{G_2} \). By Claim 32.4, we have \( X_2 \setminus A_1 = A_2 \). By \( X_2 \) is perilous, Claim 13.2 for \( J = \{2, 3, 4\} \), (13) applied for \( p_G \), \( X_2 \) and \( A_2 \cup A_3 \cup A_4 \), Definition 9.1a and Claim 18 for the crossing sets \( X' \) and \( A_1 \), gives \( p_G(X') = 0 \). Note that \( p_G(X') = p_{G_2}(X') \) and, by \( m_G \) is modular, \( X \) is perilous, Claim 41 and Definition 9.1a, we have \( m_{G_2}(X') = m_G(X') - 2 = m_G(X_2) + m_G(A_3) + m_G(A_4) - 2 = 1 + 1 + 1 - 2 = 1 \), so \( X' \) is \( (uv, u_3a_3) \)-perilous for \( p_{G_2} \) and \( m_{G_2} \). Hence, by Lemma 35,

\[
\text{no } (vu, va_3) \text{-perilous set exists with respect to } p_{G_2} \text{ and } m_{G_2}.
\]

(a) If \( X_6 \) is \( (vu, va_6) \)-perilous.

We may suppose, by Claim 56.4 for \( \mathcal{A}_3 \), that \( c(u) = c(a_6) \), \( c(v) = c(a_2) \) and \( c(a_1) = c(a_4) \). Now, by (23), (24) and Lemma 31(ii), improving \( uv \) to \( va_3, ua_5 \) is \( p_{G_2} \)-admissible, hence \( (p_{G_2}, \mathcal{P}) \)-allowed, and we are done.

(b) If \( X_6 \) is \( (uv, ua_6) \)-perilous.

i. Suppose that \( c(u) \neq c(a_4) \). Note that there exists \( j \in \{2, 6\} \) such that \( c(v) \neq c(a_j) \), and that, by Lemma 35, there is no \( (vu, va_j) \)-perilous set with respect to \( p_{G_1} \) and \( m_{G_4} \). Then, by (23) and Lemma 31(ii), improving \( uv \) to \( ua_4, va_j \) is \( p_{G_4} \)-admissible, hence \( (p_{G_4}, \mathcal{P}) \)-allowed, and we are done.

ii. Otherwise, \( c(u) = c(a_4) \), and if necessary reverse the order of the sets \( A_2, \ldots, A_6 \) to assume that \( c(a_3) \neq c(v) \). Hence, by (23), (24) and Lemma 31(ii), improving \( uv \) to \( ua_5, va_3 \) is \( (p_{G_2}, \mathcal{P}) \)-allowed, and this finishes the proof of the lemma.
2. \( v \in A_2 \). Note that \( uv \) connects consecutive sets of \( A_2 \) and \( A_3 \). Then, for \( i = 2 \) or \( i = 3 \), depending on whether \( c(u) = c(a_1) \) or \( c(u) \neq c(a_1) \), Claim 57.1 applied for \((p_{G_i}, P, m_{G_i})\), \( A_i \) and \( uv \), implies that there exists a \((p_{G_i}, P)\)-allowed improvement involving \( uv \).

\[\square\]

4.6 Splitting off theorem

We will now prove our new splitting off result. First, we state a version that we will need for the algorithm and then we state it as a characterization of the existence of complete \((p_0, P)\)-allowed splitting.

**Theorem 61.** Let \( p_0 : 2^V \to \mathbb{Z} \) be a symmetric crossing supermodular set function, \( m_0 : V \to \mathbb{Z}_+ \) a \((p_0, P)\)-allowed degree specification and \( P \) a partition of \( V \). Then one of the following exists

(a) a complete \((p_0, P)\)-allowed splitting off,

(b) a \( p_0 \)-full partition that shows that \( m_0 \) is not \( p_0 \)-legal and a \( P \)-partite graph \( F \) on \( V \) that covers \( p_0 \) with \(|E(F)| \leq \dim(p_0) - 1\),

(c) an obstacle for \((p_0, P, m_0)\).

**Proof.** We outline an algorithm that outputs one of the above possibilities. First, we perform allowed splitting off as long as possible, secondly, we perform allowed improvements as long as possible. When we get stuck, unsplitting edges is necessary. Depending on the position of the edges, distinct cases occur.

Now we show the correctness of the above algorithm.

**Step 1.** Perform arbitrary allowed splitting off as long as possible, and let \( G \) be the resulting graph. If \( m_G(V) \leq 2 \), then if necessary, perform a final allowed splitting off, and then we are done.

From now on, we suppose that \( m_G(V) \geq 4 \) and **no allowed splitting off exists**.

**Step 2.** Suppose that no \( p_G \)-admissible splitting off exists. If no allowed improvement exists, then, by Lemmas 37.9-10, we have the required partition and graph. Otherwise, there exists an allowed improvement, and by Corollary 29, after performing an allowed improvement, we can repeat it. Thus let us perform an arbitrary sequence of allowed improvements as long as there exists one, and let \( G' \) be the resulting graph. If \( m_{G'}(V) \geq 4 \), then we can stop with the required partition and graph provided by Lemmas 37.9-10. Otherwise, \( m_{G'}(V) \leq 2 \) and hence performing a final allowed splitting off if necessary, we are done.

From now on, we assume that a **\( p_G \)-admissible splitting off exists**.

**Step 3.** By (20) for \( m_G \) and Lemma 25, there exists a simple \( C_4^* \)-obstruction \( \mathcal{A} = \{A_1, \ldots, A_4\} \) for \((p_G, P, m_G)\). Since no splitting off is allowed, Lemma 49 implies that \( \mathcal{A} \) satisfies 7.3, hence \( \mathcal{A} \) is a simple \( C_4^* \)-obstacle for \((p_G, P, m_G)\).

**Step 4.** Suppose that there exists an allowed improvement. Then perform it and then, since \( m_G(V) = 4 \), perform a final allowed splitting off, and we are done.

Thus we assume that **no allowed improvement exists**.

**Step 5.** Suppose that all the edges of \( G \) connect distinct members of \( \mathcal{A} \). Then, by Lemma 57.2, we can find either a \( C_4^* \)-obstacle for \((p_0, P, m_0)\), or two edges \( e \) and \( f \) such that a complete \((p_{G-e-f}, P)\)-allowed splitting off exists, and we are done.
Let $m_0$ is not $p_0$-legal and a $\mathcal{P}$-partite graph $F$ on $V$ that covers $p_0$ with $|E(F)| \leq \dim(p_0) - 1$, or an obstacle for $(p_0, \mathcal{P}, m_0)$.

**Step 1.** Perform an arbitrary sequence of allowed splittings off as long as there exists one. Let $G$, $m'_G$ and $p_G$ be the resulting graph, degree specification and function.

**Step 2.** If there exists no $p_G$-admissible splitting off, then perform an arbitrary sequence of allowed improvements as long as there exists one and let $G'$ be the resulting graph.

(a) If $m'_G(V) \geq 4$, then Stop with the required partition and graph provided by Lemmas 37.9-10.

(b) Otherwise, perform, if necessary, a final allowed splitting off and Stop.

**Step 3.** Otherwise, by Lemmas 25 and 49, find a simple $C_4^*$-obstacle $A$ for $(p_G, \mathcal{P}, m_G)$.

**Step 4.** If there exists an allowed improvement, then perform it and then perform a final allowed splitting off and Stop.

**Step 5.** If all the edges of $G$ connect distinct members of $A$, then, by Lemma 57.2, find

(a) either a $C_4^*$-obstacle for $(p_0, \mathcal{P}, m_0)$ and Stop.

(b) or two edges such that after their unsplitting there exists a complete allowed splitting off and Stop.

**Step 6.** If all the edges of $G$ are contained in members of $A$ but no consecutive improvement exists, then, by Lemma 58.2, there exists a $C_5^*$-obstruction for $(p_0, \mathcal{P}, m_0)$.

(a) If it is a $C_5^*$-obstacle for $(p_0, \mathcal{P}, m_0)$, then Stop.

(b) Otherwise, Lemma 49 provides a complete $(p_0, \mathcal{P})$-allowed splitting off and Stop.

**Step 7.** Otherwise, by Lemma 59.1, there exists a consecutive improvement for an edge $e$, and then, by Lemma 59.2, there exists a $C_6^*$-obstacle for $(p_{G^e}, \mathcal{P}, m_{G^e})$.

(a) If $G^e = G$, then Stop with the $C_6^*$-obstacle for $(p_0, \mathcal{P}, m_0)$.

(b) Otherwise, $G^e$ contains an edge $e'$ and then, by Lemma 60 applied for $e'$, there exists a complete $(p_{G^e,e'}, \mathcal{P})$-allowed splitting off and Stop.

**Figure 1:** Splitting off algorithm

**Step 6.** Suppose that all the edges of $G$ are contained in members of $A$ but no consecutive improvement exists. Then, by Lemma 58.2, there exists a $C_5^*$-obstruction for $(p_0, \mathcal{P}, m_0)$. By Lemma 49, either it is a $C_5^*$-obstacle for $(p_0, \mathcal{P}, m_0)$, or we have a complete $(p_0, \mathcal{P})$-allowed splitting off and we are done.

We may assume, by Lemma 59.1, that a consecutive improvement exists for an edge $e$.

**Step 7.** By Lemma 59.2, there exists a $C_6^*$-obstacle for $(p_{G^e}, \mathcal{P}, m_{G^e})$. If $G^e = G$, then we have a $C_6^*$-obstacle for $(p_0, \mathcal{P}, m_0)$ and we are done. Otherwise, $G^e$ contains an edge $e'$ so Lemma 60 applies for $e'$ and we are done.

