BRAID GROUPS ARE LINEAR

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ABSTRACT. The braid groups B_n can be defined as the mapping class group of the n-punctured disc. The Lawrence-Krammer representation of the braid group is the induced action on a certain twisted second homology of the space C of unordered pairs of points in the n-punctured disc. Recently, Daan Krammer showed that this is a faithful representation in the case $n = 4$. In this paper, we show that it is faithful for all n.

1. INTRODUCTION

Let B_n denote Artin's braid group on n strands. Recently, Krammer [Kra] proved that a certain representation of the braid groups is faithful on B_4. The representation he used is essentially the same as one used by Lawrence in [Law90] to give a topological definition of a certain summand of the Jones representation. We call this representation the Lawrence-Krammer representation. In this paper, we prove the following.

Theorem 1.1. *The Lawrence-Krammer representation of B_n is faithful for all n.*

This proves that braid groups are linear, thus solving a long-standing open problem. Our proof can be seen as a sort of converse to the construction of elements of the kernel of the Burau representation given in [Moo91], [LP93] and [Big99]

1.1. **Definitions.**

Let D be an oriented disc in the complex plane. Fix a set $P \subset D$ consisting of n distinct points $p_1, \ldots, p_n$ in the interior of D. Let $\mathcal{H}(D, P)$ be the group of all homeomorphisms $h\colon D \to D$ such that $h(P) = P$ and h fixes ∂D pointwise. Let $\mathcal{I}(D, P)$ be the group of all such homeomorphisms which are isotopic to the identity relative to $P \cup \partial D$. We define the braid group B_n to be the group $\mathcal{H}(D, P)/\mathcal{I}(D, P)$. See [Bir74] for other equivalent definitions of these groups and a good introduction to their basic properties.

Let C denote the space of all unordered pairs of distinct points in $D \setminus P$. In other words,

$$C = \frac{((D \setminus P) \times (D \setminus P)) \setminus \{(x, x)\}}{(x, y) \sim (y, x)}.$$

Let d_1 and d_2 be distinct points in ∂D. Let $c_0 = \{d_1, d_2\}$ be a basepoint for C.

We now define a map ϕ from $\pi_1(C, c_0)$ to the free Abelian group with basis $\{q, t\}$. Let α be a closed curve in C based at c_0 representing an element $[\alpha]$ of $\pi_1(C, c_0)$. We can write α in the form

$$\alpha(s) = \{\alpha_1(s), \alpha_2(s)\}$$

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for some arcs α_1 and α_2 in $D \setminus P$. Let

$$a = \frac{1}{2\pi i} \sum_{j=1}^{n} \left(\int_{\alpha_1} \frac{dz}{z - p_j} + \int_{\alpha_2} \frac{dz}{z - p_j} \right).$$

Let

$$b = \frac{1}{\pi i} \int_{\alpha_1 - \alpha_2} \frac{dz}{z}.$$

Let $\phi([\alpha]) = q^a t^b$.

This definition requires some explanation. If α_1 and α_2 are closed loops then a is the sum of the winding numbers of α_1 and α_2 around each of the puncture points, and b is twice the winding number of α_1 and α_2 around each other. However α_1 and α_2 are not necessarily closed loops, but may "switch places". In this case, a is the sum of the winding numbers of the closed loop $\alpha_1\alpha_2$ around each of the puncture points, and $\alpha_1 - \alpha_2$ satisfies

$$(\alpha_1 - \alpha_2)(1) = -(\alpha_1 - \alpha_2)(0),$$

which implies that b is an odd integer.

Let $\tilde{C}$ be the covering space of C whose fundamental group is the kernel of ϕ. Choose a lift $\tilde{c}_0$ of c_0 to $\tilde{C}$. Let Λ denote the ring $\mathbf{Z}[q^{\pm 1}, t^{\pm 1}]$. The homology group $H_2(\tilde{C})$ can be considered as a Λ-module, where q and t act by covering transformations.

Let $\sigma \in B_n$. There is an induced action of σ on C. Let $\tilde{\sigma}$ be the lift of σ to a map from $\tilde{C}$ to itself which fixes $\tilde{c}_0$. This induces a Λ-module automorphism $\tilde{\sigma}_*$ of $H_2(\tilde{C})$. The Lawrence-Krammer representation is the map from B_n to $GL(H_2(\tilde{C}))$ taking $[\sigma]$ to $\tilde{\sigma}_*$.

1.2. **Outline.** In Section 2 we will define forks and noodles. These will be one-dimensional objects in the disc designed to represent elements of the second homology and cohomology of $\tilde{C}$. We define a pairing between forks and noodles, which will be preserved by any element of the kernel of the Lawrence-Krammer representation.

In Section 3 we prove that pairing between forks and noodles detects geometric intersection between the corresponding edges in the disc. We use this to show that a braid in the kernel of the Lawrence-Krammer representation must be trivial.

In Section 4 we compute the Lawrence-Krammer representation explicitly in terms of generators and basis elements.

1.3. **Notation.** If α and β are arcs in $D \setminus P$ such $\alpha(s) \neq \beta(s)$ for all $s \in I$ then we define $\{\alpha, \beta\}$ to be the arc in C given by

$$\{\alpha, \beta\}(s) = \{\alpha(s), \beta(s)\}.$$

If y is a point in $D \setminus P$ and α is an arc in $D \setminus (P \cup \{y\})$ then we define $\{\alpha, y\}$ to be the arc in C given by

$$\{\alpha, y\}(s) = \{\alpha(s), y\}.$$

The same arc can be denoted by $\{y, \alpha\}$.

