Some cases of preservation of the Pontryagin dual by taking dense subgroups

To the memory of Mel Henriksen

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Abstract

We study the compact-open topology on the character group of dense subgroups of topological abelian groups. Permanence properties concerning open subgroups, quotients and products are considered. We also present some representative examples. We prove that every compact abelian group G with $w(G) \geq \mathfrak{c}$ has a dense pseudocompact group which does not determine G; this provides a negative answer to a question posed by S. Hernández, S. Macario and the third listed author two years ago.

Keywords: topological group, character group, dense subgroup, determined group, compact covering map, pseudocompact group
2000 MSC: 22A05

1. Introduction

Throughout the paper all the topological groups considered will be Hausdorff (hence Tychonoff) and abelian. If G is a topological group then its character group $G^\wedge$ is defined as its group of continuous characters. This group is usually endowed with the compact-open topology.

☆Partially supported by MICINN of Spain MTM2009-14409-C02-01. The third author gratefully acknowledges the support from the Departamento de Física y Matemática Aplicada, Universidad de Navarra.

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The character group of a dense subgroup of an abelian topological group can be algebraically identified with that of the whole group, due to the fact that every continuous character is uniformly continuous. However the compact-open topology may be very different on the corresponding duals.

If a topological abelian group G is such that its character group equipped with the compact-open topology coincides algebraically and topologically with that of each of its dense subgroups, then G is called a *determined group.* On the other hand, if H is a dense subgroup of G and the character groups of G and H coincide when equipped with the compact-open topology, then we say that H *determines* G or that it is a *determining subgroup.* The word "coincide" means here that the restriction mapping from $G^\wedge$ to $H^\wedge$ is a topological isomorphism. Actually, we don't know of any example of a dense subgroup H of a topological abelian group G with $G^\wedge$ and $H^\wedge$ topologically isomorphic, yet with the restriction map from $G^\wedge$ to $H^\wedge$ not being a topological isomorphism.

Metrizable abelian groups are determined, as proved by the first author in [8], and independently in [3, Th. 4.10]. Earlier, in [5, Lemma 17], it was shown that every dense vector subspace determines a metrizable locally convex space. The name "determined group" appeared for the first time in Raczkowski's doctoral thesis [19] in 1998, where she proved that compact groups need not be determined.

Recently it was obtained in [16] that a compact group is determined if and only if it is metrizable. The same result with a different proof appears in [10], and for the nonabelian case in [13]. Another nontrivial result in this field is the fact that the arc component of a locally compact abelian group determines its connected component [4].

The property of being determined has very poor permanence properties. It is not in general preserved by taking closed subgroups or products, as it was noticed in [9]. The first goal of this paper is to obtain some positive results concerning the preservation of determining dense subgroups for subgroups, quotients and finite products. On the other hand we find representative examples of non determining dense subgroups. In this context a few natural questions arise: Are there special kinds of totally bounded groups which always determine its completion? What about totally dense subgroups or pseudocompact dense subgroups of compact groups? For totally dense subgroups the answer is positive, as it has been proved in [11]. For pseudocompact subgroups the question was posed in [16] and we see that the answer is negative.

Notation. The acronym LCA stands for "locally compact and abelian"; the symbol $w(G)$ denotes the weight of the topological group G. Drawing on results of Weil, we call a topological group *totally bounded* (resp. *locally bounded*) if it is a subgroup of a compact (resp. locally compact) group.

A character of a group G is an homomorphism from G into the torus $\mathbb{T}$. We identify $\mathbb{T}$ with the set $\left(-\frac{1}{2}, \frac{1}{2}\right]$ endowed with the sum modulo 1. The symbols $\mathbb{Z}$ and $\mathbb{Z}(p)$ will denote the groups of integers and of integers modulo p for some natural number $p > 1$, respectively, equipped with the discrete topology.

For every compact subset K of G and every $\varepsilon > 0$ we put $U_G(K, \varepsilon) := \{f \in G^\wedge : |f(x)| \leq \varepsilon \ \forall x \in K\}$ where $G^\wedge$ is the group of continuous characters of G. We will drop the subscript in $U_G(K, \varepsilon)$ if the ambient group is clear from the context, and we will simply write $U(K)$ instead of $U(K, \frac{1}{4})$. The sets $U_G(K)$ form a basis of neighborhoods of 0 in $G^\wedge$, as K runs over all compact subsets of G. In particular the dense subgroup H determines the group G if and only if for every compact subset $K \subseteq G$ there exists a compact $R \subseteq H$ with $U_H(R) \subset U_G(K)|_H$ (here $U_G(K)|_H$ denotes the set of all restrictions to H of elements in $U_G(K)$).

If $A \subseteq G$, we write $\langle A \rangle$ for the subgroup of G generated by A. As usually, a map f between spaces X and Y is called *compact-covering* if whenever K is a compact subspace of Y, there exists a compact subspace C of X such that $K \subseteq f(C)$. Note that every compact-covering map is surjective.

2. Positive results on permanence properties of determined groups

A non determined group is the continuous image of the same group endowed with the discrete topology, which is determined. Hence the property of being a determined group is not preserved by continuous images. We do not know whether the following weaker result is true: if $\varphi : G \to H$ is continuous and onto and N determines G, then $\varphi(N)$ determines H. In this direction, the following simple result has interesting consequences.

