Bivariate Interpolation on the Hypersphere with Thiele Type Rational Interpolants

Qiushi Wang, Shuo Tang*

School of Mathematics, Hefei University of Technology, Hefei 230009, China

Abstract

Interpolation problem on the sphere has applications in geodesy, earth science and other fields. In this paper, we present an axiomatic approach to vector-valued rational interpolation. In order to interpolate \( m + n + 1 \) points on the unit hypersphere \( S^{d-1} \) with a bivariate Thiele type rational function, the construction process of it is based on the Samelson inverse for vectors. We generalize Thiele type rational interpolants on the hypersphere of Thierry Gensane and give the numerical example to show validity of interpolant.

Keywords: Interpolation on the Sphere; Samelson Inverse; Bivariate Thiele Type Rational Interpolant

1 Introduction

As is well-known, interpolation method is an important way of constructive approximation. With years of developing, polynomial interpolation has already achieved many results while rational function interpolation, which is much more complicated, can be solved via means of using the feature of continued fraction and have achieved some promising results like in [10, 11]. Through using Samelson inverse we can apply continued fraction interpolation into vectors which provides a effective way for multi-rational value approximation. P. Wynn (1963) raised the question of rational interpolation of vectors. He noted that the \( \varepsilon \)-algorithm, applied to Thiele-type continued fraction for rational interpolation of vectors and implemented with Samelson inverse, can give exact results of vector-valued rational interpolants, and also Padé approximant is an effective way. Scholars such as P. R. Graves-Morris have given the basic theories and applications of vector-valued Padé approximants like in [7].

Just like expanding the concept of unary rational interpolation into multivariate, we expanding the vector-valued rational interpolation on hypersphere into multi-variable is very meaningful. In recent years, we have achieved enormous conclusions on sphere interpolation, like in [4, 5, 6], which gives the theoretic basis of algebraic approach to curves and surfaces on the sphere and

*Paper supported by Natural Science Foundation of China (No. 61272024) and Natural Science of Anhui Provincial College (No. KJ2013B232).

*Corresponding author.

Email address: tangshuo-2008@163.com (Shuo Tang).

The content of this paper is arranged as follow:

First, we introduce theory of vector-valued rational interpolation. Second, we provide bivariate rational interpolation to generalize Thiele type rational interpolants on the hypersphere of bivariate function. Finally, we give algorithm and application example.

2 Vector-valued Rational Interpolation

Suppose $\prod_x^\infty = \{x_0, x_1, \ldots\}$ and $\prod_y^\infty = \{y_0, y_1, \ldots\}$ are composed by real number, where $x_i \neq x_j$, $y_i \neq y_j$, $i \neq j$. A plane point set

$$\prod_{x,y}^{n,m} = \{(x_i, y_j) | i = 0, 1, \ldots, n; j = 0, 1, \ldots, m, x_i \in \prod_x^\infty, y_i \in \prod_y^\infty\}$$

its corresponding bivariate finite vector is

$$\nabla_{x,y}^{n,m} = \{v_{ij} | v_{ij} = v(x_i, y_j) \in C^d, (x_i, y_j) \in \prod_{x,y}^{n,m}\}$$

The formed rectangular mesh $\prod_{x,y}^{n,m}$ is given in the following array

$$
\begin{array}{cccc}
(x_0, y_0) & (x_1, y_0) & \cdots & (x_n, y_0) \\
(x_0, y_1) & (x_1, y_1) & \cdots & (x_n, y_1) \\
\vdots & \vdots & \ddots & \vdots \\
(x_0, y_m) & (x_1, y_m) & \cdots & (x_n, y_m)
\end{array}
$$

Then given $d$ dimensions interpolate vector $v_{ij}$ rectangular mesh $\prod_{x,y}^{n,m}$ is arranged in the following array

$$
\begin{array}{cccc}
v_{00} & v_{10} & \cdots & v_{n0} \\
v_{01} & v_{11} & \cdots & v_{n1} \\
\vdots & \vdots & \ddots & \vdots \\
v_{0m} & v_{1m} & \cdots & v_{nm}
\end{array}
$$