If x and y are elements of a group then we use the notation

$$x^y = y^{-1}xy$$

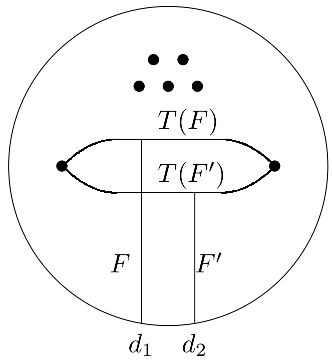

FIGURE 1. A fork F and a parallel copy F'.

and

$$[x, y] = x^{-1}y^{-1}xy.$$

Throughout this paper, I will denote the interval $[0, 1]$. Braids compose from right to left. Arcs compose from left to right.

2. Forks and Noodles

In this section we define forks and noodles and a pairing between them. The idea of using a fork to represent an element of $H_2(\tilde{C})$ is due to Krammer [Kra].

A *fork* is an embedded tree $F \subset D$ with four vertices d_1, p_i, p_j and z such that $F \cap \partial D = \{d_1\}$, $F \cap P = \{p_i, p_j\}$, and all three edges have z as a vertex. The edge containing d_1 is called the *handle* of F. The union of the other two edges is a single edge, which we call the *tine edge* of F and denote by $T(F)$.

For a given fork F we can define a *parallel copy* of F to be an embedded tree F' as shown in Figure 1. The five punctures at the top of the figure may be replaced by any number, and any orientation-preserving self-homeomorphism may be applied to the entire figure. The tine edge $T(F')$ of F' is defined analogously to that of F.

For any fork F, we define a surface $\tilde{\Sigma}(F)$ in $\tilde{C}$ as follows. Let F' be a parallel copy of F. Let z be the vertex contained in all three edges of F, and let z' be the vertex contained in all three edges of F'. We define the surface $\Sigma(F)$ to be the set of points in C which can be written in the form $\{x, y\}$, where $x \in T(F) \setminus P$ and $y \in T(F') \setminus P$. Let β_1 be an arc from d_1 to z along the handle of F and let β_2 be an arc from d_2 to z' along the handle of F'. Let $\tilde{\beta}$ be the lift of $\{\beta_1, \beta_2\}$ to $\tilde{C}$ which starts at $\tilde{c}_0$. Let $\tilde{\Sigma}(F)$ be the lift of $\Sigma(F)$ to $\tilde{C}$ which contains $\tilde{\beta}(1)$.

A *noodle* is an embedded edge $N \subset D \setminus P$ with endpoints d_1 and d_2. For a given noodle N we define the surface $\Sigma(N)$ to be the set of points $\{x, y\} \in C$ such that x and y are distinct points in N. Let $\tilde{\Sigma}(N)$ be the lift of $\Sigma(N)$ to $\tilde{C}$ which contains $\tilde{c}_0$.

Let F be a fork and let N be a noodle. We define $\langle N, F\rangle \in \Lambda$ as follows.

$$\langle N, F\rangle = \sum_{a,b\in Z} q^a t^b (q^a t^b \tilde{\Sigma}(N), \tilde{\Sigma}(F)).$$

Here $q^a t^b \tilde{\Sigma}(N)$ denotes the image of $\tilde{\Sigma}(N)$ under the covering transformation $q^a t^b$, and $(q^a t^b \tilde{\Sigma}(N), \tilde{\Sigma}(F))$ denotes the algebraic intersection number of the two surfaces in $\tilde{C}$.

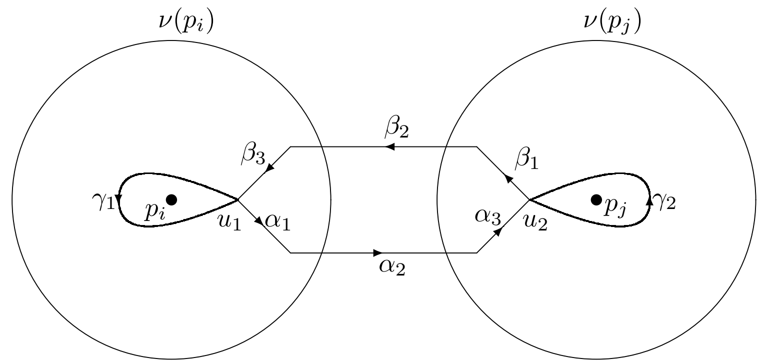

FIGURE 2. Arcs.

Lemma 2.1 (The Basic Lemma). *The pairing between forks and noodles is well-defined. Furthermore, if $[\sigma]$ lies in the kernel of the Lawrence-Krammer representation then*

$$\langle N, F\rangle = \langle N, \sigma(F)\rangle$$

for every fork F and noodle N.

2.1. **Proof of the Basic Lemma.** The problem is that one cannot necessarily define an algebraic intersection number between two properly embedded surfaces, since it might be possible to eliminate intersections by pushing them off to infinity. We overcome this problem by proving the existence of an immersed closed surface $\tilde{\Sigma}_2(F)$ which is equal to $(1-q)^2(1+qt)\tilde{\Sigma}(F)$ outside a small neighborhood of the punctures.

