Structure of the eigenspace of a Monge matrix in max-plus algebra

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Abstract

A complete description of the eigenspace structure for a given \( n \times n \) Monge matrix in a max-plus algebra is presented. Based on the description, an \( O(n^2) \) algorithm for computing the eigenspace dimension is formulated, which is faster than the previously known algorithms.

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1. Introduction

Maximum and minimum operations are involved in many optimization problems. Matrix computations using these operations were considered by a number of authors, e.g. in [1,3,5,10], and analogies of various notions from the classical linear algebra were studied. The steady states of discrete events processes correspond to eigenvectors of max-plus matrices, see [4,14], hence the characterization of the eigenspace structure is important for the applications.

In some cases, the investigation is more efficient, if the considered matrix has special properties. Many efficient solution of problems concerning Monge matrices were described in [2]. Problems connected with eigenvectors of Monge matrices were studied in papers [6–8], in which efficient algorithms for various questions were presented.

The aim of this paper is to describe the structure of the eigenspace of a Monge matrix. The techniques are used which are analogous to those known in the area of efficiently solvable cases of the Traveling Salesman Problem and other problems, and the equivalence classes connected with cycles of zero weight in the associated digraph of the given Monge matrix are computed. The structure of the eigenspace and the eigenspace dimension is completely described by this computation, which can be done in \( O(n^2) \) time, or, with some additional information available, in \( O(n) \) time.
2. Notions and notation

By a max-plus algebra we understand the algebraic structure \((G, \oplus, \otimes) = (\mathbb{R}^*, \max, +)\), where \(G = \mathbb{R}^*\) is the set of all real numbers \(\mathbb{R}\) extended by an infinite element \(\varepsilon = -\infty\), and \(\oplus, \otimes\) are the binary operations on \(\mathbb{R}\): \(\oplus = \max\) and \(\otimes = +\). The infinite element is neutral with respect to the maximum operation and absorbing with respect to addition.

The results presented in this paper for the max-plus algebra \((\mathbb{R}^*, \max, +)\) are valid also for the general notion of max-plus algebra, in which \((G, \oplus, \otimes)\) is derived in a similar way from an arbitrary divisible commutative linearly ordered group in additive notation. In the general case, the neutral element \(e \in G\) in the additive group must be used instead of \(0 \in \mathbb{R}\).

For any natural \(n > 0\), we denote \(N = \{1, 2, \ldots, n\}\). Further, we denote by \(G_n\) the set of all \(n \times n\) matrices over \(G\). The matrix operations over the max-plus algebra \(G\) are defined with respect to \(\oplus, \otimes\), formally in the same manner as the matrix operations over any field. The operation \(\otimes\) for matrices denotes the formal matrix product with operations \(\oplus = \max\) and \(\otimes = +\) replacing the usual operations \(+, \cdot\), while the operation \(\oplus\) for matrices is performed componentwise.

The problem of finding a vector \(x \in G^n\) and a value \(\lambda \in G\) satisfying

\[
A \otimes x = \lambda \otimes x
\]  

(2.1)

called is an extremal eigenproblem corresponding to the matrix \(A\), the value \(\lambda\) is called (extremal) eigenvalue, and \(x\) is called (extremal) eigenvector of \(A\). The word “extremal” is usually omitted. A survey of the results concerning various types of eigenproblems can be found in [15].

The associated digraph \(D_A\) of a matrix \(A \in G_n\) is defined as a complete arc-weighted digraph with the node set \(V = N\), and with the arc weights \(w(i, j) = a_{ij}\) for every \((i, j) \in N \times N\). If \(p\) is a path or a cycle in \(D_A\), of length \(r = |p|\), then the weight \(w(p)\) is defined as the sum of all weights of the arcs in \(p\). If \(r > 0\), then the mean weight of \(p\) is defined as \(w(p)/r\). Of all the mean weights of cycles in \(D_A\), the maximal one is denoted by \(\lambda(A)\). By Cuninghame-Green in [5], the maximal cycle mean \(\lambda(A)\) is the unique eigenvalue of \(A\). The problem of finding the eigenvalue \(\lambda(A)\) has been studied by a number of authors and several algorithms are known for solving this problem. The algorithm described by Karp in [12] has the worst-case performance \(O(n^3)\). The iterative algorithm by Howard has been reported to have on average almost linear computational complexity, though a tight upper bound has not yet been found (see [11]).

