On polygon numbers of circle graphs and distance hereditary graphs

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Abstract

Circle graphs are intersection graphs of chords in a circle and $k$-polygon graphs are the intersection graphs of chords in a convex $k$-sided polygon where each chord has its endpoints on distinct sides. Every $k$-polygon graph is a circle graph and every circle graph is a $k$-polygon graph for some $k$. The polygon number $\psi(G)$ of a circle graph $G$ is the minimum $k$ such that $G$ is a $k$-polygon graph and the polygon number of a circle representation is the minimum number of corners that must be added to the circle to produce a polygon representation. Given a circle graph $G$ and an integer $k$, determining whether $\psi(G) \leq k$ is NP-complete, while the problem is solvable in polynomial time for fixed $k$, and the polygon number of a circle representation can be computed in polynomial time [10].

In this paper, we give bounds on $\psi(G)$ when $G$ is an arbitrary circle graph – upper bounds in terms of the independence number, the clique cover number, and the number of vertices of $G$, and a lower bound in terms of the asteroidal number of $G$ – and show that $\psi(G)$ is equal to the asteroidal number of $G$ when $G$ is a connected nonclique distance hereditary graph. We then use our results to develop characterizations of distance hereditary permutation graphs. Finally, we

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describe linear time algorithms for finding the polygon number of a circle representation and for finding the asteroidal number of a distance hereditary graph, improvements over previously known algorithms for those problems.

Key words: circle graph, $k$-polygon graph, permutation graph, distance hereditary graph, polygon number, asteroidal number

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1 Introduction and Preliminaries

The $k$-polygon graphs, for $k \geq 2$, form an infinite chain of graph classes, each of which contains the class of permutation graphs, and the union of which is the class of circle graphs. For a given circle graph, the minimum value of $k$ for which the graph is a $k$-polygon graph is the polygon number of the graph. Several problems that are known to be NP-hard on circle graphs admit polynomial time algorithms for $k$-polygon graphs when $k$ is fixed, including domination and independence problems, the topological via minimization problem in circuit design [11], and vertex colouring with a fixed number of colours [22]. In addition, it has been shown in [8] that small collective additive tree spanners can be constructed efficiently for $k$-polygon graphs when $k$ is fixed. The running times of those algorithms are of the form $O(f(|V|) \cdot |V|^{g(k)})$ where $V$ is the vertex set of the input graph and $f$ and $g$ are polynomial functions. Bounds on the polygon number of a circle graph provide estimates on the running times of such algorithms and give insight into the structure of circle graphs.

In this paper, we give bounds on the polygon number of a circle graph in terms of the clique cover number, the independence number, the number of vertices, and the asteroidal number. We then show that the polygon number is equal to the asteroidal number for connected nonclique distance hereditary graphs. These results lead to characterizations of distance hereditary permutation graphs in the same vein as the characterizations of distance hereditary chordal graphs and distance hereditary comparability graphs of [17] and [7]. They also give rise to linear time algorithms for computing the polygon number of a given circle representation (an improvement over the $O(|V|^2)$ algorithm of [10]) and for computing the asteroidal number of
a distance hereditary graph (an improvement over the \( O(|V|^3 + |V|^{3/2}|E|) \) algorithm of \[19\] which handles HHD-free graphs, a superclass of distance hereditary graphs).

We begin with terminology and preliminaries for graphs and for intersection representations of circle and \( k \)-polygon graphs. Additional definitions and notation are introduced throughout the paper as needed.

### 1.1 Graph-Theoretic Terminology

All of the graphs that we consider are finite, simple, and undirected; the graph classes considered are all *hereditary*, that is, closed under taking induced subgraphs. For terms not defined here, please refer to \[14\].

In graph \( G = (V, E) \), the *neighbourhood* of vertex \( v \) is \( N_G(v) = \{ w \mid vw \in E \} \) and the *closed neighbourhood* of \( v \) is \( N_G[v] = N(v) \cup \{ v \} \). The neighbourhood of a subset \( W \) of \( V \) is \( N_G(W) = \{ v \in V - W \mid wv \in E \text{ for some } w \in W \} \). The *degree* of a vertex \( v \) in graph \( G \), denoted \( d_G(v) \), is equal to \( |N_G(v)| \). A vertex of degree 1 is called a *leaf*. In all definitions, we omit the subscript \( G \) when the context is clear.

We use \( G - V' \) and \( G - E' \) as shorthand for the subgraph of \( G \) induced by \( V - V' \), and the graph \( (V, E - E') \), respectively. A *cut vertex* of a connected graph \( G \) is a vertex \( v \) such that \( G - \{ v \} \) is disconnected.

For graph \( G \), \( \alpha(G) \) is the size of a maximum independent set and \( \kappa(G) \) is the size of a minimum clique cover. It is always the case that \( \alpha(G) \leq \kappa(G) \). Using the notation of \[19\], a set \( A \subseteq V \) is called an *asteroidal set* if for every vertex \( a \in A \), there is a path between each pair of vertices \( x, y \in A \setminus \{ a \} \) in \( G - N[a] \). The *asteroidal number*, denoted \( an(G) \), of \( G \) is the cardinality of a maximum asteroidal set of \( G \). An *asteroidal triple* (or \( AT \)) is an asteroidal set of size three.

A graph is called *AT-free* if it has no asteroidal triple or equivalently if its asteroidal number is at most two. A graph is a *comparability graph* if its edges can be transitively oriented, and a *cocomparability graph* if its complement is a comparability graph. A graph \( G \) is a *distance hereditary graph* if, for every connected induced subgraph \( H \) of \( G \), the distance between each pair of vertices of \( H \) is the same in \( H \) as it is in \( G \). We refer the reader to \[3\] for more information about graph classes.

The *intersection graph* of a finite collection of sets is the graph containing one vertex for each set in which two vertices are joined by an edge if and only if the intersection of the corresponding sets is not empty.
A graph is a circle graph if it is the intersection graph of a set of chords of a circle. For \( k \geq 3 \), a graph is a \( k \)-polygon graph if it is the intersection graph of chords inside a convex polygon with \( k \) sides such that each chord has its endpoints on two distinct sides of the polygon. A graph \( G = (V, E) \) where \( V = \{v_1, \ldots, v_n\} \) is a permutation graph if there exists a permutation \( \pi \) of \( \{1, 2, \ldots, n\} \) such that \( v_i v_j \in E \) if and only if \( (i - j) \cdot (\pi_i^{-1} - \pi_j^{-1}) < 0 \). Equivalently, permutation graphs are the intersection graphs of straight line segments connecting two parallel lines. For reasons that will be made evident in Section 1.2, permutation graphs are considered to be 2-polygon graphs, thereby allowing us to define the set of \( k \)-polygon graphs for any \( k \geq 2 \). With these definitions in hand, it is easy to see that:

\[
\text{permutation graphs} \equiv \text{2-polygon graphs} \subset \text{3-polygon graphs} \subset \ldots \subset \sum_{k=2}^{\infty} k\text{-polygon graphs} \equiv \text{circle graphs}
\]

The polygon number of a circle graph \( G \), \( \psi(G) \) (called \( k(G) \) in [10]) is the minimum value of \( k \) such that \( G \) is a \( k \)-polygon graph. In [10], Elmallah and Stewart showed that the problem of determining if \( \psi(G) \leq k \) for some given circle graph \( G \) and an integer \( k \) is NP-complete, and they gave a polynomial time algorithm for solving the problem when \( k \) is a fixed integer. They also showed the following relationship between the polygon number of a disconnected graph and the polygon numbers of its connected components.

**Theorem 1** [10] For any circle graph \( G \) with connected components \( G_1, G_2, \ldots, G_r \),

\[
\psi(G) = \left( \sum_{i=1}^{r} \psi(G_i) \right) - 2(r - 1).
\]

### 1.2 Intersection Representations

An intersection representation of a graph \( G = (V, E) \) where \( V = \{v_1, v_2, \ldots, v_n\} \) is a collection of sets \( S = \{s_1, s_2, \ldots, s_n\} \) such that for all \( 1 \leq i, j \leq n, i \neq j \), \( v_i v_j \in E \) if and only if \( s_i \cap s_j \neq \emptyset \). In general, \( S \) may be a multiset but in this paper, for reasons that will be made clear later in this section, we restrict our attention to intersection representations that consist of distinct elements.

In an intersection representation \( S = \{s_1, s_2, \ldots, s_n\} \), the neighbourhood of \( s_i \in S \) is \( N_S(s_i) = \{s_j \in S \mid i \neq j \text{ and } s_i \cap s_j \neq \emptyset\} \) and the degree of \( s_i \)
is $d_S(s_i) = |N_S(s_i)|$. The subscript $S$ may be omitted when the context is clear.