Let F be a fork. Let the endpoints of $T(F)$ be p_i and p_j. Let $\nu(p_i)$ and $\nu(p_j)$ be disjoint ϵ-neighborhoods of p_i and p_j respectively such that $\nu(p_k) \cap P = \{p_k\}$ for $k = i, j$. Let U be the set of $\{x, y\} \in C$ such that at least one of x and y lies in $\nu(p_i) \cup \nu(p_j)$. Fix a basepoint $u_0 = \{u_1, u_2\} \in U$, where $u_1 \in \nu(p_i)$ and $u_2 \in \nu(p_j)$. Let $\tilde{U}$ be the pre-image of U in $\tilde{C}$. Choose a lift $\tilde{u}_0$ of u_0 to $\tilde{C}$. We start by analyzing $\pi_1(U, u_0)$.

Using the arcs shown in Figure 2, we define the following elements of $\pi_1(U, u_0)$.

$$\begin{aligned}
a_1 &= \{\gamma_1, u_2\},\\
a_2 &= \{u_1, \gamma_2\},\\
b_1 &= \{\alpha_1, \beta_1\beta_2\beta_3\}\{\alpha_2\alpha_3, u_1\},\\
b_2 &= \{\alpha_1\alpha_2\alpha_3, \beta_1\}\{u_2, \beta_2\beta_3\}.
\end{aligned}$$

Note that b_1 and b_2 are homotopic in C, but not in U.

The following relations hold in $\pi_1(U)$.

$$[a_1, a_2] = 1, \tag{1}$$

$$[a_1, b_1a_1b_1] = 1, \tag{2}$$

$$[a_2, b_2a_2b_2] = 1. \tag{3}$$

The first of these is obvious. The second follows from the fact that $b_1a_1b_1$ is equal in $\pi_1(U)$ to $\{u_1, \delta\}$, where δ is a curve based at u_2 which passes counterclockwise around p_1 and u_1. The third follows by a similar argument.

We now analyze $\pi_1(\tilde{U}, \tilde{u}_0)$. Let $i\colon U \to C$ be the inclusion map and let i_* be the induced map on fundamental groups. Then $\pi_1(\tilde{U})$ is the kernel of the map ϕi_*. We define the following elements of $\pi_1(\tilde{U}, \tilde{u}_0)$.

$$\begin{aligned} a &= a_2^{-1} a_1, \\ b &= b_2^{-1} b_1, \\ c &= a_1^{-1} b_1^{-1} a_1 b_1, \\ d &= a_2^{-1} b_2^{-1} a_2 b_2. \end{aligned}$$

If $x \in \pi_1(\tilde{U}, \tilde{u}_0)$ and $y \in \pi_1(U, u_0)$ then the conjugate $x^y = y^{-1}xy$ is also an element of $\pi_1(\tilde{U}, \tilde{u}_0)$. The following relations hold in $\pi_1(\tilde{U}, \tilde{u}_0)$.

$$a^{a_1} = a, \tag{4}$$

$$c^{b_1 a_1} c = 1, \tag{5}$$

$$d^{b_2 a_2} d = 1, \tag{6}$$

$$d b a^{b_1} = a b^{a_1} c. \tag{7}$$

To see this, rewrite these relations in terms of a_1, a_2, b_1 and b_2. The first three translate into equations (1) to (3). The fourth translates into a trivial identity.

If $x \in \pi_1(\tilde{U})$, let $[x]$ denote the corresponding element of $H_1(\tilde{U})$. Note that if $x \in \pi_1(\tilde{U})$ and $y \in \pi_1(U)$ then $[x^y] = \phi(y)^{-1}[x]$. The relations given in equations (4) to (7) give rise to the following relations in $H_1(\tilde{U})$.

$$\begin{aligned} (q^{-1} - 1)[a] &= 0, \\ (q^{-1}t^{-1} + 1)[c] &= 0, \\ (q^{-1}t^{-1} + 1)[d] &= 0, \\ (q^{-1} - 1)[b] &= (q^{-1} - 1)[a] - [c] + [d]. \end{aligned}$$

Combining these relations, we obtain the following.

$$(1 - q)^2(1 + qt)[b] = 0.$$

Let $[\tilde{\Sigma}(F)]$ be the element of $H_2(\tilde{C}, \tilde{U})$ represented by $\tilde{\Sigma}(F)$. The long exact sequence of relative homology gives us the following exact sequence of Λ-modules

$$H_2(\tilde{C}) \xrightarrow{j_*} H_2(\tilde{C}, \tilde{U}) \xrightarrow{\partial} H_1(\tilde{U}).$$

But $\partial[\tilde{\Sigma}(F)] = [b]$. It follows that

$$(1 - q)^2(1 + qt)[\tilde{\Sigma}(F)] = j_*[\tilde{\Sigma}_2(F)]$$

for some $[\tilde{\Sigma}_2(F)] \in H_2(\tilde{C})$ represented by some closed immersed surface $\tilde{\Sigma}_2(F)$.

Let N be a noodle. Choose $\nu(p_i)$ and $\nu(p_j)$ small enough so as not to intersect N. Then

$$(1 - q)^2(1 + qt)\langle N, F\rangle = \sum_{a,b \in \mathbf{Z}} q^a t^b (q^a t^b \tilde{\Sigma}(N), \tilde{\Sigma}_2(F)).$$

Now $(q^a t^b \tilde{\Sigma}(N), \tilde{\Sigma}_2(F))$ is the algebraic intersection number between a properly embedded surface and an immersed closed surface, so is well-defined.

Suppose σ is an element of the kernel of the Lawrence-Krammer representation. Then $\sigma(\tilde{\Sigma}_2(F))$ and $\tilde{\Sigma}_2(F)$ represent the same element of homology, so have the same algebraic intersection with any $q^a t^b \tilde{\Sigma}(N)$. Thus $\langle N, \sigma(F)\rangle = \langle N, F\rangle$.

