

Generalized Solutions for Quantum Mechanical Oscillator on Kähler Conifold

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Abstract

We study the possible generalized boundary conditions and the corresponding solutions for the quantum mechanical oscillator model on Kähler conifold. We perform it by self-adjoint extension of the initial domain of the effective radial Hamiltonian. Remarkable effect of this generalized boundary condition is that at certain boundary condition the orbital angular momentum degeneracy is restored! We also recover the known spectrum in our formulation, which of course correspond to some other boundary condition.

1 Introduction

Quantum oscillator is an important model in various branches of physics, i.e, quantum mechanics, quantum field theory, string theory and gravity due to its exact solvability and overcomplete symmetry. The symmetry is manifested through angular momentum degeneracy of the energy spectrum. It is also possible to separate the differential equation with respect to variables in few coordinate systems. The overcomplete symmetry let the harmonic oscillator to remain exactly solvable even after some deformation of the potential is made. So the symmetry is the prime issue which gives the the harmonic oscillator such a status in different fields of study. But in quantum mechanical oscillator on Kähler conifold [1], proposed in Ref. [2], on the other hand this symmetry is generally broken. It is an important model nevertheless, because it is solvable and it is defined on Kähler conifold, which is a curved space. In string theory and gravity Kähler space [3, 4] gets immense importance. It is a four dimensional quantum oscillator on the (ν, ϵ) parametric family of Kähler conifolds related to the complex projective space CP^2 for $\nu = 1$ and $\epsilon = 1$ and four dimensional Lobacewsky space \mathcal{L}_2 for $\nu = 1$ and $\epsilon = -1$.

Now the question is whether it is possible to retain the degeneracy of angular momentum in the energy spectrum of the oscillator defined in Ref. [2]. The answer is yes! In our present work we are going to address this issue. We will basically perform an one parameter family of self-adjoint extension [5] of the initial domain of the radial Hamiltonian of the harmonic oscillator [2] by von Neumann method [5]. This will help us to construct a generalized boundary condition. We will show that for a particular value of the extension parameter we can in fact recover the angular momentum degeneracy in the energy spectrum. Not only that, it is also possible to get the previously obtained result [2] for another value of the parameter and other results.

However, the importance of self-adjointness of an operator is far fundamental. As we know evolution of a quantum system is dictated by unitary group and the generator of this group is the Hamiltonian

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itself. According to Stone's theorem [5] generators of unitary group (in this case Hamiltonian) should be self-adjoint. So for a non self-adjoint operator we should search for a self-adjoint extension if possible. If the system has many self-adjoint extensions then different self-adjoint extensions should unveil different physics for the system.

The paper is organized as follows: In Sec. 2, we discuss about the quantum mechanical oscillator on Kähler manifold. In Sec. 3, we perform the self-adjoint extension of the radial Hamiltonian and we make some observations for some particular value of the extension parameter ω_0 . Here we show that it is actually possible to retain degeneracy in the energy spectrum (symmetry of the system). We discuss in Sec. 4.

2 Quantum mechanical oscillator on Kähler manifold

The Hamiltonian for the system is given by

$$\widehat{\mathcal{H}} = -\hbar^2 g^{a\bar{b}} \partial_a \partial_{\bar{b}} + V_{osc}, \quad (2.1)$$

where the metric is of the form

$$g_{a\bar{b}} = \frac{\nu r_0^2 (z\bar{z})^{\nu-1}}{2(1 + \epsilon(z\bar{z})^\nu)} \left(\delta_{ab} - \frac{1 - \nu + \epsilon(z\bar{z})^\nu}{z\bar{z}(1 + \epsilon(z\bar{z})^\nu)} \bar{z}^a z^b \right), \quad (2.2)$$

and the oscillator potential is given by

$$V_{osc} = \omega^2 g^{\bar{a}b} \partial_{\bar{a}} K \partial_b K = \frac{\omega^2 r_0^2}{2} (z\bar{z})^\nu. \quad (2.3)$$

K the potential of the Kähler structure is given by

$$K = \frac{r_0^2}{2\epsilon} \log(1 + \epsilon(z\bar{z})^\nu), \quad \nu > 0; \quad \epsilon = \pm 1, \quad (2.4)$$