A set of chords of a circle of which $G$ is the intersection graph is called a circle representation for $G$. For example, Figure 1(a) shows a circle representation for $C_5$, the five vertex chordless cycle, where each chord $c_i$ has endpoints at points labelled as $e_i$ and $e'_i$. We will use the following notation to express the relative order of chord endpoints and other points on the circle. Given $\ell$ distinct points on a circle (each of which may or may not be a chord endpoint), we use $\langle p_0, p_1, \ldots, p_{\ell-1} \rangle$ to mean that in a clockwise traversal of the circle beginning at point $p_0$ and ending before $p_0$ is encountered a second time, points $p_0, \ldots, p_{\ell-1}$ are encountered in that order, although points other than $p_0, \ldots, p_{\ell-1}$ may also be encountered. For example, in Figure 1(a), $e'_0, e_1, e_3$, and $e'_4$ may be referred to as being in the order given by $\langle e'_4, e_1, e'_0, e_3 \rangle$. Two points $p_0$ and $p_1$ divide the circle into the two arcs: $(p_0, p_1)$, the open arc that is traced in a clockwise traversal of the circle beginning at $p_0$ and ending at $p_1$, and $(p_1, p_0)$ which is defined analogously. For chord $c$ with endpoints at points $e$ and $e'$ we say that $(e, e')$ and $(e', e)$ are the arcs of $c$. An arc of the circle is defined to be empty if it does not contain both endpoints of any chord. A chord that has an empty arc is said to be peripheral and we use $C_\emptyset$ to denote the set of peripheral chords of a set of chords $C$. For example, all chords shown in Figure 1(a) are peripheral, and of the arcs of $c_0$, $(e_0, e'_0)$ is empty and $(e'_0, e_0)$ is not empty. Since the graphs that we consider are finite, every circle representation containing at least two chords has two or more

![Examples of circle and polygon representations](image-url)
peripheral chords.

**Observation 2** A chord has two empty arcs if and only if it represents a universal vertex in the graph. If a circle representation has such a chord then the graph is a permutation graph.

**Observation 3** The empty arcs of two distinct peripheral chords in a circle representation contain a common point if and only if the chords intersect.

A *k*-polygon representation of a graph \( G \) is a set of chords of a \( k \)-sided convex polygon where each chord has its endpoints on two distinct sides, such that \( G \) is the intersection graph of the chords. A circle representation for graph \( G \) can be transformed into a \( k \)-polygon representation for \( G \) (for large enough \( k \)) by appropriately adding a set of points \( T \), where \( |T| = k \). These points represent the corners of the polygon, and will be referred to as such. A chord is said to be satisfied by the added corners if it has a corner in each of its arcs. If all chords in the representation are satisfied then we call the added corners a satisfying set of corners and the polygon representation can then be constructed by straightening the arcs between consecutive corners. This means that for any two of the added corners, denoted \( \tau_0 \in T \) and \( \tau_1 \in T \), such that \( (\tau_0, \tau_1) \) does not contain any other corner location in \( T \), the arc will be replaced by a straight line on which the relative ordering of endpoints remains the same. In doing so, the set of chord crossings in the representation does not change. Applying this transformation for each such pair of neighbouring corners in \( T \) results in a \( k \)-polygon representation for \( G \). For this reason, we will normally view a \( k \)-polygon representation as a circle representation with the addition of \( k \) corners. For example, Figure 1(b) shows a 3-polygon representation of \( C_5 \) which was constructed by adding three corners – labelled \( \tau_0, \tau_1, \) and \( \tau_2 \) – to the circle representation shown in Figure 1(a). \( T = \{\tau_0, \tau_1, \tau_2\} \) is a satisfying set of corners as is any superset of \( T \). Figure 1(c) then shows the result of the arc straightening procedure described above.

A *permutation representation* for a graph \( G \) is a set of straight line segments connecting two parallel lines, the intersection graph of which is \( G \). If a satisfying set of corners for a circle representation contains only two distinct corners \( \tau_0 \) and \( \tau_1 \), then a permutation representation can be constructed by replacing \( (\tau_0, \tau_1) \) and \( (\tau_1, \tau_0) \) by parallel lines and maintaining the relative ordering of the endpoints on these new lines. It is this transformation and its
inverse that allow us to refer to permutation graphs as 2-polygon graphs. We will use the terms permutation representation and 2-polygon representation interchangeably.

Without loss of generality, we assume that chord and line segment endpoints are distinct in circle, k-polygon, and permutation representations, and that chord endpoints are distinct from the corners in k-polygon representations. Thus, the intersection representations that we consider will be sets of distinct elements.

The definition of a satisfying set of corners immediately leads to the following observations (where Observation 5 follows from the fact that if a set of corners satisfies a set of chords, then that set of corners satisfies any subset of those chords):

**Observation 4** Consider a circle representation for graph \( G \) to which corners have been added at distinct points \( \tau_0, \tau_1, \ldots, \tau_{k-1} \), where \( k \geq 2 \). The result is a k-polygon representation for \( G \) if and only if the arcs \((\tau_i, \tau_{i+1})\), for \( 0 \leq i < k - 1 \), and \((\tau_{k-1}, \tau_0)\) are all empty.

**Observation 5** For any circle graph \( G \) and any induced subgraph \( H \) of \( G \), \( \psi(H) \leq \psi(G) \).

By the definition of peripheral chords, any chord in \( C - C_\emptyset \) contains a chord from \( C_\emptyset \) in each of its arcs. This implies the next observation.

**Observation 6** If a set of corners satisfies the peripheral chords of a representation, then it satisfies all chords of the representation.

Different circle representations for a graph may require different numbers of corners to be added to produce polygon representations. The polygon number of a circle representation \( C \), denoted by \( \psi_r(C) \), is the minimum number of corners that must be added to \( C \) in order to satisfy all chords of \( C \).

**Observation 7** For any circle graph \( G \),

\[
\psi(G) = \min_{C \text{ where } C \text{ is a circle representation for } G} \psi_r(C).
\]

By Observation 2, if any chord of \( C \) has two empty arcs then \( \psi_r(C) = 2 \). Therefore:
Observation 8 If $\psi_r(C) > 2$ then every chord of $C$ has a nonempty arc.

A quadratic algorithm to compute the polygon number of a circle representation is given in [10]; in Section 4 we describe a linear time algorithm for that problem.

2 Bounds on the polygon number

In this section, we give upper bounds on $\psi(G)$ in terms of $\kappa(G)$, $\alpha(G)$, and the number of vertices of $G$, and a lower bound in terms of $\alpha n(G)$.

The first bound relies on properties of proper circular-arc graphs. A circular-arc graph is the intersection graph of arcs of a circle; a collection of arcs of which the intersection graph is $G$ is a circular-arc representation for $G$. A circular-arc graph is a proper circular-arc graph if it has a proper circular-arc representation, i.e. a circular-arc representation in which no arc is contained in any other. Proper circular-arc graphs are circle graphs since every proper circular-arc graph has a proper circular-arc representation in which no two arcs cover the circle, and any such representation can be transformed into a circle representation in which all chords are peripheral by replacing each arc with a chord joining its endpoints (see Theorem 8.18 of [14] and Observation 3). Conversely, a circle representation containing only peripheral chords can be transformed into a proper circular-arc representation by replacing each chord with one of its empty arcs. This leads to the following observation:

Observation 9 Graph $G$ is a proper circular-arc graph if and only if it has a circle representation in which all chords are peripheral.

We now identify a subset of the chords of an arbitrary circle representation $C$ such that the clique cover number of the intersection graph of the subset dictates $\psi_r(C)$.

Theorem 10 Let $C$ be a circle representation for a graph $G$ that is not a clique. Then $\psi_r(C) = \kappa(G_{C_\emptyset})$ where $G_{C_\emptyset}$ is the intersection graph of $C_\emptyset$.

Proof. Let $C_\emptyset$ be the peripheral chords of $C$ where each $c_i \in C_\emptyset$ has endpoints $e_i$ and $e'_i$. The graph $G_{C_\emptyset}$ is a proper circular-arc graph by Observation 9. If $C = C_\emptyset$, then $G_{C_\emptyset}$ is not a clique by the conditions of the theorem. If
If $C \neq C_\emptyset$, then $C_\emptyset$ contains at least two peripheral chords – one in each arc of a non-peripheral chord of $C$ – that do not intersect each other; therefore $G_{C_\emptyset}$ is not a clique. Thus in either case, $\kappa(G_{C_\emptyset}) \geq 2$.

Consider a set of $\psi_r(C)$ corners whose addition to $C$ produces a $\psi_r(C)$-polygon representation for $G$. Each empty arc of a peripheral chord contains a corner and, for each corner, the set of peripheral chords with an empty arc that contains that corner forms a clique by Observation 3. Therefore, $\kappa(G_{C_\emptyset}) \leq \psi_r(C)$.

In the remainder of the proof we show that $\psi_r(C) \leq \kappa(G_{C_\emptyset})$. If $G$ has a universal vertex then the chord associated with that vertex has two empty arcs and therefore $\psi_r(G) = 2 \leq \kappa(G_{C_\emptyset})$ by Observation 4. From now on, we assume that $G$ has no universal vertex.

Tucker [21] showed that in any circular-arc representation for a circular-arc graph with clique cover number two, there are two points on the circle such that every arc contains one or both of the points. Thus, if $\kappa(G_{C_\emptyset}) = 2$, there are two such points with respect to the empty arcs of $C_\emptyset$. No empty arc of a chord $c \in C_\emptyset$ can contain both of the points, as that would contradict that the chord with both endpoints in the nonempty arc of $c$ (which exists since $G$ has no universal vertex) contains one of the points. Therefore, adding corners at those two points yields a $\kappa(G_{C_\emptyset})$-polygon representation for $G_{C_\emptyset}$, which is also a $\kappa(G_{C_\emptyset})$-polygon representation for $G$ by Observation 6. Therefore, $\psi_r(C) \leq \kappa(G_{C_\emptyset})$ if $\kappa(G_{C_\emptyset}) = 2$.