For \(B \in G_n\) we denote by \(A(B)\) the matrix \(B \oplus B^{(2)} \oplus \cdots \oplus B^{(n)}\) where \(B^{(s)}\) stands for the \(s\)-fold iterated product \(B \otimes B \otimes \cdots \otimes B\). Further, we denote \(A_{ij} = -\lambda(A) \otimes A\) (here we have a formal product of a scalar value \(-\lambda(A)\) and a matrix \(A\), i.e. \([A_{ij}]_{ij} = -\lambda(A) + a_{ij}\) for any \((i, j) \in N \times N\)). It is shown in [5] that the matrix \(A(A_{ij})\) contains at least one column, the diagonal element of which is 0 and every such a column is an eigenvector (so-called: fundamental eigenvector) of the matrix \(A\). Moreover, every eigenvector of \(A\) can be expressed as a linear combination of fundamental eigenvectors.

Let \(A(A_{ij}) = (\delta_{ij})\). It follows from the definition of \(A(A_{ij})\) that \(\delta_{ij}\) is the maximal weight of a path from \(i\) to \(j\) in \(D_{A_{ij}}\). Hence, \(A(A_{ij})\) can be computed in \(O(n^3)\) time, using the Floyd–Warshall algorithm [13]. In this way, a complete set of fundamental eigenvectors can be found by at most \(O(n^3)\) operations. However, if we wish to compute only one single eigenvector of \(A\), no better algorithm than \(O(n^3)\) is known for matrices of a general type. In the special case, when the matrix \(A\) is Monge, the above computations can be performed in a more efficient way.

**Definition 2.1.** We say that a matrix \(A = (a_{ij}) \in G_n\) is Monge if

\[
a_{ij} + a_{kl} \leq a_{il} + a_{kj} \quad \text{for all } i < k, j < l.
\]

It has been shown in [7] that the eigenvalue \(\lambda(A)\) of a Monge matrix can be found in \(O(n^2)\) time. In [8], an \(O(n^2)\) algorithm was presented for computing a single eigenvector of a Monge matrix.

3. Cycles of zero weight

Our goal in the following sections will be to show that the equivalence classes of fundamental eigenvectors of a given Monge matrix (the number of which is the eigenspace dimension) can be computed in \(O(n^2)\) time. The maximal cycle mean \(\lambda(A)\) is the unique eigenvalue of a matrix \(A \in G_n\), therefore if \(x \in G^n\) is an eigenvector of \(A\) satisfying the
Theorem 3.1. If 
\[ \lambda = \lambda(A) \]. Then we have
\[ -\lambda(A) \otimes A \otimes x = -\lambda(A) \otimes x \]
which is equivalent to
\[ A_{\lambda} \otimes x = 0 \otimes x. \] (3.1)
Hence, \( x \) is also an eigenvector of \( A_{\lambda} \) and \( \lambda(A_{\lambda}) = 0 \). Therefore, in further investigations we may without any loss of generality assume that \( \lambda(A) = 0 \) and \( A_{\lambda} = A \).

Let us denote by \( g_1, g_2, \ldots, g_n \) all columns of \( A(A) \). We shall say that vectors \( g_j, g_k \) are equivalent, if there is \( x \in G \) such that \( g_j = x \otimes g_k \). It has been shown in [5] that a vector \( g_j \) is fundamental if and only if the node \( j \) in the associated digraph \( D_A \) lies in a cycle \( c \) with \( w(c) = 0 \) (shortly: in a zero-cycle). Such a node \( j \) is called an eigennode. The set of all eigennodes of the matrix \( A \) will be denoted by \( E_A \). Moreover, vectors \( g_j, g_k \) are equivalent if and only if the vertices \( j, k \) are contained in a common zero-cycle in \( D_A \). The eigenspace dimension of matrix \( A \) is the maximal number of non-equivalent fundamental eigenvectors, i.e. the maximal number of non-trivial highly connected components in the sense of the following definition (introduced in [6]).

**Definition 3.1 (Galavec [6]).** Let \( A \in G_n \) with \( \lambda(A) = 0 \). We say that the eigennodes \( i, j \in E_A \) are equivalent, if they are contained in a common zero-cycle. By a **highly connected component** \( \mathcal{X} \) in the associated digraph \( D_A \) we mean a maximal subdigraph with the property that any two nodes in \( \mathcal{X} \) are equivalent. For \( i \in E_A \), the highly connected component containing \( i \) is denoted by \( \mathcal{X}[i] \). A highly connected component is non-trivial, if it contains at least one zero-cycle of positive length. The set of all non-trivial highly connected components in \( D_A \) will be denoted by \( \text{HCC}^*(A) \).

An important role in our investigations will be played by zero-cycles of lengths 1 and 2. They will be referred to as zero-loops (length 1) and zero-eyes (length 2). As the starting point we use the following theorems which can easily be proved by the arguments used in the proof of Theorem 2.1 in [7].