In order to investigate the maximum possible solutions of the problem we need to consider the eigenvalue problem, which is

$$\widehat{\mathcal{H}}\Psi = E\Psi, \quad (2.5)$$

Equation (2.5) can be separated out in spherical coordinates

$$\begin{aligned} z^1 &= r^{\frac{1}{\nu}} \cos \frac{\beta}{2} \exp \left[\frac{i}{2}(\alpha + \gamma) \right], \\ z^2 &= -ir^{\frac{1}{\nu}} \sin \frac{\beta}{2} \exp \left[-\frac{i}{2}(\alpha - \gamma) \right], \end{aligned} \quad (2.6)$$

if we consider the trial wavefunction of the form

$$\Psi = \psi(r) D_{m,s}^j(\alpha, \beta, \gamma). \quad (2.7)$$

Here $\alpha \in [0, 2\pi)$, $\beta \in [0, \pi]$ and $\gamma \in [0, 4\pi)$, and r is a dimensionless radial coordinate taking values in the interval $[0, \infty)$ for $\epsilon = +1$, and in $[0, 1]$ for $\epsilon = -1$. In the Wigner function $D_{m,s}^j(\alpha, \beta, \gamma)$ j , m denote orbital and azimuthal quantum numbers and corresponding operators are $\widehat{\mathcal{J}}^2$, $\widehat{\mathcal{J}}_3$ respectively, while s is the eigenvalue of the operator $\widehat{\mathcal{J}}_0$.

$$\widehat{\mathcal{J}}_0 \Psi = s\Psi, \quad (2.8)$$

$$\widehat{\mathcal{J}}^2 \Psi = j(j+1)\Psi, \quad \widehat{\mathcal{J}}_3 \Psi = m\Psi, \quad (2.9)$$

$$m, s = -j, -j+1, \dots, j-1, j \quad \text{where } j = 0, 1/2, 1, \dots \quad (2.10)$$

The volume element reads

$$dV_{(4)} = \frac{\nu^2 r_0^4}{32} \frac{r^3}{(1 + \epsilon r^2)^3} \sin \beta dr d\alpha d\beta d\gamma. \quad (2.11)$$

Separating the differential equation we get the radial eigenvalue equation of the form

$$H(r)\psi(r) = E\psi(r), \quad (2.12)$$

where

$$H(r) = -\frac{\hbar^2(1+\epsilon r^2)^2}{2r_0^2} \left[\frac{d^2}{dr^2} + \frac{3+\epsilon r^2}{1+\epsilon r^2} \frac{1}{r} \frac{d}{dr} + \frac{\epsilon \omega^2 r_0^4}{\hbar^2(1+\epsilon r^2)^2} - \frac{\epsilon \delta^2}{1+\epsilon r^2} - \frac{4\nu j(j+1)+4(1-\nu)s^2}{\nu^2 r^2(1+\epsilon r^2)} \right] \quad (2.13)$$

We now move to the next section to discuss the self-adjointness of the radial Hamiltonian $H(r)$ of Eq.(2.13).

3 Self-adjointness of the radial Hamiltonian

The effective radial Hamiltonian $H(r)$, Eq.(2.13) is formally self-adjoint, but formal self-adjointness does not mean that it is self-adjoint on a given domain [6]. This operator $H(r)$ belongs to unbounded differential operator defined on a Hilbert space. As we have mentioned in our introduction that we will perform self-adjoint extension of the operator $H(r)$ by von Neumann's method [5], so for the shake of completeness here we briefly review the von Neumann method.

Let us consider an unbounded differential operator T defined over a Hilbert space \mathcal{H} and consider a domain $D(T) \subset \mathcal{H}$ for the operator T such that it becomes symmetric on the domain $D(T) \subset \mathcal{H}$. Note that the operator T is called symmetric or Hermitian if $(T\phi, \chi) = (\phi, T\chi) \quad \forall \phi, \chi \in D(T)$, where (\cdot, \cdot) is the inner product defined over the Hilbert space \mathcal{H} . Let $D(T^\dagger)$ is the domain of the corresponding adjoint operator T^\dagger . The operator T is self-adjoint iff $T = T^\dagger$ and $D(T) = D(T^\dagger)$.

