Hypergraph Convexities

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Abstract Convexity Theory: Axioms

Let $V$ be a finite nonempty set and $L$ a system of subsets of $V$. The pair $(V, L)$ is a (finite) **convexity space** if:

(K1) $L$ contains $\emptyset$ and $V$, and

(K2) $L$ is closed under intersection.

The members of $L$ are called **convex sets** and, given a subset $X$ of $V$, ‘the’ smallest member of $L$ containing $X$ is called the **convex hull** of $X$.

**Remark** When $(V, L)$ is a connected space (in topological sense)

(K3) Every convex set is connected.

Let $V$ be a finite nonempty set and $\sigma$ an operator from $\mathcal{P}(V)$ to $\mathcal{P}(V)$. The pair $(V, \sigma)$ is a **closure space** if $\sigma$ is a **closure operator** in that it enjoys the following three properties:

(C1) $X \subseteq \sigma(X)$,

(C2) if $X \subseteq Y$ then $\sigma(X) \subseteq \sigma(Y)$, and

(C3) $\sigma(\sigma(X)) = \sigma(X)$.

— Given a convexity space $(V, L)$ the operator that maps subsets of $V$ to their convex hulls is a closure operator.

— Given a closure space $(V, \sigma)$ such that $\sigma(\emptyset) = \emptyset$, the pair $(V, L)$, where $L = \{X \in \mathcal{P}(V): \sigma(X) = X\}$, is a convexity space.
**Abstract Convexity Theory: Convex Geometries**

Let \((V, L)\) be a (finite) convexity space. An element \(v\) of a convex set \(X\) is an *extreme point* of \(X\) if \(X \setminus \{v\}\) is a convex set.

A convexity space \((V, L)\) is a *convex geometry* (or defines an “antimatroid”) if it satisfies the following condition:

(Minkowski-Krein-Milman property) *Every convex set is the hull of the set of its extreme points.*

\[\text{the triangle is the convex hull of the set of its vertices}\]
\[\text{the circle is the convex hull of its circumference}\]
Convexities in graphs

Let $\pi$ be a path type (e.g., shortest paths, chordless paths, paths).

A vertex set $X$ is $\pi$-convex if $X$ contains all vertices on every $\pi$-path joining two vertices in $X$.

<table>
<thead>
<tr>
<th>$\pi$</th>
<th>convexity</th>
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<tbody>
<tr>
<td>shortest path</td>
<td>geodetic ($g$-convexity)</td>
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<tr>
<td>chordless path</td>
<td>monophonic ($m$-convexity)</td>
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(a, c) is chord of the path (a, b, c, d)
\textbf{\textit{m-convexity vs. g-convexity}} (1)

\begin{itemize}
  \item \textit{m-convexity is finer than g-convexity}
    
    (Every \textit{m-convex} set is \textit{g-convex} but the converse need not hold)
  \item \textit{g-convexity = m-convexity in distance-hereditary graphs} (*)
  \item \textit{m-convexity is a convexity geometry in chordal graphs} (**) 
    \textit{g-convexity is a convexity geometry in Ptolemaic graphs} (***)
\end{itemize}

(*) A graph is \textit{distance-hereditary} if every two vertices have the same distance in every connected induced subgraph containing both.

(**) A graph is \textit{chordal} (or “triangulated”) if every cycle of length $\geq 4$ has a chord.

(***) A graph is \textit{Ptolemaic} if it is chordal and distance-hereditary (or is chordal and contains no induced “3-fan”).
chordal graphs
Ptolemaic graphs
distance-hereditary graphs

$g$-convexity = $m$-convexity

$g$-convexity = convex geometry

$g$-convexity = convex geometry
Hypergraphs

A hypergraph is a (possibly empty) set $H$ of nonempty sets; the members of $H$ are called the (hyper)edges of $H$ and their union is called the vertex set of $H$ denoted by $V(H)$. A hypergraph is a (simple) graph if its edges have all cardinality 2.

Two vertices are adjacent in $H$ if they belong together to some edge of $H$. A clique of $H$ is a nonempty set of pairwise adjacent vertices of $H$. A partial edge of $H$ is a nonempty subset of some edge of $H$. (Of course, every partial edge is a clique of $H$.)

A hypergraph $H$ is conformal if every clique of $H$ is a partial edge of $H$.

The 2-section of $H$ is the graph $H[2]$ on $V(H)$ where two vertices are adjacent iff they are so in $H$.

A hypergraph $H$ is chordal if $H[2]$ is chordal.

A hypergraph $H$ is $\alpha$-acyclic (or acyclic or decomposable) if $H$ is conformal and chordal.
Degrees of acyclicity

- conformal hypergraphs
- $\alpha$-acyclic hypergraphs
- $\beta$-acyclic hypergraphs
- $\gamma$-acyclic hypergraphs
- quasi-acyclic hypergraphs
- chordal hypergraphs
Examples

An $\alpha$-acyclic hypergraph

Two quasi-acyclic hypergraphs
Convexities in hypergraphs: preliminaries

Let $X$ be a vertex set in $H$. By $≡_X$ we denote the equivalence relation between edges of $H$ defined as follows: $A ≡_X B$ if

- $A = B$ or
- $(A \cap B) \setminus X \neq \emptyset$ or
- there exists $C$ such that $(A \cap C) \setminus X \neq \emptyset$ and $(C \cap B) \setminus X \neq \emptyset$.