To complete the proof, suppose that $\kappa(G_{C_\emptyset}) \geq 3$ and consider a minimum clique partition for $G_{C_\emptyset}$. If every clique in the partition corresponds to a set of chords whose empty arcs contain a point in common, then adding corners at the common points gives a $\kappa(G_{C_\emptyset})$-polygon representation for $G_{C_\emptyset}$ and therefore for $G$. Now suppose that this is not true, and so there is some clique in the partition – corresponding to the set of chords $K$ – such that the empty arcs of the chords of $K$ do not all contain a point in common. Let $c_f$ and $c_g$ be chords in $K$ such that the intersection of the empty arcs of these chords does not contain an endpoint from any of the other chords in $K$. Such a pair of chords exists since $K$ is finite. Note that we will use the convention that $(e_i, e'_i)$ is the empty arc of chord $c_i$ in this proof, though this convention is not assumed in the remainder of this paper unless otherwise stated. As such, the empty arcs of $c_f$ and $c_g$ are $(e_f, e'_f)$ and $(e_g, e'_g)$, respectively, and we can assume that $(e_f, e_g, e'_f, e'_g)$ without loss of generality. This means that $(e_g, e'_f)$ is the arc corresponding to the intersection of the empty arcs of $c_f$ and $c_g$. By the choice of $c_f$ and $c_g$, the empty arc of each other chord of $K$
Figure 2: Three pairwise crossing peripheral chords whose empty arcs (shown) do not contain a point in common

either contains or is disjoint from the arc \((e_g, e'_f)\). Since we assumed that the empty arcs of chords in \(K\) do not all have a point in common, there must be at least one chord in \(K\) whose empty arc is disjoint from \((e_g, e'_f)\). Let \(c_h\) be such a chord. Since \(c_h\) intersects both \(c_f\) and \(c_g\), it must be that \(c_f, c_g,\) and \(c_h\) cover the circle and \(\langle e_f, e'_h, e_g, e'_f, e_h, e'_g \rangle\) holds. This situation is depicted in Figure 2.

Now consider some \(c_i \in C_{\emptyset} - \{c_f, c_g, c_h\}\) with empty arc \((e_i, e'_i)\), the existence of which is guaranteed by the fact that \(\kappa(G_{C_{\emptyset}}) \geq 3\). If \(e_i \in (e_f, e_g)\) then \(e'_i \in (e'_f, e_f)\) or else the fact that \((e_f, e'_f)\) is empty is contradicted. As a result, \((e_i, e'_i)\) contains \((e_g, e'_f)\). Similarly, if \(e_i \in (e_g, e_h)\) then \((e_i, e'_i)\) contains \((e_h, e'_g)\), and if \(e_i \in (e_h, e_f)\) then \((e_i, e'_i)\) contains \((e_f, e'_h)\).

Therefore, all chords in \(C_{\emptyset}\) have an empty arc containing at least one of \((e_f, e'_h), (e_g, e'_f), \) or \((e_h, e'_g)\). Thus, we can partition \(C_{\emptyset}\) into three sets such that the empty arcs of all chords in the same set contain the same one of these intervals. By Observation \(3\) each partition forms a clique. Putting a corner in each interval yields a 3-polygon representation for \(C\), and therefore \(\psi_r(C) \leq \kappa(G_{C_{\emptyset}})\).

In Section 4, we will show how Theorem 10 and the algorithm of [18] lead
to a linear time algorithm for the polygon number of a circle representation.

Now Theorem 10 combined with Observation 7 and the fact that $G_{C_0}$ is an induced subgraph of $G$ implies:

**Corollary 11** For any nonclique circle graph $G$, $\psi(G) \leq \kappa(G)$.

By Corollary 11, cobipartite circle graphs are permutation graphs and this, combined with the fact that permutation graphs are closed under complement, implies a result of [2] that any bipartite graph whose complement is a circle graph is itself a circle graph. Cobipartite circle graphs show that the bound of Corollary 11 is tight, while permutation graphs demonstrate that $\kappa(G)$ can be arbitrarily larger than $\psi(G)$.

We now turn our attention to bounds on $\psi(G)$ in terms of $\alpha(G)$ and the number of vertices of $G$. First we introduce the concept of an independent set in series and a technical result (Theorem 14) that leads to the next two bounds.

**Definition 12** A sequence $c_0, c_1, \ldots, c_{\ell-1}$ of $\ell \geq 1$ chords in a circle is an $\ell$-independent set in series if there is a point $x$ on the circle, such that, in a clockwise traversal of the circle starting at $x$ and ending before $x$ is encountered a second time, both endpoints of $c_i$ are encountered before both endpoints of $c_{i+1}$ for all $0 \leq i < \ell - 1$.

A single chord is a 1-independent set in series and a pair of non-crossing chords forms a 2-independent set in series. Note that if a circle representation contains an $\ell$-independent set in series, then it contains an $\ell$-independent set in series composed entirely of peripheral chords. Since, in a polygon representation, each peripheral chord must have a corner in its empty arc, we have the following.

**Observation 13** For any circle representation $C$,

$$\psi_r(C) \geq \max\{ \ell \mid C \text{ has an } \ell\text{-independent set in series} \}.$$ 

For the next results, we define a labelling of the corners and sides of a $k$-polygon representation as follows. We use $\tau_0, \ldots, \tau_{k-1}$ to denote the $k$ corners, where $\tau_0$ is an arbitrary corner and $\langle \tau_0, \ldots, \tau_{k-1} \rangle$. We also use $s_{i-1}$ and $s_i$ to denote the sides (in clockwise order, arithmetic modulo $k$) that meet at $\tau_i$. We say that a chord is in a corner $\tau$ if its endpoints are on the two sides that
meet at \( \tau \). If \( k = 2 \) then all chords are in both corners; for \( k > 2 \), each chord is in at most one corner. We use \( C_\tau \) to denote the set of chords of \( C \) that are in corner \( \tau \). A chord endpoint \( e \) is said to be close to a corner \( \tau \) if \( e \) is on one of the sides that meet at \( \tau \) and there is no other endpoint on that side between \( e \) and \( \tau \). We say that a chord \( c \) is close to a corner \( \tau \) if at least one of its endpoints is close to \( \tau \). An extreme chord in a polygon representation is one that is close to a corner. See Figure 3 for examples of these concepts.

Next, we examine the size and structure of independent sets in series in polygon representations that have the minimum number of sides. In order to do so, we need to define a polygon representation transformation called sliding. Sliding refers to moving an endpoint in relation to a corner, while still maintaining the relative ordering of all endpoints. For example, consider a \( k \)-polygon representation of some graph \( G \) containing chord \( c_0 \) with endpoints \( e_0 \) and \( e'_0 \) and corners \( \tau_i \) and \( \tau_{i+1} \) such that \( \langle e_0, \tau_i, \tau_{i+1}, e'_0 \rangle \). If \( e_0 \) is close to \( \tau_i \) on side \( s_{i-1} \), then \( e_0 \) can slide clockwise around \( \tau_i \) resulting in the configuration \( \langle \tau_i, e_0, \tau_{i+1}, e'_0 \rangle \). See Figure 3 for an example of a slide transformation.

This new configuration is still a valid \( k \)-polygon representation of \( G \) since the chord crossings have been maintained, and all chords remain satisfied. In this way, an arbitrary number of endpoints can slide around a corner, provided the slide does not result in two endpoints of the same chord being on the same side. Counterclockwise slides are defined analogously.

The proof of the next theorem shows how to slide the chords of an ar-
Figure 4: If \( e_0 \) is close to \( \tau_i \) on side \( s_{i-1} \) and endpoint \( e_0' \) is not on side \( s_i \), as in the configuration on the left, then \( e_0 \) can slide clockwise around \( \tau_i \) resulting in the configuration on the right.

Theorem 14

Let \( C \) be a circle representation for graph \( G \) such that \( \psi_r(C) = \psi(G) = k \). Then \( C \) contains an independent set in series of size \( k - 1 \). Furthermore, a set of \( k \) corners \( \tau_0, \ldots, \tau_{k-1} \) can be added to \( C \) such that all chords in \( C \) are satisfied and, where \( c_i \) is the chord whose endpoint on \( s_i \) is close to \( \tau_i \) for all \( 0 \leq i \leq k - 1 \):

1. \( c_i \) is in \( \tau_i \) for all \( 1 \leq i \leq k - 1 \), and

2. one of the following holds:

   (a) \( c_0 \) is in \( \tau_0 \) and \( c_0, \ldots, c_{k-1} \) form a \( k \)-independent set in series.

   (b) \( c_1, \ldots, c_{k-1} \) form a \((k-1)\)-independent set in series, and if \( k > 2 \) then all chords in \( \tau_0 \) intersect \( c_1 \) and do not intersect \( c_{k-1} \), while if \( k = 2 \) then \( G \) is a clique.

Proof. The theorem is trivially true if \( G \) is a clique since then \( k = 2 \). Thus, in the remainder of the proof, we assume that \( G \) is not a clique. We first consider the case in which \( k = 2 \) and so \( G \) is a permutation graph that is not a clique. Let \( C \) be a circle representation for \( G \) with \( \psi_r(C) = 2 \), and let \( \tau_0 \) and
τ₁ be corners which, when added to C, form a permutation representation for G. Let e₀ (respectively e₁) be the first endpoint encountered in a clockwise traversal of side s₀ (s₁) whose chord does not correspond to a universal vertex of G. Such endpoints exist because G is not a clique. Let c₀ (c₁) be the chord whose endpoint is e₀ (e₁). If e₀ and e₁ belong to the same chord, that chord would correspond to a universal vertex, contradicting the choices of e₀ and e₁. Thus, c₀ ≠ c₁. Furthermore, c₀ and c₁ do not cross each other by the choice of e₀ and e₁ as the first endpoints satisfying the conditions.