We now state the criteria of self-adjointness of a symmetric operator T according to von Neumann method. We need to find out the the deficiency subspaces (it is actually a null space) $D^\pm \equiv \text{Ker}(i \mp T)$ and the deficiency indices $n^\pm(T) \equiv \dim(D^\pm)$. Depending upon n^\pm , T is classified as [5]:

- 1) T is essentially self-adjoint if $n^+ = n^- = 0$.
- 2) T has a n -parameter family of self-adjoint extension if $n^+ = n^- = n \neq 0$.
- 3) T has no self-adjoint extension if $n^+ \neq n^-$. In this case T is called maximally symmetric.

We now return to the discussion of our effective radial differential operator $H(r)$. This operator is symmetric in the domain

$$D(H(r)) = \{ \phi(r) : \phi(r) = \phi'(r) = 0, \text{ absolutely continuous, square integrable over the full range with measure } d\mu \}. \quad (3.1)$$

where $d\mu = \frac{r^3}{(1+\epsilon r^2)^3} dr$, $\phi'(r)$ is the derivative of $\phi(r)$ with respect to r . The domain of the adjoint operator $H^\dagger(r)$, whose differential expression is same as $H(r)$ due to formal self-adjointness, is given by

$$D^\dagger(H(r)) = \{ \phi(r) : \text{absolutely continuous, square integrable over the full range with measure } d\mu \}, \quad (3.2)$$

$H(r)$ is obviously not self-adjoint [5], because

$$D(H(r)) \neq D(H^\dagger(r)) \quad (3.3)$$

So we may ask whether there is any possible self-adjoint extension [5] for the problem? To answer this question we need to investigate whether there is any square-integrable solution for the differential equations

$$H(r)^\dagger \phi^\pm = \pm i \phi^\pm \quad (3.4)$$

Eq. (3.4) can be transformed into Hypergeometric [7] differential equation upon transformation

$$r = \begin{cases} \tan \theta, & \text{for } \epsilon = 1; \\ \tanh \theta, & \text{for } \epsilon = -1; \end{cases} \quad (3.5)$$

and taking the trial solution of the form

$$\phi^\pm = \begin{cases} \sin^{j_1-1} \theta \cos^\delta \theta \psi^\pm, & \text{for } \epsilon = 1; \\ \sinh^{j_1-1} \theta \cosh^{-\delta-2a^\pm} \theta \psi^\pm, & \text{for } \epsilon = -1. \end{cases} \quad (3.6)$$

The transformed differential equation is given by

$$t(1-t) \frac{d^2 \psi^\pm}{dt^2} + [c - (a^\pm + b^\pm + 1)t] \frac{d\psi^\pm}{dt} - a^\pm b^\pm \psi^\pm, \quad (3.7)$$

where

$$a^\pm = \frac{1}{2} \left(1 + j_1 + \epsilon \delta - \sqrt{\frac{\pm 2r_0^2 i}{\epsilon \hbar^2} + 4 + \frac{\omega^2 r_0^4}{\epsilon^2 \hbar^2}} \right), b^\pm = \begin{cases} -a^\pm + \delta + j_1 + 1, & \text{for } \epsilon = 1; \\ a^\pm + \delta, & \text{for } \epsilon = -1; \end{cases}$$

$$c = j_1 + 1, j_1^2 = \frac{4j(j+1)}{\nu} + 1 - \frac{4(\nu-1)s^2}{\nu^2}, \delta^2 = \frac{4s^2}{\nu^2} + \frac{\omega^2 r_0^4}{\hbar^2},$$

$$t = \begin{cases} \sin^2 \theta, & \text{for } \epsilon = 1; \\ \tanh^2 \theta & \text{for } \epsilon = -1. \end{cases} \quad (3.8)$$

The squareintegrable solutions of the deficiency space, apart from normalization is given by

$$\phi^\pm = \begin{cases} Dt^{\frac{c-2}{2}} (1-t)^{\frac{b^\pm+a^\pm-c}{2}} {}_2F_1(a^\pm, b^\pm; c; t), & \text{for } \epsilon = 1; \\ D\left(\frac{t}{1-t}\right)^{\frac{c-2}{2}} \left(\frac{1}{1-t}\right)^{-\delta-2a^\pm} {}_2F_1(a^\pm, b^\pm; c; t) & \text{for } \epsilon = -1, \end{cases} \quad (3.9)$$

where ${}_2F_1$ is the Hypergeometric function [7].