The previous theorem gives at once the main result of the section.

**Theorem 62.** Let $p_0 : 2^V \to \mathbb{Z}$ be a symmetric crossing supermodular set function, $m_0 : V \to \mathbb{Z}_+$ a degree specification and $\mathcal{P}$ a partition of $V$. There exists a complete $(p_0, \mathcal{P})$-allowed splitting
off if and only if $m_0$ is $(p_0, \mathcal{P})$-allowed and $p_0$-legal, and there exists no obstacle for $(p_0, \mathcal{P}, m_0)$.

Proof. The necessity of the conditions follows by Lemmas 38 and 54. To show the sufficiency, let us suppose that $m_0$ is $(p_0, \mathcal{P})$-allowed and $p_0$-legal, and there exists no obstacle for $(p_0, \mathcal{P}, m_0)$. Then, by Theorem 61, there exists a complete $(p_0, \mathcal{P})$-allowed splitting off. \hfill \Box

5 Minimization version

In this section, we are given a symmetric crossing supermodular function $p$ on $V$ and a partition $\mathcal{P} = \{P_1, \ldots, P_r\}$ of $V$. We show how to find algorithmically a $\mathcal{P}$-partite graph covering $p$ with a minimum number of edges.

First, we provide the lower bound. Secondly, we explain how to find a minimum $(p, \mathcal{P})$-allowed degree specification. Then, we describe the instances for which the lower bound may not be achieved. Finally, we prove our main result, see Theorem 71.

5.1 Lower bound

Let $OPT(p, \mathcal{P})$ be the minimum number of edges of a $\mathcal{P}$-partite a graph that covers $p$.

Definition 63. Let $\Phi$ be the maximum of the following values.

$$\alpha_p = \max\{\frac{1}{2} \sum_{x \in X} p(X) : X \text{ subpartition of } V\},$$

$$\beta_p = \max\{\sum_{Y \in \mathcal{Y}} p(Y) : Y \text{ subpartition of } P, P \in \mathcal{P}\},$$

$$\dim(p) - 1 = \max\{|V| : V \text{ p-full partition of } V\} - 1.$$

Lemma 64. $OPT(p, \mathcal{P}) \geq \Phi$.

Proof. Let $G = (V, E)$ be a $\mathcal{P}$-partite graph that covers $p$ and that has $OPT(p, \mathcal{P})$ edges. First, let $X$ be a subpartition of $V$ such that $\alpha_p = \frac{1}{2} \sum_{X \in \mathcal{X}} p(X)$. Since an edge of $E$ connects at most two sets of $X$, applying (5) gives $|E| \geq \frac{1}{2} \sum_{X \in \mathcal{X}} d_G(X) \geq \frac{1}{2} \sum_{X \in \mathcal{X}} p(X) = \alpha_p$. Secondly, let $\mathcal{Y}$ be a subpartition of some $P \in \mathcal{P}$ such that $\beta_p = \sum_{Y \in \mathcal{Y}} p(Y)$. Since $G$ is $\mathcal{P}$-partite, an edge of $E$ enters at most one set of $\mathcal{Y}$, thus applying (5) gives $|E| \geq \sum_{Y \in \mathcal{Y}} d_G(Y) \geq \sum_{Y \in \mathcal{Y}} p(Y) = \beta_p$. By the above inequalities and by Lemma 2.1, $OPT(p, \mathcal{P}) = |E| \geq \max\{\alpha_p, \beta_p, \dim(p) - 1\} = \Phi$. \hfill \Box

5.2 Extension

We start this section with some algorithmic arguments. Since we do not want to rely on the results of Benczúr and Frank [4], we do not know how to calculate $dim(p)$. On the other hand, we want to present an algorithmic proof of the result so we will not use $\Phi$, that depends on $dim(p)$, in the extension phase.

A degree-specification $m$ is called an extension for $(p, \mathcal{P})$ if $m$ is $p$-admissible, $\mathcal{P}$-feasible and satisfies

$$\frac{1}{2} m(V) = \max\{\alpha_p, \beta_p\}. \quad (25)$$

An extension always exists, and we describe below how to find one. The algorithm is formulated in a way such that any extension can be its output (with appropriate choices). Recall that for $u \in V$, $X_u$ is the minimal tight set containing $u$ (if $u$ is not contained in a tight set then $X_u := V$).

Lemma 65. The Extension Algorithm outputs an extension.
Proof. Let $m_i$ be the degree specification obtained after Step $i$ in the above algorithm.

- By Step 1, $m_1$ (and then $m_2$) is $p$-admissible. Hence, by Claim 16, $m_{3a}$ is $p$-admissible and then so is $m_4$.

- By Step 2, $m_2(V)$ is even. If $m_2$ satisfies (20), then $m_4 = m_2$ is $\mathcal{P}$-feasible. Otherwise, there exists a $P \in \mathcal{P}$ such that $m_2(P) > \frac{1}{2} m_2(V)$. Then either in Step 3a $m_2(P)$ decreases to $\frac{1}{2} m_2(V)$, that is $m_4(P) = m_{3a}(P) = \frac{1}{2} m_2(V) = \frac{1}{2} m_4(V)$, or in Step 3b $m_2(V)$ increases to $2 m_{3a}(P)$, that is, by $v \in V \setminus P$, $m_4(P) = m_{3b}(P) = m_{3a}(P) = \frac{1}{2} m_{3b}(V) = \frac{1}{2} m_4(V)$. In both cases, $m_4$ is $\mathcal{P}$-feasible.

- It remains to show that (25) is satisfied. For some subpartition $\mathcal{Y}$ of some $P \in \mathcal{P}$, by $m_4$ is $p$-admissible and $\mathcal{P}$-feasible, we have $\beta_p = \sum_{Y \in \mathcal{Y}} p(Y) \leq \sum_{Y \in \mathcal{Y}} m_4(Y) \leq m_4(P) \leq \frac{1}{2} m_4(V)$. By parity, Theorem 1 and (9), $\frac{1}{2} m_2(V) = \lceil \frac{1}{2} m_1(V) \rceil = \lceil \frac{1}{2} \sigma_p \rceil = \alpha_p$. It follows that $\max \{ \alpha_p, \beta_p \} \leq \frac{1}{2} m_4(V)$.

If $m_4(V) = m_2(V)$, then $\max \{ \alpha_p, \beta_p \} \leq \frac{1}{2} m_4(V) = \frac{1}{2} m_2(V) = \alpha_p \leq \max \{ \alpha_p, \beta_p \}$, and we are done.

Otherwise, for some $P' \in \mathcal{P}$, $m_4(V) = m_{3b}(V) = m_{3a}(V) + (2 m_{3a}(P') - m_{3a}(V)) = 2 m_{3a}(P')$. Note that the set $D$ of the $m_{3a}$-positive elements of $P'$ is precisely the set of the $m_{3b}$-positive elements of $P'$. We have, by definition, $m_{3a}(D) = m_{3a}(P')$. Since the algorithm executed Step 3b, every element of $D$ belongs to a tight set. Hence, by Claim 15.2 applied to $D$, there exists a subpartition $\mathcal{Z}$ of $P'$ such that $\sum_{Z \in \mathcal{Z}} p(Z) \geq m_{3a}(D)$. By the definition of $\beta_p$, we have $\beta_p \geq \sum_{Z \in \mathcal{Z}} p(Z)$. Then $\max \{ \alpha_p, \beta_p \} \geq \beta_p \geq \sum_{Z \in \mathcal{Z}} p(Z) \geq m_{3a}(D) = m_{3a}(P') = \frac{1}{2} m_4(V) \geq \max \{ \alpha_p, \beta_p \}$ and we are done.

Note that if at least one of the conditions of Step 2 or 3b holds, then $m(V) > \sigma_p$, therefore there is no obstacle for $(p, \mathcal{P}, m)$.

5.3 Configurations

In this section we describe the functions and the partitions for which the lower bound may not be achieved. They may be classified in three different types of structures, called configurations. Two of them are natural generalizations of the configurations arising for graphs and hypergraphs. A new kind of configuration arises, it exists only in the abstract form of the problem. A configuration is a partition $\mathcal{A}$ of $V$ satisfying two kinds of conditions. First, the $p$-values of the
sets in the partition $\mathcal{A}$ and the lower bound $\Phi$ satisfy strict conditions. Secondly, the partition $\mathcal{A}$ is intimately related to the partition $\mathcal{P}$.

**Definition 66.** A partition $\mathcal{A} = \{A_1, \ldots, A_4\}$ of $V$ is a $C^*_4$-configuration for $(p, \mathcal{P})$ if

1. $\mathcal{A}$ is a $C^*_4$-construction for $p$,
2. $\alpha_p = \max\{\alpha_p, \beta_p\}$,
3. there exist $\ell \in \{1, 2\}$ and $P \in \mathcal{P}$ such that $(A_\ell, A_{\ell + 2})$ is a $P$-pair.

**Definition 67.** A partition $\mathcal{A} = \{A_1, \ldots, A_4, B_1, \ldots, B_t\}$ of $V$ ($t \geq 1$) is a $C^*_5$-configuration for $(p, \mathcal{P})$ if

1. $\mathcal{A}$ is a $C^*_5$-construction for $p$,
2. $\alpha_p = \max\{\alpha_p, \beta_p\}$,
3. (a) either there exist $\ell \in \{1, 2\}$ and $P \in \mathcal{P}$ such that $(A_\ell, A_{\ell + 2}, B_1, \ldots, B_t)$ is a $P$-subpartition,

$$\text{(b) or there exist } j_0 \in \{1, \ldots, t\} \text{ and distinct } P_{k_1}, P_{k_2} \in \mathcal{P} \text{ such that for } i = 1, 2, \{A_i, A_{i + 2} \cup \{B_j : j \neq j_0\}\} \text{ is a } P_{k_i}\text{-subpartition.}$$

A $C^*_5$-configuration is of type 1 (respectively type 2) if $3a$ (resp. $3b$) is satisfied.