**Theorem 3.1.** If \( A \in G_n \) is a Monge matrix with \( \lambda(A) = 0 \), then every eigennode in \( D_A \) is contained in a zero-loop or in a zero-eye.

**Theorem 3.2.** If \( A \in G_n \) is a Monge matrix with \( \lambda(A) = 0 \), then any equivalent eigennodes \( i, j \) in \( D_A \) can be connected by a concatenation of zero-eyes.

In the rest of this section we shall assume that \( A \in G_n \) is a fixed Monge matrix with \( \lambda(A) = 0 \). The above theorems imply that in order to describe the structure of \( \text{HCC}^*(A) \), we may reduce our considerations to zero-loops and zero-eyes in \( D_A \). Using the Monge property of \( A \), we prove several useful lemmas. To avoid too many indices, we use the notation \( a(i, j) \) instead of \( a_{ij} \).

**Lemma 3.3.** Let \( i, j \in N \) with \( i < j \). If \( (i, i) \) and \( (j, j) \) are zero-loops, then \( (i, j, i) \) is a zero-eye.

**Proof.** By assumption, we have \( a(i, i) = a(j, j) = 0 \). Using the Monge property and the assumption that the maximal weight of a cycle in \( D_A \) is \( \lambda(A) = 0 \), we get
\[ 0 = a(i, i) + a(j, j) \leq a(i, j) + a(j, i) \leq 0 \]
which implies \( a(i, j) + a(j, i) = 0 \). Hence, \( (i, j, i) \) is a zero-eye. \( \square \)

The assertion of Lemma 3.3 is schematically depicted in Fig. 1. Nodes are identified with the diagonal elements of the matrix, zero-loops are denoted by circles and each zero-eye consists of two hooked vectors.

**Lemma 3.4.** Let \( i, j, k \in N \) with \( i < j < k \) or \( i > j > k \). If \( (i, i) \) is a zero-eye and \( (k, k) \) is a zero-loop, then \( (i, k, i) \) and \( (j, k, j) \) are zero-eyes.
Proof. By the Monge property, we have the inequalities
\[ a(i, j) + a(k, k) \leq a(k, j) + a(i, k), \]  
(3.3)
\[ a(j, i) + a(k, k) \leq a(k, i) + a(j, k). \]  
(3.4)
As the maximal weight of any cycle in \( \mathcal{D}_A \) is 0, we have
\[ a(i, k) + a(k, i) \leq 0, \]  
(3.5)
\[ a(j, k) + a(k, j) \leq 0. \]  
(3.6)
Adding the inequalities (3.3), (3.4) and using the fact that \((i, j, i)\) is a zero-eye and \((k, k)\) is a zero-loop, we get, in view of (3.5) and (3.6),
\[ 0 \leq a(i, k) + a(k, i) + a(j, k) + a(k, j) \leq 0, \]
\[ 0 = a(i, k) + a(k, i) + a(j, k) + a(k, j) = 0. \]
Hence, \((i, k, i)\) and \((j, j)\) are zero-eyes. □

The assertion of Lemma 3.4 is schematically depicted in Fig. 2.

Lemma 3.5. Let \(i, j, k \in \mathbb{N}\) with \(i < j < k\). If \((i, j, i)\) and \((j, k, j)\) are zero-eyes, then \((i, k, i)\) is a zero-eye and \((j, j)\) is a zero-loop.
Proof. The Monge property gives the inequalities

\[ a(i, j) + a(j, k) \leq a(j, j) + a(k, i), \quad (3.7) \]
\[ a(j, i) + a(k, j) \leq a(k, i) + a(j, j). \quad (3.8) \]

Adding the inequalities (3.7), (3.8) and using the assumptions

\[ a(i, j) + a(j, i) = 0, \]
\[ a(j, k) + a(k, j) = 0. \]

we get, by analogous arguments as in the previous proof,

\[ 0 \leq a(i, k) + a(k, i) + 2a(j, j) \leq 0 \]

which implies \( a(i, k) + a(k, i) = 0 \) and \( a(j, j) = 0 \). Hence, \((i, k, i)\) is a zero-eye and \((j, j)\) is a zero-loop. □

A schematic picture of Lemma 3.5 can be found in Fig. 3.

Lemma 3.6. Let \( i, j, k, l \in \mathbb{N} \) with \( i < j < k < l \) (or \( i \leq k < j \leq l \)). If \((i, j, i)\) and \((k, l, k)\) are zero-eyes, then \((i, l, i)\) and \((j, k, j)\) are zero-eyes, as well.