The Samelson inverse of a bivariate vector $v(x, y) \in \mathbb{R}^d$:

$$v^{-1}(x, y) = v^*(x, y) / \|v(x, y)\|^2$$
where $v^*(x, y)$ is the conjugate vector of $v(x, y)$ and $|v(x, y)|^2$ is the norm of $v(x, y)$. Let

$$R(x, y) = \frac{N(x, y)}{D(x, y)} = \frac{(N_1(x, y), \ldots, N_d(x, y))}{D(x, y)}$$

(5)

where $N_k(x, y)(k = 1, 2, \ldots, d)$ and $D(x, y)$ are bivariate polynomials. The problem of bivariate vector-valued rational interpolation is to look for a vector-valued rational function $R(x, y)$ which satisfied interpolation conditions

$$R(x_i, y_j) = N(x_i, y_j) \quad D(x_i, y_j) = v(x_i, y_j), (x_i, y_j) \in \prod_{x,y}^{n,m}$$

The following formula is the bivariate Thiele type vector-valued rational interpolation which is of order $(n, m)$:

$$R_{n, m}(x, y) = L_0(y) + \frac{x - x_0}{L_1(y) + \cdots + \frac{x-x_{n-1}}{L_n(y)}}$$

(6)

where

$$L_i(y) = b_{i,0} + \frac{y - y_0}{b_{i,1} + \cdots + \frac{y-y_{n-1}}{b_{i,m}}}, i = 0, 1, \ldots, n$$

(7)

**Definition 1** [1] If $R(x, y) = N(x, y)/D(x, y)$ satisfied the following conditions

1. $\deg N_j \leq l, j = 1, 2, \cdots, d$, and exist one $j_0(1 \leq j_0 \leq d)$ s.t. $\deg N_{j_0} = l$;
2. $\deg D = m$, where $R(x, y)$ is of type $[l/m]$.

If $N_k(x, y)(k = 1, 2, \cdots, d)$ and $D(x, y)$ polynomials have no nonconstant common factor, then $R(x, y)$ is irreducible.

As is well known Characteristic Theorem and Uniqueness Theorem are the most important theorems about interpolant problem.

**Lemma 1** [1] (Characteristic Theorem)

The generalized inverse rational interpolation $R(x, y)$ given in (6) is of type $[(mn+m+n)/(mn+n+m)]$ when $m$ and $n$ are even; $R(x, y)$ is of type $[(mn+m+n)/(mn+n+m-1)]$ when $m$ and $n$ are not all even. We can prove the follow Uniqueness Theorem by Characteristic Theorem.

**Lemma 2** [1] (Uniqueness Theorem)

If two vector-valued rational functions $R(x, y)$ and $r(x, y)$ are given arbitrarily, which satisfied interpolation conditions $R(x_i, y_j) = r(x_i, y_j) = v_{ij}, (x_i, y_j) \in \prod_{x,y}^{n,m}$ and have the same type, then we get $R(x, y) \equiv r(x, y)$.

### 3 Bivariate Vector-valued Rational Interpolation on the Hypersphere

Before we introduce the problem of bivariate vector-valued rational interpolation on the hypersphere, we consider to introduce the problem of interpolating points of the unit hypersphere $S^{d-1}$.
by a univariate rational curve lying on $S^{d-1}$. Let $2n + 1$ vectors $v_0, v_1, \cdots, v_n \in R^d$ of norm 1 and $2n + 1$ parameter values $x_0, x_1, \cdots, x_{2n}$ (pairwise distinct) be given. A vector-valued function

\[ R(x) = \frac{N(x)}{D(x)} = \frac{(N_1(x), \cdots, N_d(x))}{D(x)} \]

(8)

where $N_i$ and $D$ are polynomials is called a solution of the problem IP if

\[ \forall x \in R, |R(x)| = 1, \forall i \in 0, \cdots, n, R(x_i) = v_i \]

(9)

**Lemma 3** [2] If the generalized inverse rational interpolant $R(x)$ of $2n + 1$ points of the hypersphere $S^{d-1}$ exists and is of type $S^{d-1}$, then $R(x)$ lies on the hypersphere $S^{d-1}$.