**Definition 68.** A partition $\mathcal{A} = \{A_1, \ldots, A_6\}$ of $V$ is a $C^*_6$-configuration for $(p, \mathcal{P})$ if

1. $\mathcal{A}$ is a $C^*_6$-construction for $p$,
2. there exist distinct $P_{k_i} \in \mathcal{P}$ such that $(A_i, A_{i + 3})$ is a $P_{k_i}$-pair for $i = 1, 2, 3$.

A configuration is a $C^*_4$- or a $C^*_5$- or a $C^*_6$-configuration.

The following lemma explains why the condition $\alpha_p = \max\{\alpha_p, \beta_p\}$ does not exist for $C^*_6$-configurations and why the third lower bound $\text{dim}(p) - 1$ does not appear in the definition of configurations.

**Lemma 69.** If a configuration exists for $(p, \mathcal{P})$, then

1. $\alpha_p = \max\{\alpha_p, \beta_p\}$,
2. $\alpha_p = \Phi$.

**Proof.** Let $\mathcal{A}$ be a configuration for $(p, \mathcal{P})$.

1. If $\mathcal{A}$ is a $C^*_4$- or a $C^*_5$-configuration, then Definition 66.2 or 67.2 implies 69.1. Let $\mathcal{A}$ be a $C^*_5$-configuration. By (9) and Definitions 68.1 and 9.1d, $\alpha_p = \lceil \frac{1}{2} \sigma_p \rceil = 3$. By Definitions 68.2 and 9.1a, for $i = 1, 2, 3$, there exist distinct $P_{k_i} \in \mathcal{P}$ and subpartitions $X_j$ of $A_j \cap P_{k_i}$ for $j = i, i + 3$ such that $\sum_{X \in X_j \cup X_{i + 3}} p(X) = p(A_i) + p(A_{i + 3}) = 2$. Let $\mathcal{P} \in \mathcal{P}$ and $Y$ subpartition of $\mathcal{P}$ such that $\beta_p = \sum_{Y \in Y} \sigma(Y)$. Without loss of generality we may suppose that $P_{k_1} \neq P_{k_2}$. Then $Z := X_1 \cup X_4 \cup X_2 \cup X_5 \cup Y$ is a subpartition of $V$ and hence $6 = \sigma_p \geq \sum_{Z \in Z} \sigma(Z) = \sum_{X \in X_j \cup X_5} p(X) + \sum_{X \in X_j \cup X_2} p(X) + \sum_{Y \in Y} \sigma(Y) = 2 + 2 + \beta_p$ that is $\alpha_p = 3 > 2 \geq \beta_p$ and 69.1 follows.

2. By Theorem 1, let $m$ be a minimal $p$-admissible degree specification. Then, by Definition 66.1 or 67.1 or 68.1, $\mathcal{A}$ is an obstruction for $(p, m)$. By Lemma 48, there exists a complete $p$-admissible splitting off, let $E$ be the resulting set of edges. By Lemma 2.1 and Observation 10, we have $\text{dim}(p) - 1 \leq |E| = \frac{1}{2} m(V) = \frac{1}{2} \sigma_p = \alpha_p$. Then, by 69.1, $\alpha_p \geq \Phi = \max\{\alpha_p, \beta_p, \text{dim}(p) - 1\} \geq \alpha_p$ as required.
Proof. (of sufficiency): Let \( \mathcal{A} \) be a configuration and \( m \) an extension for \( (p, \mathcal{P}) \).

1. By Definitions 66.1, 67.1 and 68.1, \( \mathcal{A} \) is a construction for \( p \) of the same type, and hence Definitions 7.1, 8.1 and 9.1 are satisfied.

2. Since \( m \) is an extension for \( (p, \mathcal{P}) \), \( m \) is \( p \)-admissible and \( \frac{1}{2}m(V) = \max\{\alpha_p, \beta_p\} \). By Lemma 69.1, \( \max\{\alpha_p, \beta_p\} = \alpha_p \). By Lemma 69.2, \( \alpha_p = \Phi \). By Lemma 2.1 and Observation 10, \( \alpha_p = \frac{1}{2}\sigma_p \). It follows that \( \sigma_p = m(V) \) so \( m \) is minimal and also \( m(V) = 2\Phi \). Thus, Definitions 7.2, 8.2 and 9.2 are satisfied.

3. Since \( m \) is an extension for \( (p, \mathcal{P}) \), \( m \) is \( \mathcal{P} \)-feasible. We consider the three different configurations separately.

(a) If \( \mathcal{A} \) is a \( C^*_1 \)-configuration. Then, there exist \( \ell \in \{1, 2\} \) and \( \mathcal{P} \in \mathcal{P} \) such that 

\[
(A_\ell, A_{\ell+2}) \text{ is a } \mathcal{P} \text{-pair, that is there exists a subpartition } X_i \text{ of } A_i \cap \mathcal{P} \text{ such that } \sum_{X \in X_i} p(X) = p(A_i) \text{ for } i = \ell, \ell + 2.
\]

By Lemma 69.2, Definitions 67.2, 8.1f, 67.3b, \( m \) is \( p \)-admissible and \( \mathcal{P} \)-feasible and \( m(V) = 2\Phi \), we have \( \Phi = \frac{1}{2}\sigma_p = p(A_\ell) + p(A_{\ell+2}) = \sum_{X \in X_\ell \cup X_{\ell+2}} p(X) \leq \sum_{X \in X_\ell} m(X) \leq m(P \cap (\bigcup X \in X) \leq \frac{1}{2}m(V) = \Phi \). It follows that the \( m \)-positive elements of \( A_\ell \cup A_{\ell+2} \) are the \( m \)-positive elements of \( P \), so Definition 7.3 is satisfied.

(b) If \( \mathcal{A} \) is a \( C^*_1 \)-configuration.

i. If it is of type 1, that is, there exist \( \ell \in \{1, 2\} \) and \( \mathcal{P} \in \mathcal{P} \) such that 

\[
X = \{A_\ell, A_{\ell+2}, B_1, \ldots, B_t \} \text{ is a } \mathcal{P} \text{-subpartition, then, by Definitions 67.2, 8.1f, 67.3a,}
\]

\( m \) is \( p \)-admissible and \( \mathcal{P} \)-feasible and \( m(V) = 2\Phi \), we have \( \Phi = \frac{1}{2}\sigma_p = t + 2 = \sum_{X \in X_\ell} 1 = \sum_{X \in X} p(X') \leq \sum_{X \in X} m(X') \leq m(P \cap (\bigcup X \in X) \leq \frac{1}{2}m(V) = \Phi \). It follows that each set of \( \mathcal{P} \) contains an \( m \)-positive element of \( \mathcal{P} \), so Definition 8.3a is satisfied.

ii. If it is of type 2, that is, there exist \( j_0 \in \{1, \ldots, t\} \) and distinct \( P_{k_1}, P_{k_2} \in \mathcal{P} \) such \( X_1 = \{A_1, A_3\} \cup \{B_j : j \neq j_0\} \) and \( X_2 = \{A_2, A_4\} \cup \{B_j : j \neq j_0\} \) are \( P_{k_1} \)- and \( P_{k_2} \)-subpartitions, then, by Definitions 67.2, 8.1f, 67.3b, \( m \) is \( p \)-admissible, 8.1b and \( m(V) = 2\Phi \), we have \( 2\Phi - 2 = \sigma_p - 2 = 2t + 2 = \sum_{X \in X_{\ell+1}} 1 = \sum_{X \in X} p(X') + \sum_{X \in X} m(X') \leq \sum_{X \in X} m(X') \leq m(P_{k_1} \cap (\bigcup X \in X_1) + m(P_{k_2} \cap (\bigcup X \in X_2)) \leq m(V - B_{j_0}) = m(V) - 2 = 2\Phi - 2 \).

It follows that \( A_\ell, A_{\ell+2}, B_j \) for \( j \in \{1, \ldots, t\} \setminus j_0 \) contains an \( m \)-positive element of \( P_{k_i} \) for \( i = 1, 2 \), so Definition 8.3b is satisfied.

(c) If \( \mathcal{A} \) is a \( C^*_3 \)-configuration. Then there exist, for \( i = 1, 2, 3 \), distinct \( P_{k_i} \in \mathcal{P} \) such that \( (A_i, A_{i+3}) \) is a \( P_{k_i} \)-pair, that is, for \( j = i, i + 3 \), there exists a subpartition \( X'_j \) of \( A_j \cap P_{k_i} \) such that \( \sum_{X \in X'_j} p(X) = p(A_j) \). By Lemma 69.2, Definitions 9.1d, 68.2, \( m \) is \( p \)-admissible and \( m(V) = 2\Phi \), we have \( 2\Phi = \sigma_p = \sum_{i=1}^3 (p(A_i) + p(A_{i+3})) = \sum_{i=1}^3 \sum_{X \in X_i \cup X_{i+3}} p(X) \leq \sum_{i=1}^3 \sum_{X \in X_i \cup X_{i+3}} m(X) \leq \sum_{i=1}^3 m(P_{k_i} \cap (A_i \cup A_{i+3})) \leq \sum_{i=1}^3 m(A_i \cup A_{i+3}) = m(V) = 2\Phi \). It follows that the \( m \)-positive elements of \( A_i \cup A_{i+3} \) are the \( m \)-positive elements of \( P_{k_i} \) for \( i = 1, 2, 3 \), so Definition 9.3 is satisfied.
Proof of necessity: Let us suppose that no configuration exists for \((p, \mathcal{P})\). By Lemma 65, there exists an extension \(m\) for \((p, \mathcal{P})\). If no obstacle exists for \((p, \mathcal{P}, m)\), then the lemma is proved. Suppose that there is an obstacle \(\mathcal{A}\) for \((p, \mathcal{P}, m)\).