2.2. **Alternative proofs.** There are many possible approaches to proving the Basic Lemma. The proof given above is a compromise of sorts, since it proves the existence of an appropriate element of $H_2(\tilde{C})$, but does so in a non-constructive way.

It is possible to give a more constructive proof which uses an explicit computation of $H_2(\tilde{C})$. One obtains a concrete description of an immersed genus two surface which can be seen to be the same as $(1-q)^2(1+qt)\tilde{\Sigma}(F)$ away from the puncture points. This is perhaps best done in private, since the details are only convincing to the person who figures them out. Some details of a computation of $H_2(\tilde{C})$ will be given in Section 4. See also [Law90], where similar methods are used to calculate the middle homology of the space of *ordered* k-tuples of distinct points in the n-times punctured disc, where k can be any positive integer.

It is tempting to seek a less constructive proof which makes no reference to $\tilde{\Sigma}_2(F)$. It is intuitively obvious that the problem of pushing intersections off to infinity does not arise in the context of forks and noodles. However this line of reasoning runs into some technical difficulties which I feel distract from the true nature of the problem at hand. A proof that B_n acts faithfully on $H_2(\tilde{C})$ should refer to an element of $H_2(\tilde{C})$.

It is possible to prove that braid groups are linear without reference to C, let alone $H_2(\tilde{C})$. The Lawrence-Krammer representation can be defined to be the action of B_n on a Λ-module consisting of formal linear combinations of forks subject to certain relations, as described by Krammer in [Kra]. The pairing $\langle N, F\rangle$ can be defined solely in terms of winding numbers. One must check that this pairing respects the relations between forks. The Basic Lemma then follows immediately. The rest of the proof that the Lawrence-Krammer representation is faithful proceeds virtually unchanged.

3. The representation is faithful

In this section, we prove that the Lawrence-Krammer representation is faithful. The main ingredient in the proof is the following lemma.

Lemma 3.1 (The Key Lemma). *Let N be a noodle and let F be a fork. Then $\langle N, F\rangle = 0$ if and only if $T(F)$ is isotopic relative to $\partial D \cup P$ to an arc which is disjoint from N.*

3.1. **Computing the pairing.** We now describe how to compute the pairing between a give noodle and fork. Let N be a noodle and let F be a fork. By applying a preliminary isotopy, we can assume that $T(F)$ intersects N transversely at a finite number of points, which we label $z_1, \dots, z_l$. Let F' be a parallel copy of F. Choose F' so that $T(F')$ intersects N transversely at $z'_1, \dots, z'_l$, where z_i and z'_i are joined by a short arc in N which meets no other z_j or z'_j. Let z be the vertex which lies in all three edges of F, and let z' be the vertex which lies in all three edges of F'.

For every pair $\{z_i, z'_j\}$ there exist unique integers $a_{i,j}$ and $b_{i,j}$ such that $\tilde{\Sigma}(F)$ intersects $q^{a_{i,j}}t^{b_{i,j}}\tilde{\Sigma}(N)$ at a point in the fiber of $\{z_i, z'_j\}$. Let $\epsilon_{i,j}$ denote the sign of that intersection. Let $m_{i,j} = q^{a_{i,j}}t^{b_{i,j}}$. Then

$$\langle N, F\rangle = \sum_{i=1}^{l}\sum_{j=1}^{l} \epsilon_{i,j} m_{i,j}. \tag{8}$$

We can compute $m_{i,j}$ as follows. Define the following embedded arcs in $D \setminus P$.

- α_1 from d_1 to z along the handle of F,
- α_2 from d_2 to z' along the handle of F',
- β_1 from z to z_i along $T(F)$,
- β_2 from z' to z'_j along $T(F')$,
- γ_1 from z_i to d_k along N, where $k = 1, 2$ is such that γ_1 does not pass through z'_j,
- γ_2 from z'_j to $d_{k'}$ along N, where $k' = 1, 2$ is such that γ_2 does not pass through z_i.

Let

$$\delta_{i,j} = \{\alpha_1, \alpha_2\}\{\beta_1, \beta_2\}\{\gamma_1, \gamma_2\}.$$

Then

$$m_{i,j} = \phi(\delta_{i,j}).$$

We can calculate the exponent $a_{i,j}$ even more explicitly. For $k = 1, \ldots, l$, let ζ_k be the arc from d_1 to z_k along F, then back to d_1 along N. Let a_k be the sum of the winding numbers of ζ_k around each of the puncture points. Let ζ be the arc from d_1 to d_2 along N, and then back to d_1 moving clockwise along ∂D. Let a be the sum of the winding numbers of ζ around the puncture points.

Claim 3.2. *$a_{i,j} = a_i + a_j + a$.*

Proof. Let ζ'_k be the arc from d_2 to z'_k along F', then back to d_2 along N. Let a'_k be the sum of the winding numbers of ζ'_k around each of the puncture points. If z_i is closer to d_1 than z'_j along N then $a_{i,j} = a_i + a'_j$. If not, then $a_{i,j} = a'_i + a_j$. But $a'_k = a_k + a$ for all $k = 1, \ldots, l$, so in either case $a_{i,j} = a_i + a_j + a$. □

In order to compute $\epsilon_{i,j}$ we need to choose orientations for $\Sigma(F)$, $\Sigma(N)$ and C. We choose the orientation on C induced by the product orientation on $D \times D$. We orient $\Sigma(F)$ and $\Sigma(N)$ as follows.