Now consider the representation obtained when all chord endpoints that occur between τ₀ and e₀, but not e₀, slide counterclockwise around τ₀ and chord endpoints that occur between τ₁ and e₁, but not e₁, slide counterclockwise around τ₁. For any non-universal vertex, the corresponding endpoints would remain on the same sides that they were on in the original representation. Thus, these chords are satisfied. Any endpoints that slide correspond to universal vertices whose corresponding endpoints both slide. Since the corresponding chords have their endpoints again on different sides, these chords too are satisfied. Therefore, the new placement of chord endpoints and corners yields a valid permutation representation. Moreover, this representation satisfies conditions 1 and 2(a) of the theorem.

Now consider a k-polygon representation for G obtained by adding k corners to C, where k ≥ 3. We will show how to alter the representation by sliding some of the chord endpoints around corners to obtain a k-polygon representation.
representation for $G$ that has the required properties. The alteration proceeds as follows. Visit the corners of the representation in the order $\tau_1, \tau_2, \ldots, \tau_{k-1}$; while visiting $\tau_i$ perform the following:

Let $e_i$ be the first endpoint of a chord encountered in a clockwise traversal of $s_i$ such that the other endpoint of the chord is on $s_{i-1}$. Such a chord must exist because otherwise the corner $\tau_i$ could be removed, contradicting that the representation contains the minimum number of corners. Additionally, such a chord is peripheral or else we contradict the choice of $e_i$. Now, slide all endpoints that occur between $\tau_i$ and $e_i$ (but not $e_i$) counterclockwise around the corner $\tau_i$ onto $s_{i-1}$.

The above operation results in a $k$-polygon representation for $G$, since all chord crossings remain the same as in the original representation and all chords remain satisfied. Moreover, in the new representation $c_i$ is close to $\tau_i$ for all $1 \leq i \leq k-1$.

For all $i$, $1 \leq i \leq k-1$, let $c_i$ be the chord whose endpoint is $e_i$. We claim that the chords $c_1, c_2, \ldots, c_{k-1}$ form an independent set in series of size $k-1$ in $C$. To see this, first notice that $c_i \neq c_j$ for all $1 \leq j < i \leq k-1$. Since $c_j$ has endpoints on sides $s_{j-1}$ and $s_j$ and $c_i$ has endpoints on sides $s_{i-1}$ and $s_i$, and $i \neq j$, if $c_i = c_j$ it must be that $i = j - 1$ and $j = i - 1$. This means that $k = 2$, which contradicts our choice of $G$.

Next, we show that $c_i$ does not cross $c_j$, for all $1 \leq j < i \leq k-1$. In order for the chords to cross, it must be that $j = i - 1$ and $c_i$’s endpoint occurs before $c_j$’s endpoint in a clockwise traversal of side $s_j$. But after the sliding operation at $\tau_j$, $c_j$’s endpoint is the closest endpoint to $\tau_j$ on side $s_j$, a contradiction. Finally, the chords $c_1, \ldots, c_{k-1}$ form an independent set in series since each $c_i$ has endpoints on sides $s_{i-1}$ and $s_i$ for all $1 \leq i \leq k-1$.

Now, let $e_0$ be the first endpoint of a chord in $\tau_0$ encountered in a clockwise traversal of $s_0$. Such a chord must exist as otherwise $\tau_0$ could be removed. Let $c_0$ be the chord that has $e_0$ as an endpoint and let $e_0'$ be the other endpoint of $c_0$. Since the endpoints of $c_0$ are on sides $s_{k-1}$ and $s_0$, chord $c_0$ cannot be identical to any of $c_1, \ldots, c_{k-1}$ because this would imply that $k = 2$, contradicting the choice of $G$. Endpoint $e_0'$ must come after $e_{k-1}$ in a clockwise traversal of $s_{k-1}$ or else $e_0'$ would have been slid onto $s_{k-2}$ when $\tau_{k-2}$ was visited. Therefore, $c_0$ does not cross $c_{k-1}$. In addition, $c_0$ does not cross any of $c_2, \ldots, c_{k-2}$ since it does not have an endpoint on the same side as any of those chords. If $c_0$ does not cross $c_1$ then $c_0, \ldots, c_{k-1}$ is a $k$-independent
set in series, and we can slide all endpoints that occur between \( \tau_0 \) and \( e_0 \) (but not \( e_0 \)) counterclockwise around the corner \( \tau_0 \) onto \( s_{k-1} \). If \( c_0 \) crosses \( c_1 \) then, since endpoints of all chords in \( \tau_0 \) are clockwise of \( e_0 \) on \( s_0 \), all chords in \( \tau_0 \) cross \( c_1 \). Finally, no chord in \( \tau_0 \) crosses \( c_{k-1} \) since \( c_{k-1} \)'s endpoint on \( s_{k-1} \) is close to \( \tau_{k-1} \). \( \square \)

The previous theorem directly implies the following bound, the tightness of which is demonstrated by odd chordless cycles on five or more vertices. Permutation graphs show that \( \psi(G) \) and \( \alpha(G) + 1 \) can be arbitrarily far apart.

**Corollary 15** For any circle graph \( G \), \( \psi(G) \leq \alpha(G) + 1 \).

Any chordless cycle on \( n \) vertices, \( C_n \), satisfies \( \psi(C_n) = \lceil n/2 \rceil \); furthermore, as the following theorem shows, \( \lceil n/2 \rceil \) is an upper bound on the minimum number of sides in a polygon representation for any circle graph on three or more vertices.

**Theorem 16** For any circle graph \( G \) on \( n \geq 3 \) vertices, \( \psi(G) \leq \lceil n/2 \rceil \).

**Proof.** We consider only graphs without isolated vertices since removing such vertices does not affect \( \psi(G) \). The theorem is trivially true if \( \psi(G) = 2 \), so let \( G \) be a circle graph on \( n \geq 3 \) vertices such that \( k = \psi(G) \geq 3 \), and let \( C \) be a \( k \)-polygon representation for \( G \) that satisfies the conditions of Theorem 14. We will show that in each of the two cases described in Theorem 14, \( n \geq 2k - 1 \).

**Case 1:** For all \( 0 \leq i \leq k - 1 \), the chord \( c_i \) whose endpoint on \( s_i \) is close to \( \tau_i \), is in \( \tau_i \), and \( c_0, \ldots, c_{k-1} \) form a \( k \)-independent set in series (1 and 2(a) of Theorem 14).

For each corner, we assign a chord of \( C - \{c_0, \ldots, c_{k-1}\} \) to be counted in that and no other corner, implying that \(|C - \{c_0, \ldots, c_{k-1}\}| \geq k \). The assigned chords together with \( \{c_0, \ldots, c_{k-1}\} \) form a set of size \( 2k \), and the result follows. The chords are assigned in a two-stage process. If at any point in the process, a corner has a chord assigned to it then that corner is said to be finished; otherwise, it is unfinished. All corners are initially unfinished. The two stages are:

1. **Stage 1.** While there exists an unfinished corner \( \tau_i \) where \( 0 \leq i \leq k - 1 \), such that \( N(c_i) \cup C_{\tau_i} - \{c_i\} \) contains exactly one unassigned chord, assign that chord to \( \tau_i \).
Stage 2. Arbitrarily assign one unique and unassigned chord of $C - \{c_0, \ldots, c_{k-1}\}$ to each unfinished corner.

We will show that the above process correctly assigns a unique chord from $C - \{c_0, \ldots, c_{k-1}\}$ to each of the $k$ corners. First, we observe that during Stage 1, no chord is assigned to more than one corner, exactly one corner is finished in each iteration, and once a corner is finished it remains that way for the rest of the process. Consequences of this observation are that Stage 1 terminates and that for each corner $\tau_i$ that remains unfinished when Stage 1 terminates, $N(c_i) \cup C_{\tau_i} - \{c_i\}$ either contains no unassigned chords, or it contains at least two unassigned chords. To rule out the former, we require the following definition. Two $k$-polygon representations for a graph $G$ are said to agree at corners $X \subseteq \{\tau_0, \tau_1, \ldots, \tau_{k-1}\}$ if all chord endpoints occur in the same cyclic order, all corners in $\{\tau_0, \tau_1, \ldots, \tau_{k-1}\}$ occur in the same cyclic order, and each corner in $X$ occurs at exactly the same position (with respect to the chord endpoints) in the two representations.

We now introduce the following two invariants and prove by induction that they hold after each iteration of Stage 1:

(I) In every $k$-polygon representation for $G$ that agrees with $C$ at the finished corners, the chord assigned in the iteration just completed is close to the corner to which it was assigned and not close to any other corner.

(II) There is an unassigned chord of $C - \{c_0, \ldots, c_{k-1}\}$ close to each unfinished corner.

First, consider the statements prior to the first iteration. At this point, all corners are unfinished and all chords are unassigned, and so (II) is guaranteed since $G$ has no isolated vertices, while (I) is vacuously true.

Now assume that the invariants are true after all of the first $j \geq 0$ iterations. If there is no unfinished corner $\tau_i$ in which $N(c_i) \cup C_{\tau_i} - \{c_i\}$ contains exactly one unassigned chord, then Stage 1 terminates and the invariants are true after all iterations. Instead, suppose that $d_i$ is the chord assigned to corner $\tau_i$ in iteration $j+1$. By the inductive hypothesis on (II), there is a chord $d'_i$ in $C - \{c_0, \ldots, c_{k-1}\}$ close to $\tau_i$ that was unassigned at the beginning of iteration $j+1$. Since our assumption that $C$ satisfies the conditions of Theorem 14 requires that the endpoint of $c_i$ that is close to $\tau_i$ is on $s_i$, the endpoint of $d'_i$ which is close to $\tau_i$ must be on $s_{i-1}$ and in the empty arc of $c_i$. Therefore $d'_i$ is in $N(c_i)$. Since $N(c_i) \cup C_{\tau_i} - \{c_i\}$ contains exactly one unassigned chord at the beginning of iteration $j+1$, it must be that $d_i = d'_i$.
and so $d_i$ is close to $\tau_i$ with an endpoint on $s_{i-1}$, and $d_i$ is in $N(c_i)$.