1. By Definitions 7.1, 8.1 and 9.1, \(\mathcal{A}\) is a construction for \(p\) of the same type, and hence Definitions 66.1, 67.1 and 68.1 are satisfied.

2. Since, by Definitions 7.2, 8.2 and 9.2, the extension \(m\) is minimally \(p\)-admissible, we have, by Observation 10, \(\max\{\alpha_p, \beta_p\} = \frac{1}{2}m(V) = \frac{1}{2}\sigma_p = \alpha_p\), so Definitions 66.2 and 67.2 are satisfied.

3. Since no configuration exists for \((p, \mathcal{P})\), Definition 66.3 (resp. 67.3 and 68.2) does not hold for \(\mathcal{A}\).

Throughout this proof, we will very often replace \(m\) by \(m' = m - \chi_{\{u\}} + \chi_{\{u'\}}\) for \(u' \in X_u\). We use that, by Claim 16, \(m'\) is \(p\)-admissible and that, by Lemma 43, \(X_u \subseteq A' \in \mathcal{A}\).

Below, we treat each obstacle separately. We will replace the extension \(m\) by an extension \(m'\) so that no obstacle exists for \((p, \mathcal{P}, m')\), arguing as follows. Suppose an obstruction exists for \((p, m')\). If \(\mathcal{A}\) fits the hypothesis of Lemma 44, that is \(\mathcal{A}\) is any obstacle but a non simple \(C_4^*\)-obstacle for \((p, \mathcal{P}, m)\), then \(\mathcal{A}\) is the unique obstruction for \((p, m)\), hence, by construction, the unique one for \((p, m')\). In this case we will choose \(m'\) so that \(\mathcal{A}\) does not satisfy the color condition for \((p, \mathcal{P}, m')\), ensuring that no obstacle exists for \((p, \mathcal{P}, m')\). Otherwise, \(\mathcal{A}\) is a \(C_4^*\)-obstacle that is not simple for \((p, \mathcal{P}, m)\), and if an obstruction exists for \((p, m')\), it is either \(\mathcal{A}\) or \(\mathcal{A}' = \{A_1, A_3, A_2, A_4\}\). We will choose \(m'\) so that \(\mathcal{A}\) is not an obstacle for \((p, \mathcal{P}, m')\), and show that \(\mathcal{A}'\) is not an obstruction for \((p, m')\) (see case (a)). Again, no obstacle will exist for \((p, \mathcal{P}, m')\).

(a) If \(\mathcal{A}\) is a \(C_4^*\)-obstacle, then Definition 66.3 does not hold.

If for some dominating \(P \in \mathcal{P}\) and for all \(m\)-positive elements \(u \in P\), we have \(X_u \subseteq P\), then by Definition 7.3, there exists \(\ell \in \{1, 2\}\) such that the \(m\)-positive elements of \(A_\ell \cup A_{\ell+2}\) are the \(m\)-positive elements of \(P\). Hence for \(i = \ell, \ell + 2\) and for the subpartition \(X_i\) of \(A_i\) defined by the maximal elements of the laminar family \(\{X_u : u \in V_+(m) \cap A_i\}\) we have, by Lemma 43, \(\sum_{X \in \mathcal{X}_i} p(X) = \sum_{X \in \mathcal{X}_i} m(X) = m(A_i) = p(A_i)\), that is \(A_\ell \cup A_{\ell+2}\) is a \(P\)-pair so Definition 66.3 holds for \(\mathcal{A}\), a contradiction.

Thus, for every dominating \(P \in \mathcal{P}\), there exist an \(m\)-positive element \(u \in P\) and an element \(u' \in X_u \setminus P\) and then replace \(m\) by \(m' = m - \chi_{\{u\}} + \chi_{\{u'\}}\). Then Definition 7.3 does not hold for \(\mathcal{A}\) and \(m'\), that is \(\mathcal{A}\) is not an obstacle for \((p, \mathcal{P}, m')\).

If \(\mathcal{A}\) is a simple \(C_4^*\)-obstacle for \((p, \mathcal{P}, m)\), then, by Lemma 44, \(\mathcal{A}\) is the unique obstruction for \((p, m')\), thus no obstacle exists for \((p, \mathcal{P}, m')\).

Otherwise, by Lemma 43, \(\mathcal{A}\) is the unique partition of \(V\) into maximal tight sets. Thus, if an obstacle exists for \((p, \mathcal{P}, m')\), then we may suppose that it is \(\mathcal{A}' = \{A_1, A_3, A_2, A_4\}\). By Definition 7.3 for \(\mathcal{A}, m, \mathcal{P}\), there exists an index \(i\) such that \((A_i, A_{i+2})\) is an \((m, P)\)-pair for some \(P \in \mathcal{P}\), that is the \(m\)-positive elements of \(A_i \cup A_{i+2}\) are exactly the \(m\)-positive elements of \(P\).

Note that, by construction of \(m'\) and by Claim 41, we have \(m(A_j) = m'(A_j)\), for all \(j = 1, \ldots, 4\). We may assume that \(m' = m - \chi_u + \chi_{u'} - \chi_v + \chi_{v'}\), with \(u \in A_i\) (and possibly \(v = v'\)).

Let us show that we may assume \(m(A_i) = 1\). Suppose that \(m(A_i) \geq 2\). Since \(m'(A_i) = m(A_i)\), we may assume, by Definition 7.3 for \(\mathcal{A}', m', \mathcal{P}\), that \((A_{i+2}, A_{i+3})\) is an \((m', P')\)-pair for some \(P' \in \mathcal{P}\). By Observation 11 and since \((A_i, A_{i+2})\) is an
(m, P)-pair, we have \( v \neq v', v \in A_j \) for some \( j \in \{i + 2, i + 3\} \), and \( m(A_j) = 1 \). Replacing \( u \) by \( v \), we may assume \( m(A_i) = 1 \).

Now, applying Claim 41 and Definitions 7.1d and 7.2 for \( m \) and \( A', m' \), we have
\[
\frac{m(A_1) + m(A_{i+2})}{2} = \frac{1}{2} m(V) = \frac{1}{2} m'(V) = m'(A_{i+1}) + m'(A_{i+2}) = m(A_{i+1}) + m(A_{i+2}),
\]
thus \( m(A_{i+1}) = 1 = m'(A_{i+1}) \). Then, by Lemma 43 for \( m' \), Definition 7.1b for \( m \), Claim 41 for \( m \), Lemmas 42 and 19, we have
\[
2 = m(A_i \cup A_{i+1}) - p(A_i \cup A_{i+1}) = p(A_i) + p(A_{i+1}) - (m(A_i) + m(A_{i+1})) = (m(A_i) + m(A_{i+1})) - (p(A_i) + p(A_{i+1})) = 0,
\]
a contradiction.

If \( A \) is a \( C_i^2 \)-obstacle, then let \( R = B \subseteq P \). We will say that \( u \in R \) (resp. \( B \)) is red (resp. blue). We suppose in this case that \( m \) is chosen in such a way that \( m(R) + m(B) \) is minimal. Then if \( u \) is a red or a blue \( m \)-positive element, we have \( X_u \subseteq R \cup B \). There are four cases, depending on \( m(R) \) and \( m(B) \).

i. \( m(R) = \frac{m(V)}{2} - 1 = m(B) \). Then \( A \) is of type 2, and by Definition 8.3b for \( m \), every \( B_j \neq B_{j_0} \) has exactly one \( m \)-positive element, and \( B_{j_0} \) has no red and no blue \( m \)-positive element. Note that the situation is exactly the same for red and for blue. Since Definition 67.3b does not hold, we may assume that there exists a red \( m \)-positive element \( u \) and a blue element \( u' \in X_u \). Replace \( m \) by \( m' \). Then none of Definitions 8.3a and 8.3b holds for \( m' \) because \( B \) became dominating and it contains no \( m' \)-positive element of \( B_{j_0} \).

ii. \( m(R) = \frac{m(V)}{2} > m(B) + 1 \). Then \( A \) is of type 1, and by Definition 8.3a for \( m \), every \( B_j \) contains a red \( m \)-positive element. Since Definition 67.3a does not hold, there exist a red \( m \)-positive element \( u \) and a blue element \( u' \in X_u \). Replace \( m \) by \( m' \). Then \( m'(P) < \frac{m'(V)}{2} \) for all \( P \in \mathcal{P} \) so Definition 8.3a does not hold for \( m' \). If Definition 8.3b holds for \( m' \), then the two colors involved in it are red and blue. However, each set \( B_j \) contains either a red or a blue \( m' \)-positive element, thus Definition 8.3b does not hold for \( m' \).

iii. \( m(R) = \frac{m(V)}{2} = m(B) + 1 \). Since Definition 67.3a does not hold, there exist a red \( m \)-positive element \( u \) and a blue element \( u' \in X_u \). Replace \( m \) by \( m' \). Then \( m'(R) + 1 = \frac{m'(V)}{2} = m'(B) \) and the only dominating set for \( m' \) is \( B \).