Let $f\colon T(F) \to I$ and $f'\colon T(F') \to I$ be diffeomorphisms which give $T(F)$ and $T(F')$ parallel orientations. Let f_1 and f_2 be the maps from $\Sigma(F)$ to I such that if $x \in T(F) \setminus P$ and $y \in T(F') \setminus P$ then $f_1(\{x, y\}) = f(x)$ and $f_2(\{x, y\}) = f'(y)$. Then the 2-form $df_1 \wedge df_2$ defines an orientation on $\Sigma(F)$.

Let $g\colon N \to I$ be a diffeomorphism such that $g(d_1) = 0$ and $g(d_2) = 1$. Let g_1 and g_2 be the maps from $\Sigma(N)$ to I given by $g_1(\{x, y\}) = \min(g(x), g(y))$ and $g_2(\{x, y\}) = \max(g(x), g(y))$. Then the 2-form $dg_1 \wedge dg_2$ defines an orientation on $\Sigma(N)$.

Claim 3.3. *With these orientations, $\epsilon_{i,j} = -m_{i,i} m_{i,j} m_{j,j}|(q = 1, t = -1)$.*

Proof. By definition, $\epsilon_{i,j}$ is the sign of the volume form $dg_1 \wedge dg_2 \wedge df_1 \wedge df_2$ at the point $\{z_i, z'_j\}$ in C. This is determined by the following three things:

- the sign of the intersection of N with $T(F)$ at z_i,
- the sign of the intersection of N with $T(F')$ at z'_j,
- which of z_i and z'_j is closer to d_1 along N.

These in turn are determined by the following three values respectively:

- $m_{i,i}|(q = 1, t = -1)$,
- $m_{j,j}|(q = 1, t = -1)$,
- $m_{i,j}|(q = 1, t = -1)$.

To see this, observe that in general the sign of $m_{i',j'}|(q=1, t=-1)$ is determined by whether the two points switch places in the path $\delta_{i',j'}$. This in turn is determined by which of $z_{i'}$ and $z'_{j'}$ is closer to d_1 along N.

It follows from the above considerations that $\epsilon_{i,j}$ is determined by the value of the product $m_{i,i}m_{i,j}m_{j,j}$ evaluated at $q=1$ and $t=-1$. It remains to check that the sign is as claimed. This can be done by calculating $\epsilon_{i,j}$ in a specific example. $\square$

3.2. **Proof of the Key Lemma.** Let N be a noodle and let F be a fork. By applying a preliminary isotopy, we can assume that $T(F)$ intersects N transversely at a finite number of points, which we label $z_1, \ldots, z_l$. Further, we can assume that l is the minimal possible number of points of intersection. Let F' be a parallel copy of F. Choose F' so that and $T(F')$ intersects N transversely at $z'_1, \ldots, z'_l$, where z_i and z'_i are joined by a short arc in N which meets no other z_j or z'_j.

If $l = 0$ then clearly $\langle N, F\rangle = 0$. We now assume $l > 0$ and show that $\langle N, F\rangle \neq 0$. We use the following lexicographic ordering on the set of monomials $q^a t^b$.

Definition 3.4. We say $q^a t^b \leq q^{a'} t^{b'}$ if and only if either
- $a < a'$, or
- $a = a'$ and $b \leq b'$.

For $i, j \in \{1, \ldots, l\}$ we say that $m_{i,j}$ is *maximal* if $m_{i,j} \geq m_{i',j'}$ for all $i', j' \in \{1, \ldots, l\}$.

Claim 3.5. *If $m_{i,j}$ is maximal then $m_{i,i} = m_{j,j} = m_{i,j}$.*

Proof. Suppose $m_{i,j}$ is maximal. Then $a_{i,j}$ is maximal among all the integers $a_{k,l}$. By Claim 3.2 it follows that a_i and a_j are maximal among all the integers a_k. Thus $a_{i,i} = a_{j,j} = a_{i,j}$.

We now show that $b_{i,i} = b_{i,j}$. Suppose not. Then $b_{i,i} < b_{i,j}$. Let α be an embedded arc from z'_i to z'_j along $T(F')$. Let β be an embedded arc from z'_j to z'_i along N.

If β does not pass through the point z_i, let $\delta = \alpha\beta$ and let w be the winding number of δ around z_i. Then $b_{i,j} - b_{i,i} = 2w$.

If β does pass through z_i, first modify β in a small neighborhood of z_i so that z_i lies to its left. Next let $\delta = \alpha\beta$ and let w be the winding number of δ around z_i. Then $b_{i,j} - b_{i,i} = 2w - 1$.

In either case, w is greater than zero.

Let $D_1 = D \setminus \{z_i\}$. Let $\tilde{D}_1$ be the universal cover of D_1. Let $\tilde{\alpha}$ be a lift of α to $\tilde{D}_1$. Let $\tilde{\beta}$ be the lift of β to $\tilde{D}_1$ which starts at $\tilde{\alpha}(1)$. Let $\tilde{\gamma}$ be an embedded arc in $\tilde{D}_1$ from $\tilde{\beta}(1)$ to $\tilde{\alpha}(0)$ which intersects $\tilde{\alpha}$ and $\tilde{\beta}$ only at its endpoints. Let γ be the projection of $\tilde{\gamma}$ to D_1. Choose $\tilde{\gamma}$ so that γ does not wind around any puncture points.