The proof of (I) is completed by contradiction. To that end, assume that there exists a $k$-polygon representation $C'$ for $G$ that agrees with $C$ at all of the finished corners including $\tau_i$, and in which $d_i$ is close to some corner $\tau_\ell$, where $\ell \neq i$. The proof is handled in two cases, depending on the degree of $c_i$ in $C$.

**Case $d(c_i) > 1$:**

Let $f$ be the chord in $N(c_i) - \{d_i\}$ whose endpoint is next to $d_i$'s endpoint in the empty arc of $c_i$ (in both $C$ and $C'$). Since $d_i$ was the only unassigned chord in $N(c_i)$ at the beginning of iteration $j+1$, $f$ must have been assigned to some corner in a previous iteration. If the endpoint of $d_i$ that is on $s_{i-1}$ can be slid to $s_i$ then the endpoint of $f$ will be close to $\tau_i$. If the resulting $k$-polygon representation still agrees with $C$ at the corners that were finished just after $f$ was assigned, then the fact that the endpoint of $f$ is close to $\tau_i$ in addition to the corner it was assigned to is a contradiction to the induction hypothesis on (I). Therefore, we need to show that the endpoint of $d_i$ which is on $s_{i-1}$ can be slid to $s_i$ such that the resulting $k$-polygon representation still agrees with $C$ at the corners that were finished immediately after $f$ was assigned.

If $d_i$ does not already have an endpoint on $s_i$, then this slide can be made immediately. If $d_i$ does have an endpoint on $s_i$, then it is necessarily close to $\tau_{i+1}$ by our assumption that $d_i$ is close to some corner other than $\tau_i$. Since this means that $N(c_{i+1})$ contains at least one unassigned chord prior to $d_i$ being assigned, $\tau_{i+1}$ is not finished in the first $j$ iterations. We can therefore slide the endpoint of $d_i$ on $s_i$ to $s_{i+1}$ and then slide the endpoint of $d_i$ on $s_{i-1}$ to $s_i$. The result is a valid $k$-polygon representation that agrees with $C$ at the corners that were finished when $f$ was assigned, but in which $f$ is close to two different corners. Therefore, (I) holds if $d(c_i) > 1$.

**Case $d(c_i) = 1$:**

In this case, $C'$ can be altered by moving $c_i$ from being in corner $\tau_i$ to being in $\tau_\ell$. This is possible since $d_i$ is close to $\tau_\ell$, and $d_i$ is $c_i$'s only neighbour. The result is a $k$-polygon representation $C''$ for $G$ in which $C''_{\tau_i} = C'_{\tau_i} - \{c_i\}$.

Now if $C''_{\tau_i} = \emptyset$, the corner $\tau_i$ can be eliminated from $C''$ since there are no chords with endpoints on both $s_{i-1}$ and $s_i$. The result is a $(k-1)$-polygon representation for $G$, which contradicts the fact that $k = \psi(G)$.

From now on, we assume that $C''_{\tau_i} \neq \emptyset$. Each chord in $C''_{\tau_i}$ is either $d_i$ or a chord that was previously assigned to a corner that it is close to in $C$ (and
also in $C'$ and $C''$, by the choices of $C'$ and $C''$). This is true by the description of Stage 1, the induction hypothesis on (I) and (II), and the fact that $C$ and $C'$ agree at $\tau_i$ and all corners finished before $\tau_i$. By the choice of $C$ according to the conditions of Theorem 14, the only corner other than $\tau_i$ that a chord in $C''$ can be close to is $\tau_{i+1}$ and therefore there is only one chord in $C''_{\tau_i}$.

If the chord in $C''_{\tau_i}$ is $d_i$ then we slide one of $d_i$'s endpoints from $s_i$ to $s_{i+1}$ and the other endpoint from $s_{i-1}$ to $s_i$. The corner $\tau_i$ can then be eliminated to produce a $(k-1)$-polygon representation for $G$, contradicting the fact that $k = \psi(G)$.

Otherwise, let $f$ be the chord in $C''_{\tau_i}$. Then $C''$ can be altered as follows. First, $d_i$'s endpoint close to $\tau_i$ on side $s_{i-1}$ can be slid around $\tau_i$ to $s_i$. All endpoints on $s_{i-1}$ between $f$'s endpoint and $d_i$'s endpoint (but not $f$'s endpoint) can then slide around $\tau_i$ to $s_i$. This set of slides is possible since none of the corresponding chords is in $C''_{\tau_i}$ because, as previously seen, $C''_{\tau_i} = \{f\}$. Finally, both endpoints of $f$ can now be moved – one from $s_i$ to $s_{i+1}$ and one from $s_{i-1}$ to $s_i$ – to create a $k$-polygon representation for $G$ in which there are no chords in corner $\tau_i$. This means that $\tau_i$ can be eliminated to produce a $(k-1)$-polygon representation for $G$, which is a contradiction of the fact that $k = \psi(G)$. Thus, (I) holds if $d(c_i) = 1$.

This concludes the proof of (I). We now turn our attention to (II). For each corner that remains unfinished after the $(j+1)$st iteration, (II) was true before the assignment of $d_i$ to $\tau_i$ by the inductive hypothesis, and (II) remains true after the assignment of $d_i$ since, by (I), $d_i$ is not close to any corner other than $\tau_i$. Therefore (II) holds.

We now show that after Stage 1 is complete, there are still enough unassigned chords in $C - \{c_0,\ldots, c_{k-1}\}$ so that one can be assigned to each of the remaining unfinished corners. Notice that for each corner $\tau_i$ that remains unfinished at the end of Stage 1, $N(c_i) \cup C_{\tau_i} - \{c_i\}$ contains at least two unassigned chords (by (II) and the fact that Stage 1 has terminated). We can therefore conclude that the number of unassigned chords is greater than or equal to the number of unfinished corners since each unassigned chord of $N(c_j) \cup C_{\tau_i} - \{c_j\}$ appears in at most one other set $N(c_j) \cup C_{\tau_j} - \{c_j\}$, where $j \neq i$. Therefore, $|C - \{c_0,\ldots, c_{k-1}\}| \geq k$, and so along with $k$ chords in $\{c_0,\ldots, c_{k-1}\}$, we conclude that $n \geq 2k$ in Case 1.

**Case 2:** For all $1 \leq i \leq k-1$, the chord $c_i$ whose endpoint on $s_i$ is close to $\tau_i$ is in $\tau_i$. In addition, $c_1,\ldots, c_{k-1}$ form a $(k-1)$-independent set in series, and all chords in $\tau_0$ are adjacent to $c_1$ and not adjacent to $c_{k-1}$ (1 and 2(b)
of Theorem [14].

Let $C'$ be $C$ with the addition of one chord in $\tau_0$ with endpoints just clockwise of the endpoint of $c_{k-1}$ on side $s_{k-1}$ and just counterclockwise of the endpoint of $c_1$ on $s_0$. Now, $C'$ satisfies conditions 1 and 2(a) of Theorem [14] Thus, by Case 1, it contains at least $2k$ chords and therefore $C$ contains at least $2k - 1$ chords. \(\square\)

We close this section with a lower bound on $\psi(G)$.

**Theorem 17** For any circle graph $G = (V, E)$, $\psi(G) \geq an(G)$.

**Proof.** If $an(G) \leq 2$, the statement is true since $\psi(G) \geq 2$ for all circle graphs. Instead, consider the case where $an(G) \geq 3$. Let $C$ be a $\psi(G)$-polygon representation of $G$, $A$ be an asteroidal set of $G$ where $|A| = an(G)$, and $C_A \subseteq C$ be the set of chords corresponding to vertices in $A$. Consider any vertex $v \in A$, corresponding to chord $c_v$ with endpoints $e_v$ and $e'_v$.

Let $a, b \in A - \{v\}$ correspond to chords $c_a$ and $c_b$, respectively. By the definition of $A$, neither intersects $c_v$. Therefore, both endpoints of $c_a$, which are denoted as $e_a$ and $e'_a$, must be in the same arc of $c_v$. The same is true for the endpoints $e_b$ and $e'_b$, which are defined analogously.

Without loss of generality, assume that $e_a, e'_a \in (e_v, e'_v)$ and $e_b, e'_b \in (e'_v, e_v)$. By the definition of $A$, there must be some path from $a$ to $b$ consisting only of vertices in $V - N(v)$. However, at least one vertex of any path from $a$ to $b$ must correspond to a chord with one endpoint in $(e_v, e'_v)$ and one in $(e'_v, e_v)$. This chord intersects $c_v$, and so the corresponding vertex is in $N(v)$. Therefore, by contradiction, $e_a, e'_a, e_b, e'_b$ will necessarily be contained in the same arc of $c_v$.