A. If \( u \) is in some \( A_i \), then the \( m' \)-positive element \( u' \) of \( A_i \) is blue and, by Definition 8.3a for \( m \), the \( m' \)-positive element of \( A_{i+2} \) is red. Therefore, since \( B \) is dominating, neither Definition 8.3a nor Definition 8.3b holds for \( m' \).

B. Otherwise, \( u \) is in some \( B_k \). If Definition 8.3a or 8.3b holds for \( m' \), then by Definition 8.3a for \( m \), both \( A_{i+2} \) contains a red \( m \)-positive element, both \( A_{i+1} \) and \( A_{i+3} \) contains a blue \( m \)-positive element, every \( B_j \neq B_k \) contains exactly one red and one blue \( m' \)-positive element and the \( m' \)-positive element in \( B_k \) different from \( u' \) is neither blue nor red. Note that the situation is exactly the same for \( m' \) and blue and for \( m' \) and red. Since Definition 67.3b does not hold, we may assume that there exist a red \( m \)-positive element \( v \notin B_k \) and a blue element \( v' \in X_v \). Repeating case iii. for \( m \) with \( v \) and \( v' \) instead of \( u \) and \( u' \), we get an extension \( m'' \) such that \( B_k \) has two blue \( m'' \)-positive elements, and since \( B \) is dominating, we are done.

iv. \( m(R) = \frac{m(V)}{2} = m(B) \). Then \( A \) is of type 1 both for \( B \) and \( R \), and by Definition 8.3a for \( m \), both \( A_{i+2} \) contains a red \( m \)-positive element, both \( A_{i+1} \) and \( A_{i+3} \) contains a blue \( m \)-positive element, and every \( B_j \) contains exactly one red and one blue \( m \)-positive element. Since Definition 67.3a does not hold for \( R \) and for \( B \), there exist a red \( m \)-positive element \( u \), a blue element \( u' \in X_u \), a blue \( m \)-positive element \( v \) and a red element \( v' \in X_v \). Replace \( m \) by \( m'' = m - \chi_u + \chi_{u'} - \chi_v + \chi_{v'} \). Note that \( m''(R) = m''(B) = \frac{m''(V)}{2} \).
A. If \( u \) or \( v \) is in some \( A_i \), then either \( A_{i-1} \) and \( A_i \) or \( A_i \) and \( A_{i+1} \) contain \( m'' \)-positive elements of the same color, thus neither Definition 8.3a nor Definition 8.3b holds for \( m'' \).

B. Otherwise, since Definition 67.3b does not hold, we may assume that \( u \in B_i, v \in B_j \) with \( i \neq j \). Then \( B_i \) contains two blue \( m'' \)-positive elements and \( B \) is dominating, hence neither Definition 8.3a nor Definition 8.3b holds for \( m'' \).

(c) If \( \mathcal{A} \) is a \( C^*_m \)-obstacle, then Definition 68.2 does not hold. By Definition 9.3, there exist distinct \( P_{k_i} \in \mathcal{P} \) such that the \( m \)-positive elements of \( A_i \cup A_{i+3} \) are the \( m \)-positive elements of \( P_{k_i} \) for \( i = 1, 2, 3 \). If for \( i = 1, 2, 3 \), for \( j = i, i + 3 \) and for all \( m \)-positive elements \( u \in A_j \), we have \( X_u \subseteq P_{k_i} \), then \( p(X_u) = m(X_u) = m(A_j) = p(A_j) \), that is \( A_i \cup A_{i+3} \) is a \( P_{k_i} \)-pair so Definition 68.2 holds for \( \mathcal{A} \), a contradiction. Thus there exist an \( m \)-positive element \( u \in P \) and an element \( u' \in X_u \setminus P \), for some \( p \in \mathcal{P} \). Replace \( m \) by \( m' \). Then Definition 9.3 does not hold for \( m' \).

After these modifications, Definition 7.3 (resp. 8.3 and 9.3) does not hold, hence \( \mathcal{A} \) is not an obstacle for \((p, \mathcal{P}, m')\). When \( \mathcal{A} \) was any obstacle but a non simple \( C^*_m \)-obstacle for \((p, \mathcal{P}, m)\), then \( \mathcal{A} \) is the unique obstruction for \((p, m)\), hence, by construction, the unique one for \((p, m')\), thus no obstacle exists for \((p, \mathcal{P}, m')\). When \( \mathcal{A} \) is a \( C^*_m \)-obstacle that is not simple for \((p, \mathcal{P}, m)\), we used ad hoc arguments to show that no obstacle exists for \((p, \mathcal{P}, m')\), see case (a). In conclusion, \( m' \) is the desired extension.

5.4 Main theorem

By exploiting the relations between configurations and obstacles and by applying our splitting off result, we may now prove our main theorem. It states that the lower bound \( \Phi \), defined in Section 5.1, may always be achieved unless there exists a configuration, in which case one more edge is needed.

**Theorem 71.** Let \( p : 2^V \rightarrow \mathbb{Z} \) be a symmetric crossing supermodular set function and \( \mathcal{P} \) a partition of \( V \). Then the minimum number of edges of a \( \mathcal{P} \)-partite graph that covers \( p \) is \( \Phi \) unless a configuration exists, in which case it is \( \Phi + 1 \).

**Proof.** The following lemmas prove the theorem.

**Lemma 72.** \( \text{OPT}(p, \mathcal{P}) \geq \Phi \). If there exists a configuration for \((p, \mathcal{P})\), then the inequality is strict.

**Proof.** By Lemma 64, \( \text{OPT}(p, \mathcal{P}) \geq \Phi \).

Suppose there exists a configuration for \((p, \mathcal{P})\) and the inequality is not strict that is \( \text{OPT}(p, \mathcal{P}) = \Phi \). Let \( F \) be a minimum set of edges such that \((V, F)\) covers \( p \) and satisfies the partition constraint, and let \( m \) be the degree specification obtained from \( m := 0 \) by unsplitting every edge of \( F \). Note that \( m \) is \((p, \mathcal{P})\)-allowed and there exists a complete \((p, \mathcal{P})\)-allowed splitting off. By the construction of \( m \), the minimality of \( F \), \( \text{OPT}(p, \mathcal{P}) = \Phi \) and Lemma 69.2-1, \( \frac{1}{2} m(V) = |F| = \text{OPT}(p, \mathcal{P}) = \Phi = \alpha_p = \max\{\alpha_p, \beta_p\} \) so \( m \) is an extension for \((p, \mathcal{P})\). Since there is a configuration for \((p, \mathcal{P})\), by Lemma 70, there is an obstacle for \((p, \mathcal{P}, m)\). But now Theorem 61 contradicts the existence of a complete \((p, \mathcal{P})\)-allowed splitting off.

**Lemma 73.** \( \text{OPT}(p, \mathcal{P}) \leq \Phi + 1 \). If there exists no configuration for \((p, \mathcal{P})\), then the inequality is strict.

**Proof.** If there exists no configuration for \((p, \mathcal{P})\), then, by Lemma 70, there exists an extension \( m \) for \((p, \mathcal{P})\) such that no obstacle exists for \((p, \mathcal{P}, m)\). Hence \( \frac{1}{2} m(V) = \max\{\alpha_p, \beta_p\} \). By Theorem
61, there exists a $P$-partite graph $(V, F)$ that covers $p_0$ with either $|F| \leq \frac{1}{2} m(V) = \max \{\alpha_p, \beta_p\}$ or $|F| \leq \dim(p_0) - 1$. In both cases $\OPT(p, P) \leq |F| \leq \Phi$ and the strict inequality follows.

If there exists a configuration for $(p, P)$, then let $m$ be an extension for $(p, P)$. By Lemma 69, $m(V) = 2\Phi$. This implies that $m$ is $p$-legal. Replace $m$ by $m' := m + \chi_u + \chi_v$ for some $u, v$ without violating $m'(P) \leq \frac{m(V)}{2}$ for every $P \in \mathcal{P}$. Then $m'(V) = 2\Phi + 2$, $m'$ is $p$-legal and no set containing $u$ or $v$ is tight. By Claim 41, there exists no obstacle for $(p, P, m')$. Then, by Theorem 62, there exists a complete $(p, P)$-allowed splitting off and the inequality follows. 

\[ \square \]  

6 Applications

6.1 Covering of a symmetric crossing supermodular function by a graph

Our main result, Theorem 71, implies the theorem of Benczúr and Frank [4]. Indeed, let $P$ be the partition of $V$ consisting of the singletons. Then no configuration exists and $\Phi = \max \{\alpha_p, \dim(p) - 1\}$.

6.2 Partition constrained global edge-connectivity augmentation of a hypergraph

We show in this section that our main result, Theorem 71, implies the theorem of Bernáth et al. [6] about partition constrained global edge-connectivity augmentation of a hypergraph. We mention that the proof of Theorem 76 given in [6] is considerably shorter than the present paper.

Let $\mathcal{G} = (V, \mathcal{E})$ be a hypergraph. For a vertex set $X$, we denote by $\delta_{\mathcal{G}}(X)$ the set of hyperedges intersecting both $X$ and $V \setminus X$ and $d_{\mathcal{G}}(X) = |\delta_{\mathcal{G}}(X)|$. Let us denote by $d_0(X, Y)$ (respectively $d_1(X, Y)$) the number of hyperedges intersecting $X \setminus Y$ and $Y \setminus X$ and none of $X \cap Y$ and $V \setminus (X \cup Y)$ (resp. exactly one of $X \cap Y$ and $V \setminus (X \cup Y)$).