**Lemma 4** [4] Every irreducible rational curves on the sphere $S^2$ are of even degree.

What is called the problem of bivariate vector-valued rational interpolation is find a rational function $R(x, y)$ which is like (5). And $R(x, y)$ is satisfied Interpolation condition

\[ R(x, y) = 1, R(x_i, y_j) = \frac{N(x_i, y_j)}{D(x_i, y_j)} = v(x_i, y_j), |v(x_i, y_j)| = 1, (x_i, y_j) \in \prod_{x,y}^{n,m} \]

(10)

**Definition 2** If bivariate vector-valued function (8) where $L_i(y), i = 0, 1, \cdots, n$ which is in (7), and $R_{n,m}(x, y)$ is satisfied interpolation condition (10), then we call $R_{n,m}(x, y)$ is based on generalized inverse interpolation on the hypersphere $S^{d-1}$ with bivariate Thiele type vector-valued rational interpolant.

**Theorem 1** If the generalized inverse bivariate rational fraction $R_{n,m}(x, y)$ of $mn + m + n + 1$ points of the hypersphere $S^{d-1}$ exists and is of type $[mn + m + n/mn + m + n]$ (where $n$ and $m$ are evens), then $R_{n,m}(x, y)$ lies on the hypersphere $S^{d-1}$.

**Proof**

Given vector set $v_{x,y}^{n,m} = \{v_{ij}|v_{ij} = v(x_i, y_j) \in C^d, (x_i, y_j) \in \prod_{x,y}^{n,m}\}$. Let $R(x, y) = N(x, y)/D(x, y)$ is of type $[2k/2k]$ generalized inverse bivariate irreducible rational fraction (see Lemma 4), which is satisfied interpolation condition

\[ R(x_i, y_j) = N(x_i, y_j)/D(x_i, y_j) = v(x_i, y_j), \]

where $|v(x_i, y_j)| = 1, (x_i, y_j) \in \prod_{x,y}^{n,m}$.

We want to prove that for all $(x, y),

\[ |N(x, y)|^2 = D^2(x, y). \]

We know there exists a polynomial $Q(x, y)$ such that $|N(x, y)|^2 = Q(x, y)D(x, y)$, i.e.

\[ Q(x, y) = \frac{N(x, y) * N(x, y)}{D(x, y)}. \]
We have
\[
\frac{Q(x_i, y_j) * D(x_i, y_j)}{Q^2(x_i, y_j)} = \frac{N(x_i, y_j) * N(x_i, y_j)}{Q^2(x_i, y_j)} = R(x_i, y_j) * R(x_i, y_j) = v_i * v_i = 1
\]
where ∀\(i = 0, 1, \ldots, n; j = 0, 1, \ldots, m; x_i \in \prod_n^\infty, y_i \in \prod_m^\infty\).

Then we get for all \(i = 0, 1, \ldots, n; j = 0, 1, \ldots, m, \) \(Q(x_i, y_j) = D(x_i, y_j). \) As \(\text{deg} | N | = 2(mn + m + n), \) and \(\text{deg} D = mn + m + n, \) the \(\text{deg} Q = mn + m + n. \) We get \(Q = D\) and the result follows.

Then for arbitrary \((x_i, y_j),\) we can get \(|R(x, y)|^2 = 1.\)

**Remark 1:**
If the parameter \(x\) or \(y\) in Theorem 1 identically equal to zero, then bivariate Thiele type rational interpolation degenerate to univariate.

# 4 Algorithm

In order to compute \(L_l(y), l = 0, 1, \ldots, n\) in (10), giving the following algorithm:

**Algorithm 1**

1. The first step: For \(i = 0, 1, \ldots, n; j = 0, 1, \ldots, m; (x_i, y_j) \in \prod_n^m,\) define
   \[
   A_{0,0}^{(i,j)} = v_{ij}.
   \]

2. The second step: For \(i = 0, 1, \ldots, n; j = 0, 1, \ldots, m; k = 1, \ldots, j,\) define
   \[
   A_{i,k}^{(i,j)} = \frac{y_j - y_{k-1}}{A_{i,k-1}^{(i,j)} - A_{i,k-1}^{(i,1)}}
   \]
   We can get
   \[
   L_l(y) = A_{i,0}^{(i,0)} + \frac{y - y_0}{A_{i,1}^{(i,1)}} + \cdots + \frac{y - y_{m-1}}{A_{i,m}^{(i,m)}}
   \]
   In fact, we also get \(L_l(y_l)\) by Algorithm 1.