The existence of these complex eigenvalues of $H(r)^\dagger$ signifies that $H(r)$ is not self-adjoint. The solution ϕ^\pm belong to the null space D^\pm of $H(r)^\dagger \mp i$. where $D^\pm \in D^\dagger(H)$. The dimension of D^\pm are known as deficiency indices n^\pm and is defined by

$$n^\pm = \dim(D^\pm) \quad (3.10)$$

Since in our case the deficiency indices $n^+ = n^- = 1$, we can have a 1-parameter family of self-adjoint extension of $H(r)$. The selfadjoint extension of $H(r)$ is given by $H(r)^{\omega_0}$ with domain $D(H(r)^{\omega_0})$, where

$$D(H(r)^{\omega_0}) = \{\psi(r) = \phi(r) + \phi^+(r) + e^{i\omega_0} \phi^-(r) : \phi(r) \in D(H(r)), \omega_0 \in \mathbb{R}(\text{mod } 2\pi)\}. \quad (3.11)$$

The bound state solution of $H(r)^\omega$ is given by

$$\psi(r) = \begin{cases} Ct^{\frac{c-2}{2}} (1-t)^{\frac{b+a-c}{2}} {}_2F_1(a, b; c; t), & \text{for } \epsilon = 1; \\ C\left(\frac{t}{1-t}\right)^{\frac{c-2}{2}} \left(\frac{1}{1-t}\right)^{-\delta-2a} {}_2F_1(a, b; c; t), & \text{for } \epsilon = -1; \end{cases} \quad (3.12)$$

where

$$a = \frac{1}{2} \left(1 + j_1 + \epsilon \delta - \sqrt{\frac{2r_0^2 E}{\epsilon \hbar^2} + 4 + \frac{\omega^2 r_0^4}{\epsilon^2 \hbar^2}} \right), b = \begin{cases} -a + \delta + j_1 + 1, & \text{for } \epsilon = 1; \\ a + \delta & \text{for } \epsilon = -1; \end{cases}$$

$$c = j_1 + 1, t = \begin{cases} \sin^2 \theta, & \text{for } \epsilon = 1; \\ \tanh^2 \theta, & \text{for } \epsilon = -1; \end{cases} \quad (3.13)$$

and C is the normalization constant. To find out the eigenvalue we have to match the function $\psi(r)$ with the domain (3.11) at $r \rightarrow 0$. In the limit $r \rightarrow 0$,

$$\psi(r) \rightarrow \begin{cases} Ct^{\frac{c-2}{2}} (1-t)^{\frac{b+a-c}{2}} [\Gamma_1 + (1-t)^{c-a-b} \Gamma_2], & \text{for } \epsilon = 1; \\ Ct^{\frac{c-2}{2}} [\bar{\Gamma}_1 + (1-t)^{1+\frac{c}{2}} \bar{\Gamma}_2], & \text{for } \epsilon = -1; \end{cases} \quad (3.14)$$

where

$$\Gamma_1 = \frac{\Gamma(c)\Gamma(c-a-b)\Gamma(a+b-c+1)\Gamma(1-c)}{\Gamma(c-a)\Gamma(c-b)\Gamma(b-c+1)\Gamma(a-c+1)} \quad (3.15)$$

$$\Gamma_2 = \frac{\Gamma(c)\Gamma(a+b-c)\Gamma(c-a-b+1)\Gamma(1-c)}{\Gamma(a)\Gamma(b)\Gamma(1-b)\Gamma(1-a)} \quad (3.16)$$

and

$$\phi^+(r) + e^{i\omega_0} \phi^-(r) \rightarrow \begin{cases} Dt^{\frac{c-2}{2}} (1-t)^{\frac{b+a-c}{2}} [\bar{\Gamma}_1 + (1-t)^{c-a-b} \bar{\Gamma}_2], & \text{for } \epsilon = 1; \\ Dt^{\frac{c-2}{2}} [\bar{\Gamma}_1 + (1-t)^{1+\frac{c}{2}} \bar{\Gamma}_2], & \text{for } \epsilon = -1; \end{cases} \quad (3.17)$$