For an integer $k$, let $p = k - d_{\mathcal{G}}$. It is well known that $d_{\mathcal{G}}$ satisfies equality (26) for all subsets $X$ and $Y$ of $V$. By (26), for all crossing subsets $X$ and $Y$ of $V$, $p$ satisfies (27).

\[
d_{\mathcal{G}}(X) + d_{\mathcal{G}}(Y) = d_{\mathcal{G}}(X \cap Y) + d_{\mathcal{G}}(X \cup Y) + 2d_0(X, Y) + d_1(X, Y), \tag{26}
\]

\[
p(X) + p(Y) = p(X \cap Y) + p(X \cup Y) - 2d_0(X, Y) - d_1(X, Y). \tag{27}
\]

Let us recall the following two lower bounds : $\Phi' = \max \{\alpha_{\mathcal{G}}, \beta_{\mathcal{G}}, \omega_{\mathcal{G}} - 1\}$, where $\alpha_{\mathcal{G}}, \beta_{\mathcal{G}}$ and $\omega_{\mathcal{G}}$ are defined in the Introduction and $\Phi = \max \{\alpha_p, \beta_p, \dim(p) - 1\}$, where $p = k - d_{\mathcal{G}}$.

**Definition 74.** A partition $\mathcal{A} = \{A_1, \ldots, A_4\}$ of $V$ is called a $\mathcal{C}_4$-configuration of $\mathcal{G}$ if

1. $\Phi' = k - d_{\mathcal{G}}(A_1) + k - d_{\mathcal{G}}(A_3) = k - d_{\mathcal{G}}(A_2) + k - d_{\mathcal{G}}(A_4)$,

2. there exists $\mathcal{F} \subseteq \mathcal{E}$ such that

   \hspace{1cm} (a) $\mathcal{F} = \delta(A_1) \cap \delta(A_3) = \delta(A_2) \cap \delta(A_4)$,

   \hspace{1cm} (b) $k - |\mathcal{F}|$ is odd,

3. there exist $P \in \mathcal{P}$ and $\ell \in \{1, 2\}$ such that $(A_\ell, A_{\ell+2})$ is a $P$-pair.

**Definition 75.** A partition $\mathcal{A} = \{A_1, \ldots, A_6\}$ of $V$ is called a $\mathcal{C}_6$-configuration of $\mathcal{G}$ if

1. $\Phi' = 3$,

2. (a) $k - d_{\mathcal{G}}(A_i) = 1$ for $i = 1, \ldots, 6$, 

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Proof. For each $i$, let $A_i$ be a $p$-full partition of $V$ such that $p(A_i) = p(A_{i+1}) + 1$. Then we have
\[
p(A_i) + p(A_{i-1}) = 2p(A_{i-1}) + 1.
\] If $p(A_{i-1}) = 0$, then $p(A_i) = 1$, and if $p(A_{i-1}) = 1$, then $p(A_i) = 2$. Therefore, $p(A_i) = 1$ or $p(A_i) = 2$ for all $i$.

Claim 77. For each $i$, let $A_i$ be a $p$-full partition of $V$ such that $p(A_i) = p(A_{i+1}) + 1$. Then we have
\[
p(A_i) + p(A_{i-1}) = 2p(A_{i-1}) + 1.
\] If $p(A_{i-1}) = 0$, then $p(A_i) = 1$, and if $p(A_{i-1}) = 1$, then $p(A_i) = 2$. Therefore, $p(A_i) = 1$ or $p(A_i) = 2$ for all $i$.

Claim 78. A partition $A = \{A_1, \ldots, A_4\}$ of $V$ is a $C_4$-configuration if and only if it is a $C_4$-configuration for $G$.}

Proof. Note that $\alpha_G = \alpha_p$ and $\beta_G = \beta_p$.

First we show that $\Phi \geq \Phi'$. If $\omega - 1 < \Phi$, then $\Phi' = \max\{\alpha_G, \beta_G\} = \max\{\alpha_p, \beta_p\} \leq \Phi$. If $\omega - 1 \geq \Phi'$, then let $\Phi$ be the set of hyperedges of size $k - 1$ whose deletion results in a hyper graph with $\omega_G$ connected components. Then $X$ is a $p$-full partition. Indeed, if $Y$ is the union of some of the $X'_i$s, then $p(Y) = k - d_G(Y) = k - d_X(Y) \geq k - (k - 1) = 1$, and there is an index $i$ such that $p(X_i) = 1$, otherwise, we have $\omega - 1 \geq \Phi' \geq \omega \geq \frac{1}{2}\sum_{i=1}^{\omega_G}(k - d_G(X_i)) = \frac{1}{2} \sum_{i=1}^{\omega_G} p(X_i) \geq \frac{1}{2} \sum_{i=1}^{\omega_G} 2 = \omega_G$, a contradiction. Then $\Phi \geq \Phi' + 1$.

Now we show that $\Phi' \geq \Phi$. If $\dim(p) > 1$, then $\Phi' = \max\{\alpha_G, \beta_G\} = \max\{\alpha_p, \beta_p\} \leq \Phi'$. If $\dim(p) = 1$, then, by Lemma 6.3 of [4], $\omega_G \geq \dim(p)$ and hence $\Phi' \geq \omega_G - 1 \geq \dim(p) - 1 \geq \Phi$.

By the above arguments, 77 follows.

Claim 79. A partition $A = \{A_1, \ldots, A_4\}$ of $V$ is a $C_4$-configuration for $p$ if and only if it is a $C_4$-configuration for $G$.

Proof. By Definition 66.3 is the same as Definition 74.3.

By (27) applied to $A_i \cup A_{i-1}$ and $A_i \cup A_{i+1}$, Definition 7.1a is equivalent to $d_0(A_i \cup A_{i-1}, A_i \cup A_{i+1}) = d_0(A_i \cup A_{i-1}, A_i \cup A_{i+1}) = 0$ that is to $\delta(A_1) \cap \delta(A_3) = \delta(A_2) \cap \delta(A_4)$ which is 74.2b. Let $F$ be this set of hyperedges. Then, by (26) applied to $A_i$ and $A_{i+1}$, Definition 7.1a is equivalent to $k - |F| = k - d_G(A_i) - d_G(A_{i+1}) + d_G(A_i \cup A_{i+1}) + 2d_0(A_i, A_{i+1}) = p(A_i) + p(A_{i+1}) - p(A_i \cup A_{i+1}) + 2d_0(A_i, A_{i+1})$ is odd, that is to Definition 74.2b.

Suppose that Definition 74.1 is satisfied. Then, by the definition of $\Phi$, Claim 77, Definition 74.1 and the definition of $\alpha_G$, we have $\max\{\alpha_G, \beta_G, \dim(p) - 1\} = \Phi = \Phi' = \frac{1}{2}(k - d_G(A_1) + k - d_G(A_3) + k - d_G(A_2) + k - d_G(A_4)) \leq \alpha_G$. This implies that Definitions 7.1d and 66.2 hold. Suppose that $1 = p(A_i) = k - d_G(A_i)$ for $i = 1, \ldots, 4$. It also follows that $\dim(p) - 1 \leq \frac{1}{2}(k - d_G(A_1) + k - d_G(A_3) + k - d_G(A_2) + k - d_G(A_4)) = 2$. Then, by $|A| = 4$, $A$ is not a $p$-full partition. Since $1 = p(A_i)$ for $i = 1, \ldots, 4$ and, by Claim 39.4, $p(A_i \cup A_{i+1}) = 0$ for $i = 1, \ldots, 4$, it follows, by the symmetry of $p$, that $p(A_1 \cup A_3) = p(A_2 \cup A_4) \leq 0$, and Definition 7.1c follows.

Now suppose that Definitions 7.1c, 7.1d and 66.2 are satisfied. Then, by Definition 7.1d, Lemma 69.2 and Claim 77, we have $p(A_1) + p(A_3) = p(A_2) + p(A_4) = \frac{1}{2}\sigma_p = \Phi = \Phi'$, so, by $p(A_i) = k - d_G(A_i)$, 74.1 follows.

We mention that in [6] there was a fourth condition for a $C_4$-configuration, namely $k - d_G(A_i) > 0$ for $i = 1, \ldots, 4$. However, by Claim 39.2, this condition is implied by the others.
Claim 79. There exists no $C^*_5$-configuration for $p$.

Proof. Let us suppose that $A = \{A_1, \ldots, A_4, B_1, \ldots, B_t\}$ \((t \geq 1)\) is a $C^*_5$-configuration for $p$. Let $B = \bigcup_{j=1}^t B_j$, $X_i = A_i \cup B$ and $Y_i = A_i \cup A_{i+1}$. By Claim 40, $p(B) = 2$, $p(X_1) = 1$, $p(Y_1 \cup X_{i+1}) = p(X_1 \cup X_{i+1}) = 1$ and $p(X_i \cup X_{i+2}) \leq 0$. By (27) applied $X_1$ and $X_2$, we have $1 + 1 = p(X_1) + p(X_2) = p(B) + p(X_1 \cup X_2) - 2d_0(X_1, X_2) - d_1(X_1, X_2) = 2 + 1 - 2d_0(X_1, X_2) - d_1(X_1, X_2)$, that is $d_1(X_1, X_2) = 1$, thus there exists a hyperedge $e$ of $G$ that enters $A_1$, $A_2$ and exactly one of $B$ and $A_3 \cup A_4$. If $e$ enters $B$, then $d_1(Y_1, X_2) \geq 1$ and hence, by (27) applied to $Y_1$ and $X_2$, we have $1 + 1 = p(Y_1) + p(X_2) = p(A_2) + p(Y_1 \cup X_2) - 2d_0(Y_1, X_2) - d_1(Y_1, X_2) \leq 1 + 1 - 0 - 1$, a contradiction. Otherwise, $e$ enters $A_3$ or $A_4$, say $A_3$. Then $d_1(X_1, X_3) \geq 1$, and hence, by (27) applied to $X_1$ and $X_3$, we have $1 + 1 = p(X_1) + p(X_3) = p(B) + p(X_1 \cup X_3) - 2d_0(X_1, X_3) - d_1(X_1, X_3) \leq 2 + 0 - 0 - 1$, a contradiction. \(\square\)

Claim 80. A partition $A = \{A_1, \ldots, A_6\}$ of $V$ is a $C^*_6$-configuration for $p$ if and only if it is a $C_6$-configuration for $G$.