Let $\tilde{z}'_k$ be the first point on $\tilde{\alpha}$ which intersects $\tilde{\beta}$ (possibly $\tilde{\alpha}(1)$). This lies in the fiber over z'_k for some $k = 1, \ldots, l$. Let $\tilde{\alpha}'$ be the initial segment of $\tilde{\alpha}$ ending at $\tilde{z}'_k$. Let $\tilde{\beta}'$ be the final segment of $\tilde{\beta}$ starting at $\tilde{z}'_k$. Let $\tilde{\delta}' = \tilde{\alpha}'\tilde{\beta}'\tilde{\gamma}$.

Now $\tilde{\delta}'$ is a simple closed curve in $\tilde{D}_1$, so by the Jordan curve theorem it must bound a disc $\tilde{B}$. Since γ passes clockwise around z_i, there is a non-compact region to the right of $\tilde{\delta}'$, so $\tilde{\delta}'$ must pass counterclockwise around $\tilde{B}$.

Let δ' be the projection of $\tilde{\delta}'$ to D. Then $a_k - a_i$ is equal to the sum of the winding numbers of δ' around each of the points in P. This is equal to the number of points in $\tilde{B}$ which are lifts of a point on P. This must be greater than zero, since

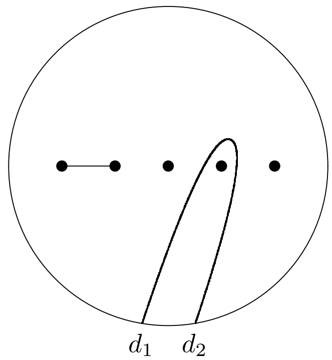

FIGURE 3. The edge E_1 and the noodle N_4

otherwise we could isotope $T(F')$ so as to have fewer points of intersection with N. Thus $a_k - a_i$ is greater than zero, contradicting the fact that a_i is maximal among all integers $a_{i'}$. Therefore our assumption that $b_{i,j} > b_{i,i}$ must have been false, so $b_{i,j} = b_{i,i}$.

The proof that $b_{i,j} = b_{j,j}$ is similar. This completes the proof of the claim. □

This claim, together with the formula for $\epsilon_{i,j}$ given in Claim 3.3, implies that if $m_{i,j}$ is maximal then $\epsilon_{i,j} = -m_{i,j}|(q = 1, t = -1)$. Thus all maximal monomials occur with the same sign in equation (8). Therefore $\langle N, F\rangle$ cannot equal zero. This completes the proof of the Key Lemma.

3.3. **Proof of the Theorem.** In this subsection we use the Basic Lemma and the Key Lemma to prove that the Lawrence-Krammer representation is faithful. The following lemma will be useful.

Lemma 3.6. *Let α and β be simple closed curves in $D \setminus P$ which intersect transversely at finitely many points. The following are equivalent.*

- *α is isotopic relative to $\partial D \cup P$ to a simple closed curve which intersects β at fewer points,*
- *α and β cobound a "digon", that is, an embedded disc in $D \setminus P$ whose boundary consists of one subarc of α and one subarc of β.*

A proof can be found in [PR99, Proposition 3.7], or [FLP91, Proposition 3.10].

Suppose $\sigma \in \mathcal{H}(D, P)$ is a homomorphism representing an element of the kernel of the Lawrence-Krammer representation. We will show that σ is isotopic relative to $\partial D \cup P$ to the identity map.

Take D to be the unit disc centered at the origin in the complex plane, and take $p_1, \dots, p_n$ to be real numbers satisfying $-1 < p_1 < \cdots < p_n < 1$. For $i = 1, \dots, n-1$, let E_i be the horizontal edge from p_i to p_{i+1}. For $i = 1, \dots, n$, let N_i be the noodle which winds around p_i and no other punctures, intersecting the real axis twice. See Figure 3.

Let F be a fork such that $T(F) = E_1$. Then $\langle N_3, F\rangle = 0$. By the Basic Lemma, $\langle N_3, \sigma(F)\rangle = 0$. By the Key Lemma, it follows that $\sigma(E_1)$ is isotopic relative to $\partial D \cup P$ to an arc which is disjoint from N_3. By composing σ with an element of $\mathcal{I}(D, P)$ if necessary, we can assume that $\sigma(E_1)$ is disjoint from N_3.

Similarly, $\sigma(E_1)$ can be isotoped so as to be disjoint from N_4. By Lemma 3.6, this isotopy can be performed by a sequence of moves which consist of eliminating digons, and hence do not introduce any new intersections with N_3. Thus we can assume that $\sigma(E_1)$ is disjoint from both N_3 and N_4.

Continuing in this way, we can assume that $\sigma(E_1)$ is disjoint from N_i for all $i = 3, \dots, n$. By applying one final isotopy relative to $\partial D \cup P$, we can assume that $\sigma(E_1) = E_1$. Note that we have not yet eliminated the possibility that σ reverses the orientation of E_1.

We can repeat the above procedure to isotope $\sigma(E_2)$ to E_2 while leaving E_1 fixed. Continuing in this way, we can assume that $\sigma(E_i) = E_i$ for all $i = 1, \dots, n-1$. It follows that σ must be isotopic relative to $\partial D \cup P$ to $(\Delta^2)^k$ for some $k \in \mathbf{Z}$, where Δ^2 is a Dehn twist about the boundary of a collar neighborhood of ∂D. However, the induced action of Δ^2 on $H_2(\tilde{C})$ is simply multiplication by $q^{2n}t^2$. Since σ acts trivially on $H_1(\tilde{C})$ we have that $k = 0$, so σ represents the trivial braid.

4. Matrices for the Lawrence-Krammer representation

In this section we give an explicit description of the Lawrence-Krammer representation in terms of matrices.