where

$$\bar{\Gamma}_1 = \frac{\Gamma(c)\Gamma(c-a^+-b^+)\Gamma(a^++b^+-c+1)\Gamma(1-c)}{\Gamma(c-a^+)\Gamma(c-b^+)\Gamma(b^+-c+1)\Gamma(a^+-c+1)} + e^{i\omega_0} \frac{\Gamma(c)\Gamma(c-a^-b^-)\Gamma(a^-+b^-+c+1)\Gamma(1-c)}{\Gamma(c-a^-)\Gamma(c-b^-)\Gamma(b^-+c+1)\Gamma(a^-+c+1)}, \quad (3.18)$$

$$\bar{\Gamma}_2 = \frac{\Gamma(c)\Gamma(a^++b^+-c)\Gamma(c-a^+-b^++1)\Gamma(1-c)}{\Gamma(a^+)\Gamma(b^+)\Gamma(1-b^+)\Gamma(1-a^+)} + e^{i\omega_0} \frac{\Gamma(c)\Gamma(a^-+b^-+c)\Gamma(c-a^-+b^-+1)\Gamma(1-c)}{\Gamma(a^-)\Gamma(b^-)\Gamma(1-b^-)\Gamma(1-a^-)} \quad (3.19)$$

Now comparing the respective coefficients in Eq. (3.14) and Eq. (3.17) we get the eigenvalue equation,

$$f(E) = \frac{\Gamma(a)\Gamma(b)\Gamma(1-b)\Gamma(1-a)}{\Gamma(c-a)\Gamma(c-b)\Gamma(b-c+1)\Gamma(a-c+1)} = \frac{\sin(c-b)\pi \sin(c-a)\pi}{\sin a\pi \sin b\pi} = \mathcal{M} \frac{\cos(\beta + \omega_0/2)}{\cos(\alpha + \omega_0/2)} \quad (3.20)$$

where

$$\Gamma(a^\pm) = \chi_1 e^{\pm i\alpha_1}, \Gamma(b^\pm) = \chi_2 e^{\pm i\alpha_2}, \Gamma(1-a^\pm) = \chi_3 e^{\pm i\alpha_3}, \Gamma(1-b^\pm) = \chi_4 e^{\pm i\alpha_4}, \quad (3.21)$$

$$\Gamma(c-a^\pm) = \lambda_1 e^{\pm i\beta_1}, \Gamma(c-b^\pm) = \lambda_2 e^{\pm i\beta_2}, \Gamma(b^\pm - c + 1) = \lambda_3 e^{\pm i\beta_3}, \Gamma(a^\pm - c + 1) = \lambda_4 e^{\pm i\beta_4}, \quad (3.22)$$

$$\mathcal{M} = \frac{\lambda_1 \lambda_2 \lambda_3 \lambda_4}{\chi_1 \chi_2 \chi_3 \chi_4}, \beta = \beta_1 + \beta_2 + \beta_3 + \beta_4, \alpha = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4. \quad (3.23)$$

The eigenvalue for general value of ω_0 can be calculated numerically. But we can immediately calculate the eigenvalue analytically at least for some values of the extension parameter ω_0 in the boundary condition. So to appreciate constructing generalized boundary condition we now investigate some special cases.

3.1 Case 1

When the right hand side of Eq. (3.20) is infinity, we get $a = \pm n$ or $b = \pm n$. $a = -n$ leads to the eigenvalue, already calculated in Ref. [2],

$$E_{n,j,s} = \begin{cases} \frac{\hbar^2}{2r_0^2} \left[(2n + j_1 + \delta + 1)^2 - 4 - \frac{\omega^2 r_0^4}{\hbar^2} \right], & \text{for } \epsilon = 1. \\ -\frac{\hbar^2}{2r_0^2} \left[(2n + j_1 - \delta + 1)^2 - 4 - \frac{\omega^2 r_0^4}{\hbar^2} \right], & \text{for } \epsilon = -1. \end{cases} \quad (3.24)$$

The radial quantum number is given by

$$n = \begin{cases} 0, 1, \dots, \infty, & \text{for } \epsilon = 1. \\ 0, 1, \dots, n^{\max} = [\delta/2 - j - 1], & \text{for } \epsilon = -1. \end{cases} \quad (3.25)$$

For $a = +n$ the energy spectrum will be the same expression (3.24), with n replaced by $-n$. For $b = +n$, the energy spectrum will be

$$E_{n,j,s} = \begin{cases} \frac{\hbar^2}{2r_0^2} \left[(2n - j_1 - 1 - \delta)^2 - 4 - \frac{\omega^2 r_0^4}{\hbar^2} \right], & \text{for } \epsilon = 1. \\ -\frac{\hbar^2}{2r_0^2} \left[(-2n + j_1 + 1 + \delta)^2 - 4 - \frac{\omega^2 r_0^4}{\hbar^2} \right], & \text{for } \epsilon = -1. \end{cases} \quad (3.26)$$

for $b = -n$, n in (3.26) will be replaced by $-n$ and radial quantum number n is given in (3.25).