The classes of the resultant partition of $H$ will be referred to as the $X$-components of $H$. (Note that every $V(H)$-component of $H$ is a trivial hypergraph and the $\emptyset$-components of $H$ are the components of $H$.)

(hyper)graph

its $X$-components for $X = \{a, b\}$
**m-convexity and c-convexity**

A vertex set $X$ is **m-convex** if, for each $X$-component $C$ of $H$, the set $X \cap V(C)$ is a clique of $H$.

**Remark.** A vertex set is $m$-convex in $H$ if and only if it is $m$-convex in $H[2]$.

A vertex set $X$ is **c-convex** if, for each $X$-component $C$ of $H$, the set $X \cap V(C)$ is a partial edge of $H$. 
An example

With $X = \{a, b, c\}$, there are four $X$-components of $H$

Since each $X \cap V(C_i)$ is a clique of $H$, $X$ is $m$-convex. Since $X \cap V(C_1)$ is not a partial edge of $H$, $X$ is not $c$-convex.
$m$-convexity vs. $c$-convexity

$c$-convexity is finer than $m$-convexity

(Every $c$-convex set is $m$-convex but the converse need not hold)

$m$-convexity = $c$-convexity in conformal hypergraphs

$m$-convexity on $H$ is a convexity geometry iff $H$ is chordal

c-convexity on $H$ is a convexity geometry iff $H$ is acyclic or quasi-acyclic
chordal hypergraphs

acyclic and quasi-acyclic hypergraphs

conformal hypergraphs
**simple-path convexity**

A *path* in $H$ is a sequence $\langle v_0, A_1, v_1, A_2, \ldots, v_{k-1}, A_k, v_k \rangle$, $k \geq 1$, where the $v_i$'s are pairwise distinct vertices of $H$, the $A_i$'s are pairwise distinct edges of $H$ and every vertex pair $\{v_{i-1}, v_i\}$ is a subset of $A_i$ for $1 \leq i \leq k$.

A path $\langle v_0, A_1, v_1, A_2, \ldots, v_{k-1}, A_k, v_k \rangle$ is *simple* if

$$A_i \cap \{v_0, v_1, \ldots, v_{k-1}, v_k\} = \{v_{i-1}, v_i\} \quad (1 \leq i \leq k).$$

A vertex set $X$ is *sp-convex* if $X$ contains all vertices on every simple path joining two vertices in $X$.

**Remark.** In a graph a vertex set is *sp-convex* if and only if it is the vertex set of a nonseparable component (or biconnected component or block) of the graph.
**sp-convexity vs. c-convexity**

**sp-convexity is finer than c-convexity**

(Every sp-convex set is c-convex but the converse need not hold)

**sp-convexity = c-convexity in \( \gamma \)-acyclic hypergraphs**

**sp-convexity on \( H \) is a convexity geometry iff \( H \) is \( \beta \)-acyclic**

**c-convexity on \( H \) is a convexity geometry iff \( H \) is \( \alpha \)-acyclic or quasi-acyclic**
acyclic or quasi-acyclic hypergraphs

\( \beta \)-acyclic hypergraphs

\( \gamma \)-acyclic hypergraphs

\[ sp\text{-convexity} = c\text{-convexity} \]

\[ c\text{-convexity} = \text{convex geometry} \]
Graham reduction for computing hulls

Let $K$ be an acyclic hypergraph, and $X$ a subset of $V(K)$. By $GR(K, X)$ we denote the resultant hypergraph of the following (linear-time) algorithm.

<table>
<thead>
<tr>
<th>Graham reduction</th>
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<tbody>
<tr>
<td>Repeatedly apply the following two operations until neither can be longer applied:</td>
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<tr>
<td>(Vertex Deletion) If $a$ does not belong to $X$ and is in exactly one edge, then delete $a$.</td>
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<tr>
<td>(Edge Deletion) If $A$ is a redundant edge, then delete $A$.</td>
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</tbody>
</table>
Computing $c$-hulls and $m$-hulls

Given hypergraph $H$, let $K$ be the (acyclic) hypergraph with edges the maximal sets of vertices of $H$ that are not separable by any partial edges of $H$. The output ($Y$) of the following algorithm gives the $c$-hull of any subset $X$ of $V(H)$.

<table>
<thead>
<tr>
<th>canonical closure</th>
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<tr>
<td><strong>Step 1.</strong> Compute $GR(K, X)$ and set $Y$ to the vertex set of $GR(K, X)$.</td>
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<tr>
<td><strong>Step 2.</strong> For every edge $A$ of $GR(K$, if $A$ is neither an edge of $K$ nor a partial edge of $H$, then set $Y := Y \cup B$ where $B$ is the edge of $K$ containing $A$.</td>
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Given hypergraph $H$, let $K$ be the (acyclic) hypergraph with edges the maximal sets of vertices of $H$ that are not separable by any cliques of $H$. The output ($Y$) of the following algorithm gives the $m$-hull of any subset $X$ of $V(H)$.

<table>
<thead>
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<td><strong>Step 1.</strong> Compute $GR(K, X)$ and set $Y$ to the vertex set of $GR(K, X)$.</td>
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<td><strong>Step 2.</strong> For every edge $A$ of $GR(K$, if $A$ is neither an edge of $K$ nor a clique of $H$, then set $Y := Y \cup B$ where $B$ is the edge of $K$ containing $A$.</td>
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References

CONVEXITY THEORY


GRAPH CONVEXITY


HYPERGRAPHS


COMPUTATIONAL ASPECTS


Database Theory


Computational Statistics