   \[
   L_l(y_l) = A_{i,0}^{(i,0)} + \frac{y_l - y_0}{A_{i,1}^{(i,1)}} + \cdots + \frac{y_l - y_{j-1}}{A_{i,j}^{(i,j)}}
   = A_{i,0}^{(i,0)} + \frac{y_l - y_0}{A_{i,1}^{(i,1)}} + \cdots + \frac{y_l - y_{j-1}}{A_{i,j}^{(i,j)}} = \cdots = A_{i,j}^{(i,0)}
   \]

3. The last step:
   \[
   R(x_i, y_j) = L_0(y_j) + \frac{x_i - x_0}{L_1(y_j)} + \cdots + \frac{x_i - x_{i-1}}{L_i(y_j)}
   = A_{0,0}^{(i,j)} + \frac{x_i - x_0}{A_{1,0}^{(i,1)}} + \cdots + \frac{x_i - x_{i-1}}{A_{i-1,0}^{(i,j)}} = \cdots = A_{0,0}^{(i,j)} = v_{ij}
   \]
Also, we have a matrix algorithm. This algorithm is widely used and easily understand.

Algorithm 2 [1]

Making convenience of computation, the matrix algorithm is similar to divided difference table which can direct compute by table.

Let \( A^{(i,j)} \) = \( v_{ij} \), \( i = 0, 1, \cdots, m \)

\[
A = \begin{pmatrix}
A^{(0,0)}_{0,0} & A^{(1,0)}_{0,0} & \cdots & A^{(n,0)}_{0,0} \\
A^{(0,1)}_{0,0} & A^{(1,1)}_{0,0} & \cdots & A^{(n,1)}_{0,0} \\
\vdots & \vdots & \ddots & \vdots \\
A^{(0,m)}_{0,0} & A^{(1,m)}_{0,0} & \cdots & A^{(n,m)}_{0,0}
\end{pmatrix} = \begin{pmatrix}
v_{00} & v_{10} & \cdots & v_{n0} \\
v_{01} & v_{11} & \cdots & v_{n1} \\
\vdots & \vdots & \ddots & \vdots \\
v_{0m} & v_{1m} & \cdots & v_{nm}
\end{pmatrix}
\]

By elementary column transformation:

\[
A^{(1,j)} = \frac{x_1 - x_0}{A^{(1,j)}_{0,0}} - A^{(0,j)}_{0,0}, j = 0, 1, \cdots, m.
\]

We get new matrix

\[
\begin{pmatrix}
A^{(0,0)}_{0,0} & A^{(1,0)}_{0,0} & \cdots & A^{(n,0)}_{0,0} \\
A^{(0,1)}_{0,0} & A^{(1,1)}_{0,0} & \cdots & A^{(n,1)}_{0,0} \\
\vdots & \vdots & \ddots & \vdots \\
A^{(0,m)}_{0,0} & A^{(1,m)}_{0,0} & \cdots & A^{(n,m)}_{0,0}
\end{pmatrix}
\]

Let \( i = 0, 1, \cdots, n - 1; j = i + 1, \cdots, n \) and let the new matrix, which is created by the \( j \)th column minus the \( i \)th column of \( A \).

\[
\begin{pmatrix}
A^{(0,0)}_{0,0} & A^{(1,0)}_{1,0} & \cdots & A^{(n,0)}_{n,0} \\
A^{(0,1)}_{0,0} & A^{(1,1)}_{1,0} & \cdots & A^{(n,1)}_{n,0} \\
\vdots & \vdots & \ddots & \vdots \\
A^{(0,m)}_{0,0} & A^{(1,m)}_{1,0} & \cdots & A^{(n,m)}_{n,0}
\end{pmatrix}
\]

Similarly, let this matrix by elementary row transformation: \( i = 0, 1, \cdots, m - 1; j = i + 1, \cdots, m \), and let the new matrix, which is created by the \( j \)th row minus the \( i \)th row of the matrix.

\[
\begin{pmatrix}
A^{(0,0)}_{0,0} & A^{(1,0)}_{0,1} & \cdots & A^{(n,0)}_{0,n} \\
A^{(0,1)}_{0,0} & A^{(1,1)}_{0,1} & \cdots & A^{(n,1)}_{0,n} \\
\vdots & \vdots & \ddots & \vdots \\
A^{(0,m)}_{0,0} & A^{(1,m)}_{0,m} & \cdots & A^{(n,m)}_{0,n}
\end{pmatrix}
\]