This condition ensures that all chords in the circle representation given only by chords $C_A$ are peripheral. Equivalently, the chords in $C_A$ form an $|A|$-independent set in series. Therefore, $\psi_r(C_A) = |A| = an(G)$. Since $\psi_r(C_A) \leq \psi_r(C)$ and $\psi_r(C) = \psi(G)$, we therefore have that $an(G) \leq \psi(G)$. \(\square\)

In the next section, we show that if $G$ is a connected distance hereditary graph that is not a clique, then $\psi(G) = an(G)$. However, in general the gap between $\psi(G)$ and $an(G)$ can be arbitrarily large. For example, the unit interval graphs depicted in Figure 6 have asteroidal number two and arbitrarily large polygon number.
3 Split decomposition and distance hereditary graphs

A module in a graph $G = (V, E)$ is a subset $M$ of $V$ such that each vertex of $V - M$ is adjacent to all or to none of the vertices of $M$; it is a nontrivial module if, in addition, $1 < |M| < |V|$. All of the modules referred to in this paper are nontrivial; we will omit the word “nontrivial” and refer to them simply as modules. Two vertices are called twins if they form a module of size two. If $M$ is a module of $G = (V, E)$ and $v$ is a vertex of $M$, then the operation of reducing module $M$ to $v$ in $G$ results in the graph $G - (M - \{v\})$.

Given graphs $G = (V, E)$ and $H = (V_H, E_H)$ with disjoint nontrivial vertex sets, and a vertex $v \in V$, the operation of substituting $v$ with $H$ in $G$ produces the graph composed of $G - \{v\}$ and $H$ and all edges between $N_G(v)$ and $V_H$. Note that $V_H$ is a module in the resulting graph. If $M$ is a module of $G$ such that $G[M]$ has a certain property, we also say that $M$ has the property.

A split of a graph $G = (V, E)$ is a partition $\{V_1, V_2\}$ of $V$ such that $|V_1| > 1$, $|V_2| > 1$, and $E \cap \{xy \mid x \in V_1 \text{ and } y \in V_2\} = \{xy \mid x \in N(V_2) \text{ and } y \in N(V_1)\}$. A graph with no split is called prime.

Cunningham and Edmonds [4] [5] defined and studied a decomposition scheme for graphs based on the concept of a split. If $\{V_1, V_2\}$ is a split of $G = (V, E)$ then $\{G_1, G_2\}$ is called a simple split decomposition of $G$, where $G_1$ is the graph $G[V_1]$ to which a new vertex $v \notin V$ has been added with $N_{G_1}(v) = N_G(V_2)$ and $G_2$ is the graph $G[V_2]$ to which vertex $v$ has been added with $N_{G_2}(v) = N_G(V_1)$; $v$ is called the marker vertex of the decomposition. A split decomposition of $G$ is $\{G\}$ or any set of graphs obtained from a split decomposition of $G$ by replacing one graph of the decomposition with the two graphs of one of its simple split decompositions. The reverse of a split

Figure 6: An infinite family of unit interval graphs with asteroidal number two and arbitrarily large polygon number

\[ \text{Figure 6: An infinite family of unit interval graphs with asteroidal number two and arbitrarily large polygon number} \]
decomposition is the *join composition*. Given graphs $G_1$ and $G_2$, each with distinguished marker vertex $v$, the *join* of $G_1$ and $G_2$ is the graph composed of graphs $G_1 - \{v\}$ and $G_2 - \{v\}$ with the addition of the edges $\{xy \mid x \in N_{G_1}(v) \text{ and } y \in N_{G_2}(v)\}$. Applying the join composition repeatedly in any order to a split decomposition for $G$ reconstructs the graph $G$.

Every connected graph has a unique split decomposition, called the *standard split decomposition*, that contains only prime graphs, cliques, and stars and is not a strict refinement of any other such split decomposition [5]. This means that applying the join composition to two graphs of a standard split decomposition that share a marker vertex does not result in a clique or a star. Equivalently, no two cliques share a marker, and if two stars share a marker then the marker is either a leaf of both stars or the centre of both stars.

A split decomposition of a graph can be viewed as a tree that contains a node for each graph of the decomposition and an edge between two nodes if the graphs share a marker vertex. In what follows, we sometimes blur the distinction between a node of a decomposition tree and the graph of the decomposition that it represents. It follows from the definition of the standard split decomposition that any subtree of the standard split decomposition tree of a graph is the standard split decomposition tree of a subgraph of the graph.

The first result of this section shows how to combine two polygon representations to produce a polygon representation of the join composition of two graphs.

**Lemma 18** Let $G_1$ and $G_2$ be the graphs of a simple split decomposition of circle graph $G$, each with distinguished marker vertex $v$. If $G_1$ has a $k_1$-polygon representation and $G_2$ has a $k_2$-polygon representation then $G$ has a $(k_1 + k_2)$-polygon representation. Furthermore, if $G_1$ has a $k_1$-polygon representation in which $v$’s chord is close to $q$ corners then $G$ has a $(k_1 + k_2 - q)$-polygon representation.

**Proof.** Let $C_1$ and $C_2$ be $k_1$ and $k_2$-polygon representations for $G_1$ and $G_2$, respectively. Let $e_1$ and $e'_1$ be the endpoints of $v$’s chord in $C_1$ and let $e_2$ and $e'_2$ be the endpoints of $v$’s chord in $C_2$. Let $W$ and $Y$ be the sequences of endpoints and corners that appear in the arcs $(e_1, e'_1)$ and $(e'_1, e_1)$ of $C_1$ respectively and let $X$ and $Z$ be defined analogously for the arcs $(e_2, e'_2)$ and $(e'_2, e_2)$ of $C_2$. Then the circular sequence of endpoints and corners given by $WXYZ$ is a $(k_1 + k_2)$-polygon representation for $G$, as illustrated in Figure 22.
Figure 7: $C_1$ and $C_2$ are combined to produce $C$, a polygon representation for the join of the graphs represented by $C_1$ and $C_2$, as described in Lemma 18. The black dots represent corners.

Let $C$ be a $(k_1 + k_2)$-polygon representation for $G$ constructed in this manner. All chords of $C$ will still be satisfied after the removal of all of the corners that $v$’s chord is close to in $C_1$. The chords of $C_2$ are still satisfied by corners that were originally in $C_2$ and any chord of $C_1$ that contained only close-to-$v$ corners in one or both of its arcs now contains corners of $C_2$ in those arcs since there must be at least one corner in each arc of $v$’s chord in $C_2$. Therefore the corners of $C_1$ that are close to $v$’s chord in $C_1$ can be deleted from $C$ to produce a $(k_1 + k_2 - q)$-polygon representation for $G$. □

Notice that in Lemma 18, $q \leq 4$ by the definition of a chord being close to a corner. Furthermore, if $k_1 = \psi(G_1)$ then $q \leq 2$. This is justified by noticing that if three or four corners are close to $v$’s chord, then all but two of them can be removed while leaving all chords of $C_1$ satisfied. For example, $q = 1$ in Figure 7 and the corner that is close to $v$’s chord in $C_1$ is labelled $\tau$.

Our strategy in the remainder of this section is to examine the polygon numbers and asteroidal numbers of graphs in terms of the standard split decomposition tree. Because only connected graphs are guaranteed to have a standard split decomposition tree, most of our results apply only to connected graphs. We study general graphs and circle graphs first and then apply the results later in the context of distance hereditary graphs. As a first step, we consider the effect of removing certain leaves of a nontrivial split decomposition tree of a graph. Note that a leaf of such a tree contains
exactly one marker vertex. We maintain this property as leaves are removed by adopting the following convention: when a leaf is removed, the other copy of its marker vertex becomes an ordinary vertex.

Lemma 19 Let $G$ be a connected graph and let $T_G$ be the standard split decomposition tree of $G$. If $x$ is a leaf of $T_G$ corresponding to a graph on vertices $V_x$ in which the marker vertex $\ell \in V_x$ is universal, then the following hold:

1. $V_x - \ell$ is a module in $G$.

2. The graph obtained by reducing $V_x - \ell$ to a vertex and renaming it $\ell$ is connected, and $T_G - \{x\}$ is its standard split decomposition tree.

Proof. Since $x$ is a leaf and $\ell$ is universal in $x$, the split of $G$ corresponding to $\ell$ shows that $V_x - \ell$ is a module in $G$. Reducing a (nontrivial) module to a single vertex does not affect whether the graph is connected; therefore the graph obtained by reducing $V_x - \ell$ to a vertex in $G$ is connected. Where $y$ is the neighbour of $x$ in $T_G$, replacing $x$ and $y$ in the standard decomposition of $G$ with the join of $x$ and $y$ and then removing all but one of the vertices of $V_x - \ell$ produces a decomposition of the graph obtained by reducing $V_x - \ell$ to a vertex in $G$. That decomposition is the same as $T_G - \{x\}$ except that the vertex of $T_G - \{x\}$ that corresponds to the one remaining vertex of $V_x - \ell$ is labelled $\ell$. Therefore, $T_G - \{x\}$ represents the graph obtained by reducing $V_x - \ell$ to a vertex in $G$ and renaming it $\ell$, and furthermore it is a standard split decomposition tree by the definition of the standard split decomposition and the fact that it is a subtree of $T_G$. □

Definition 20 Let $G$ be a connected graph and let $T_G$ be the standard split decomposition tree of $G$. The pruned decomposition tree of $G$ is the tree that is produced by starting with $T_G$ and repeatedly removing a leaf whose marker vertex is universal in the graph represented by the leaf, until no such leaves remain.