Proof. Analogously as in the previous proofs, the Monge property gives

\[ a(i, j) + a(k, l) \leq a(i, l) + a(k, j), \]
\[ a(j, i) + a(l, k) \leq a(j, k) + a(l, i). \]

By the assumptions

\[ a(i, j) + a(j, i) = 0, \]
\[ a(k, l) + a(l, k) = 0, \]

we get

\[ 0 \leq a(i, l) + a(l, i) + a(j, k) + a(k, j) \leq 0 \]

which implies

\[ a(i, l) + a(l, i) = 0, \]
\[ a(j, k) + a(k, j) = 0, \]

i.e. \((i, l, i)\) and \((j, k, j)\) are zero-eyes. □

Both versions of Lemma 3.6 are schematically shown on Figs. 4 and 5.
4. Eigenspace structure

The lemmas from Section 3 are used in this section for the investigation of the eigenspace structure of the Monge matrix \( A \). We consider two kinds of eigennodes and highly connected components. An eigennode \( i \in E_A \) is called singular, if \( (i, i) \) is a zero-loop, and it is called regular, in the opposite case. A non-trivial highly connected component \( K \in \text{HCC}^*(A) \) is called singular, if it contains at least one singular eigennode, otherwise \( K \) is called regular.

**Theorem 4.1.** Let \( A \in G_n \) be a Monge matrix with \( \lambda(A) = 0 \). If a component \( K \in \text{HCC}^*(A) \) is singular, then \( K \) contains all singular eigennodes.

**Proof.** The statement follows directly from Lemma 3.3. □

Let \( i, j, k, l \in N \) and let \( (i, j, i) \) be a zero-eye. We say that the node \( k \) lies within the zero-eye \( (i, j, i) \), (alternatively: the eye \( (i, j, i) \) encircles the node \( k \)), if \( i \leq k \leq j \), or \( i \geq k \geq j \). Further, we say that a zero-eye \( (k, l, k) \) lies within the zero-eye \( (i, j, i) \) (alternatively: \( (i, j, i) \) encircles \( (k, l, k) \)), if both nodes \( k, l \) lay within \( (i, j, i) \).

If \( K_1, K_2 \in \text{HCC}^*(A) \) and if every zero-eye in \( K_1 \) and every singular eigennode in \( K_1 \) (if there is any) lies within every zero-eye in \( K_2 \), then we say that the component \( K_1 \) lies within \( K_2 \) (alternatively: \( K_2 \) encircles \( K_1 \)) in notation: \( K_1 \prec K_2 \), or \( K_2 \succ K_1 \).

**Theorem 4.2.** Let \( A \in G_n \) be a Monge matrix with \( \lambda(A) = 0 \). If the eigennodes \( i, k \in E_n \) are not equivalent, then exactly one of the following statements holds true:

(i) the node \( i \) is regular, \( k \) is singular, and \( k \) lies within every zero-eye containing \( i \)
(ii) the node \( i \) is singular, \( k \) is regular, and \( i \) lies within every zero-eye containing \( k \)
(iii) both nodes \( i, k \) are regular, and either every zero-eye \( (k, l, k) \) lies within every zero-eye \( (i, j, i) \) or every zero-eye \( (i, j, i) \) lies within every zero-eye \( (k, l, k) \).
Proof. Theorem 4.1 says that there is at most one singular highly connected component in $D_A$, which contains all singular eigen-nodes. Eigen-nodes $i, k$ cannot be contained in a common component $\mathcal{K} \in \text{HCC}^*(A)$, because, by assumption, they are not equivalent. Therefore, the nodes $i, k$ cannot be both singular, and as a consequence, either one of the two nodes is regular and the second one is singular or both nodes are regular. We shall discuss these cases in more detail.

(i) Let us assume first that the node $i$ is regular and $k$ is singular. Then for every zero-eye $(i, j, i)$ with $i < j$ we have three subcases: (a) $i < j < k$, (b) $i < k < j$, (c) $k < i < j$. In the subcase (a), Lemma 3.4 implies that the nodes $i, k$ are equivalent, in contradiction to the assumption. The subcase (c) also leads to a contradiction, according to Lemma 3.4 with interchanged variables $i, j$. In the remaining subcase (b), the node $k$ lies within $(i, j, i)$. By a similar procedure we get the same result in the situation with $i > j$. Hence, the statement (i) holds true in this case.

(ii) It can be proved analogously, that in the second case, when $i$ is singular and $k$ is regular, the statement (ii) is fulfilled.