Proof. First we show that Definition 75.2c is a corollary of Definitions 75.2a-2b. By (27) applied to $A_1 \cup A_2$ and $A_2 \cup A_3$ and Claim 13.2, $1 + 1 = p(A_1 \cup A_2) + p(A_2 \cup A_3) = p(A_2) + p(A_1 \cup A_2 \cup A_3) - 2d_0(A_1 \cup A_2, A_2 \cup A_3) - d_1(A_1 \cup A_2, A_2 \cup A_3) \leq 1 + 1 + 0 + 0$. It follows that $d_0(A_1 \cup A_2, A_2 \cup A_3) = d_1(A_1 \cup A_2, A_2 \cup A_3) = 0$, that is every hyperedge intersecting $A_1$ and $A_3$ intersects $A_2$ and $A_5 \cup A_6$. Since this is true for every $A_i$ and $A_{i+2}$, Definition 75.2(c) follows. Then, by (26) applied to $A_1$ and $A_{i+1}$, we have $k - |F| = k - d_0(A_i) - d_0(A_{i+1}) + d(A_i \cup A_{i+1}) = p(A_i) + p(A_{i+1}) - p(A_i \cup A_{i+1}) + 2d_0(A_i, A_{i+1}) = 1 + 1 - 1 + 2d_0(A_i, A_{i+1})$ is odd, that is Definition 75.2(c)ii is satisfied. Definitions 91a-1b and 68.2 are the same as Definitions 75.2a-2b and 75.3.

Suppose that Definition 75.1 is satisfied. Then, by Claim 77, we have $3 = \Phi' = \Phi = \max\{\sigma_p, \beta_p, \dim(p) - 1\}$. This implies that $\beta_p \leq 3$ and then, $6 = \sum_{i=1}^6 p(A_i) \leq \sigma_p \leq 2\alpha_p \leq 6$, and Definition 9.1d follows. It also implies $\dim(p) \leq 4$. Since $|A| = 6$, $A$ is not a $p$-full partition, and then, by Claim 13.3, Definition 9.1c follows.

Now suppose that $A$ is a $C^*_6$-configuration. Then, by Definition 9.1d, Lemma 69.2 and Claim 77, we have $3 = \frac{1}{2}\sigma_p = \Phi = \Phi'$, so 75.1 follows. \(\square\)

By Claims 78, 79 and 80, Theorem 71 implies Theorem 76.

6.2.1 Global edge-connectivity augmentation of a hypergraph in a subset of vertices

For an integer $k$ and a subset $T$ of $V$, a hypergraph $H = (V, E)$ is called $k$-edge-connected in $T$ if $d_H(X) \geq k$ for all $X \subset V$ such that $X \cap T$ and $T - X$ are non-empty. The problem of making a given hypergraph $k$-edge-connected in $T$ by adding a minimum set of edges was solved by Benczü and Frank [4]. Theorem 71 solves the following partition constrained version of this problem: given a hypergraph $H = (V, E)$, a subset $T$ of $V$, a partition $P$ of $T$ and an integer $k$, find a graph $G = (T, E)$ with a minimum number of edges to be added to $H$ between distinct members of $P$ such that the resulting hypergraph is $k$-edge-connected in $T$. Indeed, if we define the function $p : T \rightarrow Z$ with $p(X) = \max\{k - d_H(X \cup Y) : Y \subseteq V - T\}$ for any nonempty $X \subset T$ and $p(\emptyset) = p(T) = 0$, then it was shown in [4] that $p$ is symmetric and crossing supermodular, so Theorem 71 can be applied.

7 Algorithm and complexity

In this section, we describe an algorithm that, given a symmetric crossing supermodular set function $p : 2^V \rightarrow Z$ and a partition $P$ of $V$, finds a $P$-partite a graph that covers $p$ having a minimum number of edges. We then explain in which settings the subroutines needed for the
algorithm are polynomial. Finally, we sketch why the algorithm itself is polynomial in these settings.

Throughout, $G$ will denote a graph, $p_G = p - d_G$, and $m_G$ a $p_G$-admissible degree specification.

### 7.1 Augmentation algorithm

Given a symmetric crossing supermodular set function $p : 2^V \to \mathbb{Z}$ and a partition $\mathcal{P}$ of $V$, the augmentation algorithm finds a $\mathcal{P}$-partite graph that covers $p$ having a minimum number of edges. It consists of three major steps, extension, then splitting off, and finally determining if a configuration exists.

**Augmentation algorithm:**

**Input:** A symmetric crossing supermodular function $p : 2^V \to \mathbb{Z}$ and a partition $\mathcal{P}$ of $V$.

**Output:** A $\mathcal{P}$-partite graph that covers $p$ of minimum size.

**Step 1.** Find an extension $m$ for $(p, \mathcal{P})$ applying the Extension algorithm described in Figure 2 of Section 5.2.

**Step 2.** Apply the Splitting off algorithm described in Figure 1 of Section 4.6.

**Step 3.** If it stops with a complete $(p, \mathcal{P})$-allowed splitting off, then we have found a $\mathcal{P}$-partite graph that covers $p$ having $\frac{m(V)}{2} = \max\{\alpha_p, \beta_p\}$ edges and Stop.

**Step 4.** If it stops with a $\mathcal{P}$-partite graph that covers $p$ having at most $\dim(p) - 1$ edges then Stop.

**Step 5.** Otherwise, it stops with an obstacle $\mathcal{A}$ for $(p, \mathcal{P}, m)$, then apply the proof of Lemma 70 to $\mathcal{A}$.

**Step 6.** If it finds another extension $m'$ for $(p, \mathcal{P})$ such that no obstacle exists for $(p, \mathcal{P}, m')$, then Go to Step 2.

**Step 7.** Otherwise, it finds a configuration for $(p, \mathcal{P})$. The algorithmic proof of Lemma 73 provides the desired graph with $\frac{m'(V)}{2} = \Phi + 1$ edges and Stop.

**Figure 3:** Augmentation algorithm

### 7.2 Subroutines

In this section, we give the framework in which our algorithms run in polynomial time, together with the basic bricks needed throughout. These bricks mostly concern crossing supermodular functions, operations such as splitting off, tight sets, and partition constraints.

Some of the arguments used below come from [4], adapted to the partition constrained version when necessary.

#### 7.2.1 Crossing supermodular functions

**Minimization oracle.** We assume that the function $p$ is given with an evaluation oracle and a polynomial minimization oracle. The last oracle outputs, in polynomial time, a subset of $V$ that is a solution of $\min_{\emptyset \neq X \subseteq V} \{m(X) - p_G(X)\}$, for any modular function $m$ and for any graph $G$.

Note that checking whether a degree specification $m$ is $p_G$-admissible can be done in polynomial time using these oracles.

When $p$ is *fully crossing supermodular*, that is (3) is satisfied for all crossing pairs (the positivity condition is dropped), then a polynomial minimization oracle can be implemented.
from an evaluation oracle. Indeed, in this case, the minimization of $m - p_G$ can be reduced to the minimization of a fully submodular function, which can be solved in polynomial time. For example, [8] provides a minimizer in $O(|V|^5 \gamma + |V|^6 \log |V|)$, where $\gamma$ is the time needed to call the function evaluation oracle.

Note that, in our application for hypergraphs, evaluating the function can be done in polynomial time. In fact, in this case the minimization problem can be solved with network flow techniques [3].

**Subpartition lower bound and minimal degree-specification.** Thanks to the minimization oracle, a greedy algorithm computes $\sigma_p$ and a minimal degree-specification $m$ in Theorem 1, see [7]. As a consequence, the subpartition lower bound $\alpha_p$ can be computed in polynomial time.

### 7.2.2 Operations

**Splitting off.** Deciding whether splitting off at $x,y$ is $p_G$-admissible is equivalent to checking if $\min_{X \subset V} (m_G(X) - p_G(X)) \geq 0$, where $G'$ is obtained from $G$ by splitting off at $x,y$. Therefore, this can be done in polynomial time, calling the minimization oracle once.

**Flips and improvements.** Since flips and improvements can also be described by a fixed number of unsplitting and splitting off, checking their admissibility is also polynomial.

### 7.2.3 Tight sets

**Minimal tight sets.** Here, we explain how to find, in polynomial time, a minimal tight set containing a given positive element, if it exists.

First, note that, given an $m_G$-positive element $x \in V$ and $y \in V \setminus x$, one can find, if it exists, a tight set containing $x$ but not $y$ by calling the minimization oracle for $m' + m_G - p_G$, where $m'(x) = -M$, $m'(y) = M$, $m'(z) = 0$ otherwise, and $M$ is a suitable big number. Indeed, due to $p_G$-admissibility of $m_G$, and the choice of $m'$ and $M$, we have $\min_{X \subset V} (m'(X) + m_G(X) - p_G(X)) \geq -M$, and any solution to this minimization problem contains $x$ but not $y$. Moreover, such a solution $S$ is tight if and only if $m'(S) + m_G(S) - p_G(S) = -M$.