The next three lemmas outline relationships between the properties of a given graph, its pruned decomposition tree, and the graph represented by the pruned decomposition tree.
Lemma 21 Let $G$ be a connected graph, $T_G'$ its pruned decomposition tree, and $H_G$ the graph represented by $T_G'$. Then $H_G$ is a connected induced subgraph of $G$ that can be obtained from $G$ by repeatedly reducing modules and $T_G'$ is its standard split decomposition tree. Furthermore, if $G$ is a circle graph then $H_G$ can be obtained from $G$ by repeatedly reducing permutation graph modules.

Proof. The statements follow by repeated applications of Lemma 19 and by the fact that any circle graph that has a universal vertex is a permutation graph (see Observation 2). $\square$

Lemma 22 Let $G$ be a connected graph and let $H_G$ be the graph represented by the pruned decomposition tree of $G$. Then the following hold:

1. $\alpha_n(H_G) \leq \alpha_n(G)$.
2. If $G$ is a circle graph then $\psi(H_G) = \psi(G)$.

Proof. Part (1) holds since $H_G$ is an induced subgraph of $G$ (Lemma 21) and the asteroidal number of any induced subgraph of $G$ is at most $\alpha_n(G)$ [19]. To justify (2), notice that $\psi(H_G) \leq \psi(G)$ by Observation 5. Finally, a polygon representation for $H_G$ can be extended to a polygon representation for $G$ without adding any corners by virtue of Lemmas 18 and 21, and the observation that substituting a vertex with a permutation graph is equivalent to performing a join composition where one of the graphs is a permutation graph with a universal vertex marker, and such a graph has a 2-polygon representation in which the marker is close to two corners. $\square$

Lemma 23 Let $G$ be a connected graph and let $T_G'$ be the pruned decomposition tree of $G$. Then $\alpha_n(G)$ is greater than or equal to the number of leaves of $T_G'$.

Proof. The inequality is obviously true if $T_G'$ is trivial or has two leaves. Suppose that $T_G'$ has three or more leaves. We construct a set $A$ consisting of one vertex from each leaf of $T_G'$ as follows. For each leaf, choose a non-marker vertex that is not adjacent to the marker (every leaf contains such a vertex by the definition of $T_G'$) and not adjacent to at least one neighbour of the marker if possible. Now let $A$ be the set of chosen vertices.

Let $H_G$ be the graph represented by $T_G'$. Recall that $H_G$ is an induced subgraph of $G$ (Lemma 21) and that $T_G'$ is the standard split decomposition
tree of $H_G$ (by the definition of standard split decomposition). We show by contradiction that $A$ is an asteroidal set in $H_G$. Let $a \in A$, let $x$ be the leaf of $T_G'$ from which $a$ was chosen, let $y$ be the neighbour node of $x$ in $T_G'$, and let $v$ be the marker vertex that is shared by $x$ and $y$. Since $a$ is nonadjacent to the marker of $x$, its only neighbours in $H_G$ are in $x$. Therefore no two vertices of $A$ are adjacent or have a common neighbour in $H_G$.

Suppose that $A$ is not an asteroidal set in $H_G$ and let $a,b,c \in A$ be such that $b$ and $c$ are not connected in $H_G - N[a]$. Every $b,c$-path in $H_G$ must contain a vertex of $N[a]$. By our earlier observation, all vertices of $N[a]$ appear only in node $x$ of $T_G'$. Therefore, because $x$ is a leaf of $T_G'$, every $b,c$-path must go into $x$ from $y$ and back out to $y$ from $x$ along edges of the split corresponding to the shared marker $v$.

Since all neighbours of $v$ in $y$ are adjacent in $H_G$ to all neighbours of $v$ in $x$, vertex $a$ must be adjacent to all neighbours of $v$ in $x$ as otherwise there would be a $b,c$-path passing through $x$ without including a node of $N[a]$. By the choice of $a$ this implies that all nonneighbours of $v$ in $x$ are adjacent to all neighbours of $v$ in $x$. This in turn implies that either $x$ is a star of which $v$ is a leaf, or $x$ is neither a clique nor a star and has a split. The second possibility would contradict the fact that $T_G'$ is the standard split decomposition tree of $H_G$; therefore $x$ must be a star in which the marker vertex is a leaf.

Furthermore, $v$ must be a cut vertex in $y$ or else there would be a $b,c$-path in $H_G$ avoiding $x$ and therefore $N[a]$. It is easy to see that any graph on four or more vertices that has a cut vertex also has a split and therefore $y$ is not a prime graph. Therefore, since $v$ is a cut vertex, $y$ is a star in which the marker vertex $v$ is the centre.

But then replacing $x$ and $y$ with the join composition of $x$ and $y$ in $T_G'$ yields a split decomposition of $H_G$ of which $T_G'$ is a strict refinement, contradicting the fact that $T_G'$ is the standard split decomposition tree of $H_G$. Consequently, $A$ is an asteroidal set and the statement follows by Lemma 22.

\[\square\]

The results of this section up to this point apply to general graphs and circle graphs. We now focus our attention on distance hereditary graphs, the definition of which appeared in Section 1.1 and is repeated here. A graph $G$ is a distance hereditary graph if, for every connected induced subgraph $H$ of $G$, the distance between each pair of vertices of $H$ is the same in $H$ as it is in $G$. It follows that every induced subgraph of a distance hereditary graph is a distance hereditary graph. Bandelt and Mulder showed that
a graph is distance hereditary if and only if it can be constructed from a
single vertex by repeatedly adding a leaf or substituting a vertex with a
pair of twins or, equivalently, it can be transformed to a single vertex by
repeatedly removing a leaf or reducing a pair of twins. It follows that trees
and cographs are distance hereditary graphs, and that distance hereditary
graphs are circle graphs. Hammer and Maffray [15] characterized distance
hereditary graphs as graphs whose standard split decompositions contain
only cliques and stars, which implies the next observation. A forbidden
induced subgraph characterization of distance hereditary graphs was given in
[1] and [15]: a graph is a distance hereditary graph if and only if it is (house,
hole, domino, gem)-free, that is, it does not contain an induced subgraph
isomorphic to any of the house, holes, domino, or gem graphs, which are
depicted in Figure 8.

The next observation and technical lemma lead to a characterization of
distance hereditary permutation graphs and provide a device for proving that
the polygon number of a distance hereditary graph is at most the number of
leaves of the pruned decomposition tree.

**Observation 24** Each leaf of the pruned decomposition tree of a connected
distance hereditary graph is a star with a degree-one marker vertex.

**Lemma 25** Let $G$ be a connected distance hereditary graph and let $T'_G$ be the
pruned decomposition tree of $G$. Let $x_1, x_2, \ldots, x_j$ be the nodes of a path in
$T'_G$ such that $d(x_1) = 1, d(x_i) = 2$ for all $1 < i < j$, and $d(x_j) \leq 2$. For each
$1 \leq i \leq j$, let $H_i$ be the graph represented by $T'_i = T'_G[\{x_1, \ldots, x_i\}]$. Then,
for each $1 \leq i \leq j$, $H_i$ is a connected permutation graph that contains at
most one marker vertex and has a permutation representation in which the
marker vertex, if it exists, is represented by an extreme chord.

**Proof.** Since $x_1$ is a clique or a star, the chords of a permutation repre-
sentation for it can be permuted to make any chord extreme, so the result
follows for $i = 1$. Let $i > 1$ and assume that $H_{i-1}$ satisfies the lemma. Let
$l_{i-1}$ denote the one marker vertex in $H_{i-1}$. If $x_i$ has degree two in $T'_G$, it
contains two marker vertices: $l_{i-1}$ corresponding to the split between $H_{i-1}$
and the rest of the graph and one other, denoted $l_i$, corresponding to the
split between $H_i$ and $G$ minus the vertices of $H_i$. Otherwise, it contains only
one marker vertex, $l_{i-1}$, and in this case, $i = j$ and $T'_i = T'_G$ since the path
must represent the whole tree. In either case, the result of the join of $x_i$ with
$H_{i-1}$ will have at most one marker vertex. Let $D$ be a permutation diagram for $H_{i-1}$ in which $\ell_{i-1}$ is represented by an extreme chord. The remainder of the proof shows how to extend $D$ to a permutation representation for $H_i$ that has the desired properties.

If $x_i$ is a clique, we replace $\ell_{i-1}$ in $D$ with $|x_i| - 1$ pairwise crossing chords making $\ell_i$, if it exists, an extreme chord.

If $x_i$ is a star with $\ell_{i-1}$ the centre, then $\ell_i$, if it exists, must be a leaf of the star. In $D$, we replace $\ell_{i-1}$ with $|x_i| - 1$ noncrossing chords making $\ell_i$, if it exists, an extreme chord.

If $x_i$ is a star with $\ell_{i-1}$ a leaf, then if $\ell_i$ exists it could be the centre or a leaf of $x_i$. In $D$, we replace $\ell_{i-1}$ with a single chord $p$ (representing the centre of $x_i$) and then add $|x_i| - 2$ noncrossing chords, all crossing just the chord $p$, making $\ell_i$ an extreme chord if $\ell_i$ is a leaf. Otherwise, if $\ell_i$ is the centre then it is represented by the extreme chord $p$. □

Since the 1970’s when distance hereditary graphs were first defined, an active area of research has been the study of graphs in a given graph class that are also distance hereditary graphs. To mention two examples, several characterizations are known for distance hereditary chordal graphs, known as ptolemaic graphs [17], and distance hereditary comparability graphs [7]. Our next results give characterizations of distance hereditary permutation graphs.

**Theorem 26** Let $G$ be a connected distance hereditary graph. The following are equivalent:

1. $G$ is a permutation graph.
2. $G$ is AT-free.
3. The pruned decomposition tree of $G$ is a path.