Each column of this matrix (such as the \( l \)th column) is coefficients of \( L_l(y) \).
Table 1: The value of the vectors

<table>
<thead>
<tr>
<th>v</th>
<th>x₀ = 1</th>
<th>x₁ = 10</th>
<th>x₂ = 20</th>
</tr>
</thead>
<tbody>
<tr>
<td>y₀ = 0</td>
<td>v₀₀ = (3/7, -6/7, 2)</td>
<td>v₁₀ = (-2/7, 3/7, 6)</td>
<td>v₂₀ = (6/7, 2/7, 3/7)</td>
</tr>
<tr>
<td>y₁ = 10</td>
<td>v₀₁ = (3/7, 2/7, -6/7)</td>
<td>v₁₁ = (2/7, 3/7, 6)</td>
<td>v₂₁ = (2/7, 6/7, 3/7)</td>
</tr>
<tr>
<td>y₂ = 20</td>
<td>v₀₂ = (-3/7, 2/7, 6)</td>
<td>v₁₂ = (3/7, 2/7, 6)</td>
<td>v₂₂ = (-3/7, -6/7, 2)</td>
</tr>
</tbody>
</table>

5 Application Example

We consider the nine vectors of norm 1 and the parameters defined by:

In order to find-if it exists—the unique rational fraction \( R(x, y) \) of type \( [8/8] \), which interpolates the vectors \( v_{ij} \), we apply the Algorithm 2 as described in the Introduction. We find:

\[
L₀(y) = \left( \frac{-3(y^2 + 400)}{7(y^2 + 400)}, \frac{2(3y^2 + 40y - 1200)}{7(y^2 + 400)}, \frac{-2(y^2 - 120y - 400)}{7(y^2 + 400)} \right)
\]

\[
L₁(y) = \left( \frac{35(9y^2 + 2440y + 5200)}{613y^2 - 16360y + 254800}, \frac{35(71y^2 + 920y - 46800)}{613y^2 - 16360y + 254800}, \frac{3640(y^2 - 30y + 200)}{613y^2 - 16360y + 254800} \right)
\]

and

\[
L₂(y) = \left( \frac{43627841826500011291739y^2 - 2047560669054979439141807y + 16328087977682254315350221}{187402987429836173112658y^2 - 2700109134879868408684071y + 13986854102019533053816691}, \frac{93661945270722695094623y^2 + 31733626771143423056335y + 256324743283763448460998}{187402987429836173112658y^2 - 2700109134879868408684071y + 13986854102019533053816691}, \frac{-3093662179249949733328441y^2 + 5512099123150413240431y - 3915511826539786615302137}{187402987429836173112658y^2 - 2700109134879868408684071y + 13986854102019533053816691} \right)
\]

Let \( L₀(y), L₁(y), L₂(y) \) into the following formula:

\[
R_{3,3}(x, y) = \frac{N(x, y)}{D(x, y)} = L₀(y) + \frac{x - 0}{L₁(y) + \frac{y - 10}{L₂(y)}}
\]

Then we get the bivariate Thiele type continued fraction which satisfied interpolation condition, and by tail-to-head evaluation finally get the rational curve.

Remark 2:

(a) Since by tail-to-head evaluation finally get the rational fraction is too complex, so we do not display it.

(b) There are two segments in Fig. 1. Since between \( v₂₀ \) and \( v₀₁ \) (similarly, between \( v₂₁ \) and \( v₀₂ \)) is difficult to give region of the parameters \( x \) and \( y \), so we directly take the segment though \( v₂₀ \) and \( v₀₁ \). Finally, the rational curve still satisfied interpolation condition.
6 Conclusion

In this paper, we use the Generalized Inverse Rational Interpolants which is based on the Samelson inverse for vectors to construct the bivariate Thiele type rational interpolants on the hypersphere. Let $mn + m + n + 1$ points on the unit hypersphere, we can find-if it exists-the unique solution of the problem $IP$. We generalize Thiele type rational interpolants on the hypersphere of Thierry Gensane.

References