Then, given an $m_G$-positive element $x$, one can find a minimal tight set containing $x$ by applying the above remark to $x$ and $y$, for all $y \in V \setminus x$, and then by taking the intersection of the solutions which are tight sets. By Claim 14, this provides a minimal tight set containing $x$.

**Maximal tight sets.** Arguments similar to the ones above allow one to find a maximal tight set containing a given $m_G$-positive element in polynomial time, if it exists.

First, given distinct $x,y \in V$, one can find a tight set containing both $x$ and $y$, if it exists, by calling the minimization oracle for $m'' + m_G - p_G$, where $m''(x) = m''(y) = -M$, and $0$ otherwise. Then, given $x \in V_+(m_G)$, one can find a maximal tight set containing $x$ by applying the above remark to $x$ and $y$, for all $y \in V \setminus x$, and then by taking the union of the solutions which are tight sets. By Claim 14, this provides a maximal tight set containing $x$.

**Obstructions.** We explain how to find an obstruction for $(p_G,m_G)$ in polynomial time, if it exists.

Recall that, by Lemma 43, an obstruction for $(p_G,m_G)$ is the unique partition of $V$ into maximal tight sets, in which, by Claim 39.2, Definitions 8.1a and 9.1a, every member contains an $m_G$-positive element. Applying at most $|V|$ times the algorithm that finds the maximal tight set containing a given element, one can find such a partition of $V$, if it exists. Then, since Definitions 7.1, 8.1 and 9.1 involve single sets or pairs of sets of the partition, it is straightforward to check whether one holds. Finally, by the remarks on the lower bounds of Section 7.2.1, on
can check if Definitions 7.2, 8.2 or 9.2 holds. Therefore, if an obstruction exists for \((p_G, m_G)\), then it can be found in polynomial time.

7.2.4 Partition constraints

Color condition. Note that it is immediate to check whether (20) holds.

Allowed operations. Since checking the \(p_G\)-admissibility of any operation can be done in polynomial time, the above remark implies that checking whether an operation is allowed is polynomial.

Obstacles. If an obstacle exists for \((p, P, m)\), then one can find it by, first, finding the corresponding obstruction, and then checking whether this obstruction is an obstacle (that is Definition 7.3, 8.3 or 9.3 holds). The first part is done in polynomial time by results of Section 7.2.3, and the second one is immediate seen the definitions.

Sequences of allowed splitting off. We show how to perform, in polynomial time, arbitrary allowed splitting off until there are none.

(a) Suppose there is a dominating color \(P\). Then, every allowed splitting off involves exactly one \(m_G\)-positive element of \(P\).

Let \(x\) be such an element. Given \(y \in V_+(m_G) \setminus P\), the arguments of Section 7.2.3 about minimal tight sets allows one to compute \(\omega_{x,y} = \min_{x,y \in X \subseteq V} \{m_G(X) - p_G(X)\}\) in polynomial time. Note that the maximum number of \(p_G\)-admissible (hence allowed) splitting off at \(x, y\) is \(\min\{\frac{1}{2} \omega_{x,y}, m_G(x), m_G(y)\}\). Repeating this for all \(y \in V_+(m_G) \setminus P\), one can perform in polynomial time allowed splitting off involving \(x\), until there are none.

Repeating this for every \(m_G\)-positive element of \(P\), one can perform allowed splitting off until there are none, in polynomial time.

(b) Suppose there is no dominating color. Let \(x \in V_+(m_G)\), and apply the argument of (a) to perform arbitrary allowed splitting off involving \(x\): it stops either because a dominating color appears, and then we can apply (a); or because \(x\) no longer belongs to allowed splitting off, in which case we repeat (b) with another positive element.

Since (b) is repeated at most \(|V|\) times, and both (a) and (b) are done in polynomial time, we performed allowed splitting off until there are none, in polynomial time.

7.3 Complexity

The aim of the section is to sketch why the augmentation algorithm runs in polynomial time, provided the function \(p\) is given with an evaluation oracle and a minimization oracle. Before doing so, we first sketch why the extension and splitting off algorithms are polynomial.

7.3.1 Extension algorithm

We sketch why each step of the extension algorithm can be done in polynomial time, provided the function \(p\) is given with a minimization oracle.

Step 1 is polynomial, see the Subpartition lower bound paragraph of Section 7.2.1.

Step 2 is immediate.
If we execute Step 3 as it is mentioned in Figure 2, then the algorithm is not polynomial. Indeed, Step 3 can be repeated $|V|P$ times where $P$ is the maximum value of the function $p$. To turn Step 3 polynomial we have to use the tricks explained in Section 6.6 of [2]. One has to use the minimal degree specification algorithm of [7] for different starting values, for details see [2].

Step 3b is immediate.

**Lower bounds.** As we have already mentioned in Section 5.2, we can not calculate $\text{dim}(p)$. On the other hand, the extension algorithm, as it is described above, provides in polynomial time $\max\{\alpha_p, \beta_p\}$.

### 7.3.2 Splitting off algorithm

We sketch why each step of the splitting off algorithm can be done in polynomial time, provided the function $p$ is given with a minimization oracle.

Step 1 is polynomial by the remarks of Section 7.2.4 about sequences of allowed splitting off.

Step 2 is polynomial. Indeed, by Corollary 28, we have $m_G(V) \leq |V|$. Then, since performing an improvement decreases $m_G(V)$ by 2, the number of improvements in the sequence is at most $m_G(V)/2 \leq |V|/2$. There are at most $\binom{|V|}{2}$ possible improvements at each of the $|V|/2$ steps, and checking whether an improvement is allowed is polynomial.

- If the algorithm stops at (a), then we apply Lemma 37. Let us see why it provides the required partition and graph in polynomial time. By Corollary 28.2, $\{T_w, w \in V_+(m_G)\}$ is a partition of $V$ into maximal tight sets and hence can be found by applying at most $|V|$ times the maximal tight sets subroutine. The set $X_e$ for each $e \in E(G)$ can be found in polynomial time because it is defined as the intersection of two perilous sets. These last sets can be found in polynomial time because we can decide if the degree specification after a flipping is admissible or not in polynomial time. Finally, having $U$ in hand, we get $U^*$ in polynomial time.

- Otherwise, the algorithm stops at (b) and the final step is clearly polynomial.

Step 3 consists in finding a $C_4^*$-obstacle, which is polynomial by results about obstacles of Section 7.2.4.

Step 4 is polynomial, by the same reasons Step 2 is.

Step 5 finds in polynomial time, by the proof of Lemma 57.2, either a $C_4^*$-obstacle for $(p, P, m)$ or two edges $e$ and $f$ of $G$ such that in $G_{e,f}$ there exists a complete $(p_{G_{e,f}}, P)$-allowed splitting off. Since $m_{G_{e,f}}(V) = 8$ this complete splitting off can be found in polynomial time.

Step 6 first finds a $C_5^*$-obstruction for $(p, m)$ (that exists by Lemma 58.2) in polynomial time by subroutine obstructions. Then in (a) it is checked in polynomial time whether it is a $C_5^*$-obstacle. If not, then we apply in (b) the proof of Lemma 49. This can be done in polynomial time, the first part by the results of Section 7.2.4, and the second one because the proof of Lemma 49 is inductive and needs, at each of the $t + 2 \leq |V| + 2$ steps, to find one allowed splitting off and at most one allowed flip.

Step 7 first finds a consecutive improvement, unsplits the edge $e$ involved, and then finds a $C_6^*$-obstacle after. This can be done in polynomial time by the results of Sections 7.2.2
and 7.2.4. Then, Step 7 (b) unsplit an arbitrary edge $e'$ of $G^e$ and finds a complete allowed splitting. Since $m_{G^e,e'}(V) = 8$, this can be done in polynomial time.

### 7.3.3 Augmentation algorithm

We sketch why each step of the augmentation algorithm can be done in polynomial time, provided the function $p$ is given with a minimization oracle.

Steps 1-4 can be done in polynomial time by the arguments of Sections 7.3.1 and 7.3.2.

Step 5 follows the proof of Lemma 70, in which the main algorithmic ingredients are checking color conditions, and finding one or two suitable tight sets. By the remarks of Sections 7.2.3 and 7.2.4, this can be done in polynomial time. We mention that in the proof of Lemma 70 we suppose that the extension $m$ minimizes $m(R) + m(B)$. This assumption is made in order to have a shorter proof. We note that this assumption is not essential. When this assumption is not satisfied then we will change the extension at most twice and hence the proof can be modified so that Step 5 become polynomial.

Step 6 is similar to Step 2. Note that in this case the algorithm will stop either in Step 3 or in Step 4.

Step 7 first increases $m(V)$ by two so that a complete allowed splitting off exists, and then applies the splitting off algorithm, which is polynomial.

### 8 Conclusion

In this paper we proposed an abstract form for the problem of partition constrained global edge-connectivity augmentation of a hypergraph. We provided a minimax theorem for this problem and we sketched a polynomial algorithm to find an optimal solution, when the function is given with a minimization oracle. This theorem implies the main theorems of [4] and [6], and consequently the results in [3], [2] and [10]. Our abstract form also provides a new application given in Section 6.

### References


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1 In fact, the proof of the corresponding lemma gives a more precise way to find an edge to be unsplit.