(iii) Finally, let us consider the case when both $i, k$ are regular eigen-nodes. Let us suppose that $(i, j, i), (k, l, k)$ are zero-eyes. In view of the assumption that $i, k$ are not equivalent, we get, using both versions of Lemma 3.6, two subcases which exclude each other: (a) $(k, l, k)$ lies within $(i, j, i)$, (b) $(i, j, i)$ lies within $(k, l, k)$.

Subcase (a): Let us assume that $(k, l, k)$ lies within $(i, j, i)$. Let $(i, j, i)$ be another zero-eye. Using Lemma 3.5, we see that inequalities $j < i < j_1$ or $j_1 < i < j$ lead to contradiction with the assumption that $i$ is regular. Hence there are two subsubcases: either (a1) $(k, l, k)$ lies within $(i, j, i)$ or (a2) $(i, j, i)$ lies within $(k, l, k)$. By transitivity, $(k, l, k)$ lies within $(i, j, i)$ in the subsubcase (a1). Using Lemma 3.6, we get the same result in the subsubcase (a2). In a similar way we can show that every zero-eye $(k, l, k)$ lies within $(i, j, i)$. As $(i, j, i)$ is an arbitrary zero-eye containing $i$, the first part of the statement (iii) holds true. The proof in the subcase (b) is analogous. □

Theorem 4.2 says that highly connected components in $D_A$ are linearly ordered by the relation $\prec$ and the only singular component (if there is any) is the least element in this ordering. The nested structure of the eigenspace of a Monge matrix is demonstrated by an example.

**Example 1.** Let $n = 7$, let $A, B \in G_n$ be the following Monge matrices:

$$
\begin{bmatrix}
-5 & -4 & -3 & -3 & -2 & -1 & 0 \\
-4 & -3 & -2 & -2 & -1 & 0 & -1 \\
-3 & -2 & -1 & -1 & 0 & -1 & -2 \\
-2 & -1 & 0 & 0 & -1 & -2 \\
-2 & -1 & 0 & -1 & -1 & -2 & -3 \\
-1 & 0 & -1 & -2 & -3 & -4 \\
0 & -1 & -2 & -3 & -4 & -5
\end{bmatrix} \quad \begin{bmatrix}
-5 & -4 & -3 & -3 & -2 & -1 & 0 \\
-4 & -3 & -2 & -2 & -1 & 0 & -1 \\
-3 & -2 & -1 & -1 & 0 & -1 & -2 \\
-2 & -1 & 0 & 0 & -1 & -2 & -3 \\
-2 & -1 & 0 & -1 & -1 & -2 & -3 \\
-1 & 0 & -1 & -2 & -3 & -4 \\
0 & -1 & -2 & -3 & -4 & -5
\end{bmatrix}
$$

The corresponding eigenspace structures $D_A, D_B$ are shown in Fig. 6. The matrices $A, B$ only differ at position $(4, 4)$, namely, $a_{44} = 0$ and $b_{44} = -1$. Hence, the node 4 is singular in $D_A$, while it is a non-eigen-node in $D_B$. Both eigenspaces contain three nested regular components $\mathcal{K}_1 \prec \mathcal{K}_2 \prec \mathcal{K}_3 \in \text{HCC}^*(A)$. Moreover, $D_A$ has a singular component $\mathcal{K}_0 \prec \mathcal{K}_1$ (encircled by all the regular components). On the other side, $D_B$ has no singular component.
5. Highly connected components

The components shown in Example 1 have a very simple form. Every regular component in Fig. 6 consists of two eigenvalues connected in a zero-eye, and the singular component consists of one singular eigenvalue. In this section we describe the full variety of possible forms which can be taken by highly connected components in the associated digraph \( D_A \) of a Monge matrix.

If there are non-eigenvalues in \( D_A \), then by deleting the corresponding rows and columns of the matrix \( A \) we get a Monge submatrix \( A' \), which has the same zero-cycles as the original Monge matrix. Hence, without any loss of generality we may assume in this section that every node in \( D_A \) is an eigenvalue, i.e. \( E_A = N \).

For subsets \( I, J \subset N \) the notation \((I \times J \times I)^{ze}\) denotes the set of all zero-eyes of the form \((i, j, i)\) with \( i \in I, j \in J \).

**Theorem 5.1.** Let \( A \in G_n \) be a Monge matrix with \( \lambda(A) = 0 \) and \( E_A = N \). For every regular component \( \mathcal{K} \in HCC^*(A) \) there exist non-empty disjoint intervals \( I, J \subset N \) such that the node set of \( \mathcal{K} \) is \( I \cup J \) and the set of all zero-eyes in \( \mathcal{K} \) is equal to \((I \times J \times I)^{ze}\).