Theorem 4.1. *$H_2(\tilde{C})$ is a free Λ-module of rank $\binom{n}{2}$. There is a basis*

$$\{v_{j,k} : 1 \leq j < k \leq n\}$$

on which the braid σ_i acts as follows.

$$\sigma_i(v_{j,k}) = \begin{cases} v_{j,k} & i \notin \{j-1, j, k-1, k\}, \\ qv_{i,k} + (q^2 - q)v_{i,j} + (1-q)v_{j,k} & i = j-1, \\ v_{j+1,k} & i = j \neq k-1, \\ qv_{j,i} + (1-q)v_{j,k} + (q^2-q)tv_{i,k} & i = k-1 \neq j, \\ v_{j,k+1} & i = k \\ -tq^2 v_{j,k} & i = j = k-1. \end{cases}$$

We prove this theorem by constructing a two-complex which is homotopy equivalent to C. Our methods require some geometric intuition (read: "hand-waving"), and some details are left to the reader.

For $j = 1, \dots, n$, let ξ_j be a path in D based at d_1 and passing counterclockwise around p_j, and let x_j be the arc $\{\xi_j, d_2\}$ in C. Let τ_1 be an arc from d_1 to d_2 and τ_2 an arc from d_2 to d_1 such that $\tau_1\tau_2$ is a simple closed curve enclosing no puncture points. Let y be the arc $\{\tau_1, \tau_2\}$ in C. Let $\mathcal{G} = \{x_1, \dots, x_n, y\}$.

For $1 \leq j \leq n$, let

$$r_{j,j} = [x_j, yx_jy].$$

For $1 \leq j < k \leq n$, let

$$r_{j,k} = [x_j, yx_ky^{-1}].$$

Let $\mathcal{R} = \{r_{j,k} : 1 \leq j \leq k \leq n\}$. Let K be the Cayley complex of the presentation $\langle \mathcal{G} | \mathcal{R} \rangle$. In other words, K has one vertex, one edge for each $g \in \mathcal{G}$, and one face f_r for each $r \in \mathcal{R}$, where f_r is attached to the 1-skeleton according to the word r. We will show that C is homotopy equivalent to K.

Let $\bar{C}$ be the set of *ordered* pairs of distinct points in $D \setminus P$. This is the double cover of C whose fundamental group is normally generated by $x_1, \dots, x_n$ and y^2.

Let $X_j = yx_jy^{-1}$. Let $Y = y^2$. Let $\bar{\mathcal{G}} = \{x_1, \dots, x_n, X_1, \dots, X_n, Y\}$. For $1 \leq j \leq n$, let

$$\begin{aligned} \bar{r}_{j,j} &= [x_j, X_jY], \\ \bar{r}'_{j,j} &= [X_j, Yx_j]. \end{aligned}$$

For $1 \leq j < k \leq n$, let

$$\begin{aligned} \bar{r}_{j,k} &= [x_j, X_k], \\ \bar{r}'_{j,k} &= [X_j, Y x_k Y^{-1}]. \end{aligned}$$

Let

$$\bar{\mathcal{R}} = \{\bar{r}_{j,k} : 1 \leq j \leq k \leq n\} \cup \{\bar{r}'_{j,k} : 1 \leq j \leq k \leq n\}.$$

Let $\bar{K}$ be the Cayley complex of $\langle \bar{\mathcal{G}} | \bar{\mathcal{R}} \rangle$. Then $\bar{K}$ is homotopy equivalent to the double cover of K whose fundamental group is normally generated by $x_1, \ldots, x_n$ and y^2. To show that C is homotopy equivalent to K, it suffices to show that $\bar{C}$ is homotopy equivalent to $\bar{K}$.

Let $\pi\colon \bar{C} \to D \setminus P$ be the map obtained by projection onto the first coordinate. When restricted to the interior of $\bar{C}$, this is a fiber bundle over the interior of $D \setminus P$ whose fiber is an $(n+1)$-times punctured open disc.

The base $D \setminus P$ is homotopy equivalent to a graph with one vertex and n edges corresponding to $x_1, \ldots, x_n$. The fiber is homotopy equivalent to a graph with one vertex and $n+1$ edges corresponding to $X_1, \ldots, X_n$ and Y. The fiber bundle structure of $\bar{C}$ implies that it is homotopy equivalent to the Cayley complex of a presentation $\langle \bar{\mathcal{G}} | \bar{\mathcal{R}}' \rangle$, where $\bar{\mathcal{R}}'$ is a set of relations equating Y^{x_k} and $X_j^{x_k}$ to words in $\{X_1, \ldots, X_n, Y\}$, for $j, k \in \{1, \ldots, n\}$. One can compute these relations $\bar{\mathcal{R}}'$ by explicitly manipulating arcs in $\bar{C}$. They are as follows.

$$\begin{aligned} Y^{x_k} &= X_k Y X_k^{-1}, \\ X_j^{x_k} &= \begin{cases} X_j Y X_j Y^{-1} X_j^{-1}, & j = k \\ X_k Y X_k^{-1} Y^{-1} X_j Y X_k Y^{-1} X_k^{-1}, & j < k \\ X_j & j > k. \end{cases} \end{aligned}$$

One can transform the relations $\bar{\mathcal{R}}'$ to $\bar{\mathcal{R}}$ using moves which can be realized by isotopy of the attaching maps of the faces in the Cayley complex. Thus $\bar{C}$ is homotopy equivalent to $\bar{K}$, and hence C is homotopy equivalent to K.