3.2 Case 2

We can also make the right hand side of Eq. (3.20) zero, which gives us $c - b = \pm n$ or $c - a = \pm n$. for $c - b = +n$, the energy spectrum becomes,

$$E_{n,j,s} = \begin{cases} \frac{\hbar^2}{2r_0^2} \left[(-2n + j_1 + 1 - \delta)^2 - 4 - \frac{\omega^2 r_0^4}{\hbar^2} \right], & \text{for } \epsilon = 1. \\ -\frac{\hbar^2}{2r_0^2} \left[(2n - j_1 - 1 + \delta)^2 - 4 - \frac{\omega^2 r_0^4}{\hbar^2} \right], & \text{for } \epsilon = -1. \end{cases} \quad (3.27)$$

for $c - b = -n$, n in (3.27) will be replaced by $-n$ and radial quantum number n is given in (3.25). For $c - a = n$,

$$E_{n,j,s} = \begin{cases} \frac{\hbar^2}{2r_0^2} \left[(2n - j_1 - 1 + \delta)^2 - 4 - \frac{\omega^2 r_0^4}{\hbar^2} \right], & \text{for } \epsilon = 1. \\ -\frac{\hbar^2}{2r_0^2} \left[(2n - j_1 - 1 - \delta)^2 - 4 - \frac{\omega^2 r_0^4}{\hbar^2} \right], & \text{for } \epsilon = -1. \end{cases} \quad (3.28)$$

For $c - a = -n$, n in (3.28) will be replaced by $-n$ and radial quantum number n is given in (3.25).

3.3 Case 3

On the other hand if we make the right hand side ± 1 , then we get degenerate (degenerate with respect to orbital quantum no j_1) eigenvalue. For $c - b = +n + b$ and $c - a = +n + a$, we get,

$$E_{n,s} = \frac{\hbar^2}{2r_0^2} \left[(n + \delta)^2 - 4 - \frac{\omega^2 r_0^4}{\hbar^2} \right], \text{ for } \epsilon = 1. \quad (3.29)$$

For $c - b = +n + b$ and $c - a = -n + a$ we get,

$$E_{n,s} = -\frac{\hbar^2}{2r_0^2} \left[(n + \delta)^2 - 4 - \frac{\omega^2 r_0^4}{\hbar^2} \right], \text{ for } \epsilon = -1. \quad (3.30)$$

3.4 Case 4

Even if, we can get totally degenerate eigenvalue when $c - b = c - a \pm n$ and the form of the spectrum is given by

$$E_n = \frac{\hbar^2}{2r_0^2} \left[n^2 - 4 - \frac{\omega^2 r_0^4}{\hbar^2} \right], \text{ for } \epsilon = +1. \quad (3.31)$$

For $a + b + c = \pm n$ we get,

$$E_n = -\frac{\hbar^2}{2r_0^2} \left[n^2 - 4 - \frac{\omega^2 r_0^4}{\hbar^2} \right], \text{ for } \epsilon = -1. \quad (3.32)$$

We have so far discussed the oscillator, where the dimension of the complex coordinate is $N = 2$. But we can generalize it for arbitrary dimensions $N > 1$. The arbitrary dimensional conic oscillator Hamiltonian can be constructed from conic oscillator of Ref. [8] by making the magnetic field zero. Once the oscillator Hamiltonian is given for general dimensions the rest of the work of making self-adjoint extension is exactly same as what we have done above.

4 Discussion

In conclusion, we have calculated a generalized boundary condition for the harmonic oscillator [2] and we have shown that this generalized boundary condition can restore the angular momentum degeneracy in energy spectrum for a fixed value of the extension parameter. we have also recovered the result of Ref. [2] in our work. Not only that, we have shown that it allows more solutions for different values of the extension parameter.

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