**Proof.** Let us denote by \( i_0 \) the minimal eigennode, and by \( j_1 \) the maximal eigennode in \( \mathcal{K} \). As the component is regular, the nodes \( i_0, j_1 \) are different, i.e. \( i_0 < j_1 \). Further, let us denote by \( j_0 \) the minimal eigennode in \( \mathcal{K} \) such that \((i_0, j_0, i_0)\) is a zero-eye, and dually, denote by \( i_1 \) the maximal eigennode in \( \mathcal{K} \) such that \((i_1, j_1, i_1)\) is a zero-eye.

The nodes \( j_0, i_1 \) are in the component \( \mathcal{K} \), therefore, by the definition of \( i_0, j_1 \), we have \( i_0 < j_0 < j_1 \) and \( i_0 < i_1 < j_1 \). There are three possible cases: (a) \( j_0 > i_1 \), (b) \( j_0 = i_1 \), (c) \( j_0 < i_1 \). The case (b) implies that the eigennode \( j_0 = i_1 \) is singular, in view of Lemma 3.5, which is a contradiction with regularity of \( \mathcal{K} \). In the case (c), Lemma 3.6 gives the zero-eye \((j_0, i_1, j_0)\), which together with the zero-eye \((i_0, j_0, i_0)\) leads to contradiction with the regularity, analogously as above. Hence, according to case (a), we have

\[
i_0 < i_1 < j_0 < j_1.
\]

In other words, \( I = \langle i_0, i_1 \rangle \cap N \) and \( J = \langle j_0, j_1 \rangle \cap N \) are non-empty disjoint intervals in \( N \). Moreover, by Lemma 3.6, we have zero-eyes \((i_0, j_1, i_0)\) and \((i_1, j_0, i_1)\) (the largest and the least zero-eye in \( \mathcal{K} \)).

Considering the possible cases similarly as above, it is easy to show that every zero-eye \((i, j, i)\) in \( \mathcal{K} \) encircles the least zero-eye \((i_1, j_0, i_1)\). That means \( i \in I \) and \( j \in J \), hence \((i, j, i) \in (I \times J \times I)^{ze}\). Conversely, in view of the assumption \( E_A = N \) and in view of Lemma 3.6, all eigennodes \( i \in I, j \in J \) are highly connected with \( i_0 \) and \( j_1 \), which implies that nodes \( i, j \) are in \( \mathcal{K} \) and the zero-eye \((i, j, i) \in (I \times J \times I)^{ze}\) is a zero-eye in \( \mathcal{K} \). \( \square \)

The structure of regular components in \( D_A \) is schematically shown in the next example.

**Example 2.** Let \( n = 7 \), let \( A, B \in G_7 \) be the following Monge matrices:

\[
\begin{bmatrix}
-3 & -3 & -2 & -2 & -1 & -1 & 0 \\
-3 & -3 & -2 & -2 & -1 & -1 & 0 \\
-2 & -2 & -1 & -1 & 0 & 0 & -1 \\
-1 & -1 & 0 & 0 & 0 & 0 & -1 \\
-1 & -1 & 0 & -1 & -1 & -1 & -2 \\
0 & 0 & -1 & -1 & -2 & -2 & -3 \\
0 & 0 & -1 & -2 & -2 & -3 & -3
\end{bmatrix}
, \quad
\begin{bmatrix}
-3 & -2 & -2 & -1 & -1 & 0 & 0 \\
-2 & -1 & -1 & 0 & 0 & -1 & -1 \\
-2 & -1 & -1 & 0 & 0 & -1 & -1 \\
-1 & 0 & 0 & -1 & -1 & 2 & 2 \\
-1 & 0 & 0 & -1 & -1 & 2 & 2 \\
0 & -1 & -1 & -2 & 2 & 3 & 3 \\
0 & -1 & -1 & -2 & 2 & 3 & 3
\end{bmatrix}
\]

The corresponding eigenspace structures are shown in Fig. 7. \( D_A \) consists of one singular component \( \mathcal{K}_0 \) with a single node 4, and two regular components: \( \mathcal{K}_1 \) generated by intervals \( I_1 = \langle 3, 3 \rangle, J_1 = \langle 5, 6 \rangle \) and \( \mathcal{K}_2 \) generated by intervals \( I_2 = \langle 1, 2 \rangle, J_1 = \langle 7, 7 \rangle \). The eigenspace \( D_B \) consists of two regular components, \( \mathcal{K}_1 \) with \( I_1 = \langle 2, 3 \rangle, J_1 = \langle 4, 5 \rangle \) and \( \mathcal{K}_2 \) with \( I_2 = \langle 1, 1 \rangle, J_1 = \langle 6, 7 \rangle \).