We are now ready to compute $H_2(\tilde{C})$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the free Λ-modules with bases $\{[g] : g \in \mathcal{G}\}$ and $\{f_r : r \in \mathcal{R}\}$ respectively. For any word w in $\mathcal{G}$ we define $[w] \in \mathcal{C}_1$ inductively according to the following rules

$$\begin{aligned} [1] &= 0, \\ [gw] &= [g] + \phi(g)[w], \\ [g^{-1}w] &= \phi(g)^{-1}([w] - e_g), \end{aligned}$$

for any $g \in \mathcal{G}$. Then $H_2(\tilde{C})$ is the kernel of the map $\partial\colon \mathcal{C}_2 \to \mathcal{C}_1$ given by $\partial f_r = [r]$. We calculate the following.

$$\partial f_r = \begin{cases} (1 + q^{-1}t^{-1})((1-t)[x_j] + (q-1)[y]) & \text{if } r = r_{j,j}, \\ (q^{-1} - q^{-2})(-[x_j] + t[x_k] - (q-1)[y]) & \text{if } r = r_{j,k}, \text{ where } j < k. \end{cases}$$

It is now an exercise in linear algebra to compute the kernel of this map. It is a free Λ-module with bases $\{v_{j,k} : 1 \leq j < k \leq n\}$, where

$$v_{j,k} = (q-1)f_{j,j} - (q-1)t f_{k,k} + (1-t)(1+qt)f_{j,k}.$$

We now define certain forks $F_{j,k}$ which will correspond to the basis vectors $v_{j,k}$. Let D be the unit disc centered at the origin in the complex plain. Let $p_1, \ldots, p_n$ lie on the real axis and satisfy $-1 < p_1 < \cdots < p_n < 1$. Let d_1 and d_2 lie in the lower half plane, with d_1 to the left of d_2. For each $1 \leq j < k \leq n$, let $F_{j,k}$ be a fork which lies entirely in the closed lower half plane such that the endpoints of $T(F)$ are p_j and p_k. Such an $F_{j,k}$ is uniquely determined up to isotopy by j and k, and will be called a *standard fork*.

Let $D' \subset D$ be a disc containing $F_{j,k}$ such that $D' \cap P = \{p_j, p_k\}$. Let C' be the set of unordered pairs of distinct points in D'. Let $\tilde{C}'$ be the pre-image of C' in $\tilde{C}$. We can consider $v_{j,k}$ as an element of $H_2(\tilde{C}')$, in which case it generates $H_2(\tilde{C}')$ as a Λ-module. The surface $\tilde{\Sigma}_2(F_{j,k})$ lies in $\tilde{C}'$, so must represent the homology class $\lambda v_{j,k}$ for some $\lambda \in \Lambda$. The value of λ does not depend on j and k. (Actually $\lambda = 1$, but we will not need this fact.)

To write $\sigma_i(v_{j,k})$ in terms of basis vectors, we must find a Λ-linear combination of standard forks which represents the same element of $H_2(\tilde{C})$ as the fork $\sigma_i(F_{j,k})$.

In the cases $i \notin \{j-1, j, k-1, k\}$, $i = j \neq k$, and $i = k$, there is no problem because $\sigma_i(F_{j,k})$ is a standard fork.

In the case $i = j = k-1$, the fork $\sigma_i(F_{j,k})$ has the same tine edge as $F_{j,k}$. It follows that it represents the same surface in $\tilde{C}$, up to a change in orientation and application of a covering transformation. With some thought, or by pairing with an appropriate noodle, it is not hard to check that the correct formula is $\sigma_i(v_{j,k}) = -tq^2 v_{j,k}$.

The remaining cases are $i = j-1$ and $i = k-1 \neq j$. We will use the following claim.

Claim 4.2. *$\sigma_i(v_{j,k})$ is a linear combination of basis vectors $v_{j',k'}$ which satisfy $j', k' \in \{i, i+1, j, k\}$.*

Proof. There exists a disc $D' \subset D$ such that D' contains $\sigma(F_{j,k})$, D' contains $F_{j',k'}$ for all $j', k' \in \{i, i+1, j, k\}$ with $j' < k'$, and $D' \cap P = \{p_i, p_{i+1}, p_j, p_k\}$. Let C' be the set of unordered pairs of distinct points in D'. Let $\tilde{C}'$ be the pre-image of C' in $\tilde{C}$. Then $H_2(\tilde{C}')$ is a free Λ-module with basis consisting of all $v_{j',k'}$ with $j', k' \in \{i, i+1, j, k\}$ and $j' < k'$. But $\sigma(v_{j,k})$ can be considered as an element of $H_2(\tilde{C}')$, so must be a linear combination of these basis vectors. □

In the case $i = j-1$, this claim implies that $\sigma_i(F_{j,k})$ represents the same element of $H_2(\tilde{C})$ as some Λ-linear combination of the three standard forks $F_{i,j}$, $F_{i,k}$, and $F_{j,k}$. By pairing with some appropriate noodles it is not hard to check that the correct linear combination is as stated in Theorem 4.1. Similar methods can be used to verify Theorem 4.1 in the last remaining case, $i = k-1 \neq j$. This completes the proof of Theorem 4.1.

We conclude with some remarks on the BMW representation of braid groups, defined independently by Birman and Wenzl in [BW89], and by Murakami in [Mur87]. V. Jones noticed a striking resemblance between the matrices described in Theorem 4.1 and those of a certain irreducible summand of the BMW representation. He asserted that the two representations should be the same after some renormalization. The details are worked out by Zinno in [Zin]. At present, there seems to be no deep explanation for this coincidence.

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