**Proof.** Statement (1) implies (2) by Theorem [17]. Statement (2) implies (3) by the following justification: if the pruned decomposition tree is not a path then it has three leaves which implies that $an(G) \geq 3$ by Lemma [23], contradicting that $G$ is AT-free. Finally, (3) implies (1) by Lemma [25] and Lemma [22] □

**Corollary 27** Let $G$ be any distance hereditary graph. Then $G$ is a permutation graph if and only if it is AT-free.
Figure 8: The forbidden subgraphs for distance hereditary permutation graphs: the infinite family of holes of size greater than or equal to 5, and seven additional graphs

Corollary 27 implies that the classes of distance hereditary permutation graphs, distance hereditary cocomparability graphs, and distance hereditary AT-free graphs are all the same, since permutation graphs form a subset of cocomparability graphs which in turn form a subset of AT-free graphs (see [3]).

**Corollary 28** A graph is a distance hereditary permutation graph if and only if it does not contain an induced subgraph isomorphic to any of the graphs of Figure 8.

**Proof.** As previously mentioned, a graph is a distance hereditary graph if and only if it is (house, hole, domino, gem)-free [1] [15]. Therefore, by Corollary 27, a graph is a distance hereditary permutation graph if and only if it is (house, hole, domino, gem, AT)-free. Gallai [12] gave a list of minimal ATs consisting of four infinite families of graphs and eleven additional graphs. By inspection, the only minimal ATs from Gallai’s list that do not contain a house, hole, domino, or gem as an induced subgraph are the four graphs on the right in Figure 8. □

DiStefano [7] showed that a graph is a distance hereditary comparability graph if and only if it does not contain an induced subgraph isomorphic to a house, hole, domino, gem, or net (see Figure 8). Combined with Corollaries 27 and 28, this leads to the observation that distance hereditary AT-free graphs form a proper subset of distance hereditary comparability graphs.

We next show how to construct a polygon representation for a connected distance hereditary graph that has the number of corners equal to the number of leaves of the pruned decomposition tree of the graph.
Lemma 29 Let \( G \) be a connected distance hereditary graph and let \( k \geq 2 \) be the number of leaves of the pruned decomposition tree of \( G \). Then \( \psi(G) \leq k \).

Proof. If \( k = 2 \) then, by Theorem 26, \( G \) is a permutation graph and \( \psi(G) = 2 \). Let \( k > 2 \) and assume the lemma holds for graphs whose pruned decomposition trees have fewer than \( k \) leaves. Let \( G \) be a connected distance hereditary graph where \( T_G' \) is the pruned decomposition tree of \( G \) and \( k \geq 2 \) is the number of leaves of \( T_G' \). Let \( x_1, x_2, \ldots, x_j \) be the nodes of a path in \( T_G' \) such that \( d(x_1) = 1, d(x_i) = 2 \) for all \( 1 < i \leq j \), and the neighbour of \( x_j \) (other than \( x_{j-1} \) if \( j > 1 \)) has degree greater than two. Such a path must exist since \( k > 2 \). By Lemma 25, \( H_j \), the graph represented by \( T_j = T_G'\{x_1,\ldots,x_j\} \), is a connected permutation graph, contains exactly one marker vertex, and has a permutation representation \( D \) in which the marker vertex is represented by an extreme chord. The marker vertex of \( H_j \) is the marker vertex that \( x_j \) shares with its neighbour in \( T_G' \); we refer to it as \( \ell_j \). Now consider \( T^- = T_G' - \{x_1,\ldots,x_j\} \) and let \( H^- \) be the graph represented by \( T^- \). \( T^- \) has \( k - 1 \) leaves and is the pruned decomposition tree of \( H^- \). Therefore, by the inductive assumption, \( H^- \) has a \((k - 1)\)-polygon representation. The lemma now follows by Lemma 18 and Lemma 22. \( \Box \)

Finally, we have the following characterization of the polygon numbers of distance hereditary graphs, which generalizes Theorem 26 to \( k \)-polygon graphs for \( k > 2 \).

Theorem 30 For a connected nonclique distance hereditary graph \( G \), the following parameters are equal:

1. \( \psi(G) \)
2. \( an(G) \)
3. the number of leaves in the pruned decomposition tree of \( G \)

Proof. The equalities follow from Theorem 17, Lemmas 23 and 29, and the fact that nonclique graphs have asteroidal number at least two. \( \Box \)

4 Algorithms

We now examine some algorithmic implications of our results.
In [18], Hsu and Tsai give an $O(|A|)$ algorithm for computing a minimum clique cover of the intersection graph of a given circular-arc representation where $A$ is the set of arcs. A minimal arc of a set of arcs is one that does not contain any other arc in the set. The algorithm of [18] finds a minimum size set $S$ of minimal arcs of $A$ such that every arc of $A$ contains the clockwise endpoint of some arc of $S$. The collection of $|S|$ cliques of the intersection graph of $A$, each of which consists of the vertices whose arcs contain the clockwise endpoint of one of the arcs of $S$, is a minimum clique cover for the intersection graph. Note that adding a non-minimal arc to $A$ does not increase the clique cover number of the intersection graph, since any such arc contains a minimal arc which in turn contains the clockwise endpoint of an arc of $S$.

By virtue of Theorem 10, the algorithm of [18] can be used to compute the polygon number of a circle representation in linear time, an improvement over the quadratic algorithm of [10]. Let $C$ be a circle representation for graph $G = (V, E)$. It is straightforward to detect whether $G$ has a universal vertex by a single scan of $C$ and in that case $\psi_r(C) = 2$ by Observation 2. Therefore, from now on we consider the case in which $G$ has no universal vertex. From $C$, we construct a circular-arc representation $A$ for a new graph $G'$ on $2|V|$ vertices. The set of arcs $A$ consists of both arcs of each chord of $C$ where the arc endpoints are shifted slightly so that the two arcs of a single chord do not intersect but all other intersections remain the same. By Observations 2 and 3 there is a one-to-one correspondence between the minimal arcs in $A$ and the peripheral chords of $C$, and $G_{C_\emptyset}$ is an induced subgraph of $G'$. Furthermore, the vertices of $G'$ that are not in $G_{C_\emptyset}$ are represented by non-minimal arcs in $A$. Therefore by Theorem 10 it follows that $\kappa(G') = \kappa(G_{C_\emptyset}) = \psi_r(C)$, and this parameter can be computed in $O(|C|)$ time by the algorithm of [18].

The worst-case running time of the algorithm of [10] for determining whether $\psi(G) \leq k$ for a given circle graph can be improved by incorporating the algorithm of the preceding paragraph. The running time of the algorithm of [10] on a connected circle graph $G$ is given by the sum of the times to process each node $x$ of the standard split decomposition tree $T_G$, where a circle representation $C_x$ for the graph represented by node $x$ is given. The processing time for each node is the time to compute $\psi_r(C_x)$ if $x$ is a prime leaf node, $O(4^{\min(m_x,k)}n_x)$ if $x$ is a prime non-leaf node, and $O(n_x)$ if $x$ is a clique or a star node, where $n_x$ is the number of vertices of $x$, and $m_x$ is the number of marker vertices of $x$. The overall time bound is given in [10] as $O(|V|^2 + 4^k|V|)$ since the algorithm in that paper for computing $\psi_r(C_x)$
has running time \(O(n^2)\). Using the algorithm of the previous paragraph improves the overall running time to \(O(4^k|V|)\). Although the algorithm of [10] as described above requires the input graph to be connected, disconnected graphs are easily handled within the same time bound by virtue of Theorem [10].

Note that the running time of the preprocessing step for the algorithm of [10] has also been improved since that paper was published. The standard split decomposition tree can be constructed and a circle representation computed for each node of the tree in \(O((|V| + |E|) \cdot \alpha(|V| + |E|))\) time\(^1\) by the algorithm of [13], whereas previously the most efficient algorithm known for that problem was the \(O(n^2)\) algorithm of [20].

It can be seen from the analysis of [10] as previously described, that if \(m_x\) for prime nodes is bounded by a small constant then the running time of the algorithm depends on that bound rather than on \(k\). In that case the algorithm, when run on the standard split decomposition tree of a circle graph \(G = (V, E)\) and \(k = |V|\), returns a polygon representation of \(G\) with the minimum number of corners in \(O(|V|)\) time. Therefore, \(\psi(G)\) can be computed in linear time whenever the number of markers in each prime node of the standard split decomposition tree of \(G\) is bounded by a constant, for example, for prime circle graphs, the standard split decomposition trees of which have no marker vertices, and for distance hereditary graphs, the standard split decomposition trees of which have no prime nodes.

By virtue of Theorem [30], we now have two linear time algorithms for computing \(\psi(G)\) and \(an(G)\) of a distance hereditary graph \(G\). For connected graphs, first, the algorithm of [10] runs in linear time for distance hereditary graphs as mentioned in the previous paragraph and, second, the linear time algorithm of [6] for constructing the standard split decomposition tree of a graph can be combined with a simple pruning algorithm to solve the same problem. Disconnected graphs can be handled by Theorem [1] and the observation that the asteroidal number of a disconnected graph is the maximum of the asteroidal numbers of its connected components. The most efficient algorithm previously known for the asteroidal number of a distance hereditary graph is the \(O(|V|^3 + |V|^{3/2}|E|)\) algorithm of [19] for the asteroidal number of \(G = (V, E)\) where \(G\) is in the class of HHD-free graphs, a superclass of distance hereditary graphs.

\(^1\alpha(|V| + |E|)\) denotes the inverse Ackermann function
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References


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