**Theorem 5.2.** Let \( A \in G_n \) be a Monge matrix with \( \lambda(A) = 0 \) and \( E_A = N \). If the set \( S \subseteq N \) of all singular eigennodes is non-empty, then \( S \) is an interval and there exist disjoint intervals \( I, J \subset N \) of regular eigennodes such that the
node set of the singular component $K_0$ is $I \cup S \cup J$ and the set of all zero-eyes in $K_0$ is equal to the union of sets $(I \times J \times I)^ze \cup (S \times S \times S)^ze \cup (I \times S \times I)^ze \cup (J \times S \times J)^ze$.

Remark 5.1. The intervals $I, J$ in Theorem 5.2, need not be non-empty. E.g. in Example 2, the singular component $K_0$ in $DA$ only consists of one eigennode, hence the interval $S$ is the shortest possible and both intervals $I, J$ are empty. Further possibilities are shown in Example 3.

Proof. Let us assume that $K_0$ is non-empty. We denote by $s_0$ (by $s_1$) the minimal (maximal) singular eigennode. In view of Lemma 3.5, every node in the interval $\langle s_0, s_1 \rangle$ is a singular eigennode, i.e. $S = \langle s_0, s_1 \rangle$. By Lemma 3.3, any two distinct nodes $i, j \in S$ are contained in a common zero-eye.

Further, we denote by $i_0$ (by $j_1$) the minimal (maximal) eigennode in $K_0$. Then we put $i_1 = s_0 - 1, j_0 = s_1 + 1$ and $I = \langle i_0, i_1 \rangle, J = \langle j_0, j_1 \rangle$. If $i_0 < s_0$, then $i_0 \leq i_1$ and the interval $I$ is non-empty. If the singular eigennode $s_0$ is the minimal eigennode in $K_0$, i.e. if $i_0 = s_0$, then clearly $I = \emptyset$. Similarly, the interval $J$ is non-empty if and only if $s_1 < j_1$.

By Theorem 4.2 and by assumption $E_A = N$, the node set of the singular component $K_0$ is an interval in $N$, which is equal to the union of disjoint subintervals $I \cup S \cup J$.

For any $i \in I$ there is $k \in K_0$ such that $(i, k, i)$ is a zero-eye. In view of Lemmas 3.4 and 3.5, the assumption $k \in I$ implies that $k$ is singular, which is a contradiction with the fact that all eigennodes in $I$ are regular. Hence $(i, k, i)$ belongs to $(I \times J \times I)^ze$ or to $(I \times S \times I)^ze$. It is easy to prove analogously that every zero-eye of the form $(j, k, j)$ with $j \in J$ belongs to $(J \times I \times J)^ze = (I \times J \times I)^ze$ or to $(J \times S \times J)^ze$. We have shown that every zero-eye in $K_0$ belongs to the union $(I \times J \times I)^ze \cup (S \times S \times S)^ze \cup (I \times S \times I)^ze \cup (J \times S \times J)^ze$. The converse implication is trivial. □

Example 3. Let $n = 7$, let $A, B \in G_n$ be the following Monge matrices:

$$
\begin{pmatrix}
-1 & -1 & -1 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}
= 
\begin{pmatrix}
-3 & -3 & -2 & -2 & -1 & -1 & 0 \\
-3 & -3 & -2 & -2 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & -1 & -1 & -2 \\
0 & 0 & 0 & 0 & -1 & -1 & -2 \\
0 & 0 & -1 & -1 & -2 & -2 & -3
\end{pmatrix}
$$

The eigenspace $D_A$ in Fig. 8 only consists of one singular component $K_0$ with $I = \langle 1, 3 \rangle, S = \langle 4, 5 \rangle$ and $J = \langle 6, 7 \rangle$. The eigenspace $D_B$ consists of one singular component $K_0$ with $I = \emptyset, S = \langle 3, 4 \rangle$ and $J = \langle 5, 6 \rangle$ and one regular component $K_1$ with $I_1 = \langle 1, 2 \rangle, J_1 = \langle 7, 7 \rangle$.

6. Computing the eigenspace dimension

In this section we describe an algorithm for computing the eigenspace dimension of a Monge matrix. Algorithm $A$. 
We also said that the eigenspace dimension of \( A \) is equal to the number of non-trivial highly connected components in the associated digraph \( D_A \).

There is an algorithm: 

1. Put \( d := 0 \).
2. Create a Monge matrix \( A' \in G_n' \) by deleting from \( A \) the \( i \)th row and the \( i \)th column for every \( i \in N \) with all negative values \( a(i, j) + a(j, i) < 0, j \in N \). Put \( A := A', n := n' \).
3. If there is an index \( s \in N \) with \( a(s, s) = 0 \), then denote by \( s_0 \) (by \( s_1 \)) the minimal (maximal) index with this property. Otherwise, go to step 7.
4. Put \( d := 1 \). If there is \( i \in N, i < s_0 \) with \( a(i, s_1) + a(s_1, i) = 0 \), then denote by \( i^* \) the minimal such index, otherwise put \( i^* := s_0 \). If there is \( j \in N, j > s_1 \) with \( a(j, s_0) + a(s_0, j) = 0 \), then denote by \( j^* \) the maximal such index, otherwise put \( j^* := s_1 \).
5. If \( i^* = 1 \) and \( j^* = n \), then go to step 10.
6. Create a Monge matrix \( A'' \in G_n'' \) by deleting from \( A \) the \( k \)th row and the \( k \)th column for every \( k \in N, i^* \leq k \leq j^* \). Put \( A := A'', n := n'' \).
7. Put \( d := d + 1 \). Denote by \( j^* \) the minimal index \( k \in N \) with \( a(1, k) + a(k, 1) = 0 \) and denote by \( i^* \) the maximal index \( k \in N \) with \( a(n, k) + a(k, n) = 0 \).
8. If \( i^* + 1 = j^* \), then go to step 10.
9. Denote by \( A \in G_n \) the submatrix only consisting of elements \( a(k, l) \) with \( i^* < k < j^* \) and \( i^* < l < j^* \). Go to step 7.
10. Stop.

\textbf{Theorem 6.1.} There is an algorithm, which for every \( n \times n \) Monge matrix \( A \) over a max-plus algebra \( G \) computes the eigenspace dimension of \( A \) in \( O(n^2) \) time. If it is known in advance that every node in the associated digraph \( D_A \) is an eigennode, then the eigenspace dimension of \( A \) can be computed in \( O(n) \) time.

\textbf{Proof.} As we noticed in the Introduction, the eigenspace dimension is not changed by the assumption \( \lambda(A) = 0 \). We also said that the eigenspace dimension of \( A \) is equal to the number of non-trivial highly connected components \( \mathcal{X} \in \text{HCC}^*(A) \). Hence, it is sufficient to show that the algorithm \( \mathcal{A} \) defined above works properly and computes the desired output in time \( O(n^2) \).

In the first step of \( \mathcal{A} \), the output variable \( d \) is initialized to value 0. In step 2, the rows and the columns corresponding to all non-eigennodes are deleted from \( A \). The remaining rows and columns form a Monge matrix (denoted again as \( A \)). It is clear that the reduced matrix has the same eigenspace dimension as the original input matrix.

In step 3, the algorithm verifies whether there is a singular component in \( D_A \). If not, then the computation goes to the main cycle in steps 7–9. If \( D_A \) contains some singular eigennodes (at least one), then they all belong to the only singular component, which may contain also some regular eigennodes, according to Theorem 5.2. The nodes in the singular component form an interval \( [i^*, j^*] \) computed in step 4. If this interval coincides with the interval \( [1, n] \), then there are no regular components in \( D_A \) and the algorithms stops in step 10.

The number of regular components is computed in the main cycle. The cycle starts in step 7 by computing disjoint intervals \( I = (1, i^*) \) and \( J = (j^*, n) \). According to Theorem 5.1, the intervals \( I, J \) determine the maximal regular component in the sense of the relation \( \prec \). If the set of eigennodes \( I \cup J \) coincides with the interval \( [1, n] \), then the algorithms stops in step 10. Otherwise, the eigennodes in \( I \cup J \) are deleted and the main cycle goes back to step 7.
Let us evaluate the computational complexity of Algorithm A. It is easy to see that the computations in steps 1–6 are made only once and they all can be performed in $O(n)$ time, with the exception of step 2 which requires quadratic time. The total time needed for steps 7–10 is bounded by $O(n)$, because the searches of $i^*$ and $j^*$ in step 7 go in one direction and visit each node only once.

It was shown above that step 2 is the only one with the quadratic computational complexity. Hence, if we possess an extra information that every node is an eigennode, then step 2 can be omitted and the computational complexity is linear. □

**Remark 6.1.** We may notice that in each run of step 9, it is sufficient to adjust the enumeration of rows and columns, what can be done in total $O(n)$ time.

**References**