Proposition 1. *Let $\varphi : G \to H$ be a continuous compact-covering homomorphism between the topological groups G and H. If N determines G, then $\varphi(N)$ determines H. If in addition φ is open and G is determined, then H is determined.*

PROOF. By continuity $\varphi(N)$ is dense in H. Because $\varphi : G \to H$ is compact-covering, if $K \subseteq H$ is compact, there is a compact $C \subseteq G$ with $K \subseteq \varphi(C)$. Since N determines G there is a compact $C_1 \subseteq N$ with $U(C_1) \subseteq U(C)$. We claim that $U(\varphi(C_1)) \subseteq U(K)$. Let $g \in K$ and $f \in U(\varphi(C_1))$. Then $f \circ \varphi \in U(C_1)$. Let $c \in C$ be such that $g = \varphi(c)$. Then

$$\frac{1}{4} \geq |f \circ \varphi(c)| = |f(g)|,$$

as required.

To prove the last part, note that if D is a dense subgroup of H, then $N := \varphi^{-1}(D)$ is dense in G, since φ is open.

This result generalizes (3.14) of [9].

Remark 1. *(a) In Proposition 1, the compact-covering and openness requirements on φ are not necessary, as witnessed by the following example: Let G be a metrizable group with proper dense subgroups. In particular G is not discrete. Let G_d denote the same group endowed with the discrete topology. Consider the projection $\pi_1 : G_d \times G \to G_d$ and the homomorphism $\varphi : G_d \times G \to G$, where $\pi_1(x, y) = \varphi(x, y) = x$ for every $(x, y) \in G \times G$. The map φ is continuous and onto, both $G_d \times G$ and G have proper dense subgroups and are determined, and if N determines $G_d \times G$ then $\varphi(N)$ determines G (actually for every dense subgroup N of $G_d \times G$ the subgroup $\pi_1(N)$ of G_d must be dense, hence $\pi_1(N) = G = \varphi(N)$). However φ is neither compact-covering nor open.*

(b) With the notation of Proposition 1, $\varphi(N)$ might determine H with N dense in G but not determining; a nontrivial example (with $H \neq \{0\}$) follows: In $G := \{0, 1\}^{\aleph_1}$ there is a countable dense not determining subgroup ([16] or [10]). However, if φ denotes the projection over countably many coordinates, then φ is open and compact-covering with $\varphi(N)$ determining $H := \varphi(G)$.

A space that is a G_δ in at least one of its compactifications is called *Čech-complete*.

Corollary 2. *Let G be a determined topological abelian group and H a closed subgroup of G. Suppose that one of the following conditions holds:*

(a) H is discrete.

(b) H is compact.

(c) G is Čech-complete.

Then the quotient group G/H is determined as well.

PROOF. If H is a closed subgroup of G, the quotient map $\varphi : G \to G/H$ is always continuous and open. It is known that under any of these three extra assumptions, the quotient map is compact covering as well: for (a) it is straightforward; for (b) it is a corollary of [17, 5.24(a)]; and for (c) it is proved in [2]. Hence this result follows from Proposition 1.

Even for a (proper) compact subgroup K of a topological group G, the quotient G/K can be determined but G not, as witnessed by $G := \{0,1\}^{\aleph_0} \times \{0,1\}^{\aleph_1}$, with $K := \{(0)_{\aleph_0}\} \times \{0,1\}^{\aleph_1}$. However, as the next Proposition shows, when the quotient G/K is determined, it is easy to prove that subgroups of the form $\varphi^{-1}(D)$, with D a dense subgroup of G/K, determine G.

Proposition 3. *Let G be a topological group, H a compact subgroup of G and $\varphi : G \longrightarrow G/H$ the natural map. If D is a dense subgroup of G/H which determines G/H, then $\varphi^{-1}(D)$ determines G.*

PROOF. Take a compact set C_1 in G. Then $\varphi(C_1)$ is compact in G/H; since D determines G/H, we can find another compact subset C_2 of D such that $U(C_2) \subset U(\varphi(C_1))$. Since C_2 is compact in D, $\varphi^{-1}(C_2)$ is compact ([17, 5.24(a)]) and it is contained in $\varphi^{-1}(D)$.

Let us see that $U(\varphi^{-1}(C_2)) \subset U(C_1)$: take an element $\chi \in U(\varphi^{-1}(C_2))$ and $g \in C_1$. We can suppose $0 \in C_2$, hence $H \subseteq \ker\chi$ and we can decompose χ as $\psi \circ \varphi$ for some $\psi \in (G/H)^\wedge$. Since $\chi \in U(\varphi^{-1}(C_2))$, $\psi \in U(C_2)$ but $U(C_2) \subset U(\varphi(C_1))$, hence $|\psi(\varphi(g))| \leq 1/4$ and therefore $\chi \in U(C_1)$.

Corollary 4.

1. *Let $\{G_i : i \in I\}$ be a family of topological abelian groups. If the product $\prod_{i\in I} G_i$ is determined, then $\prod_{j\in J} G_j$ is determined for every $J \subset I$.*
2. *If G is determined and H is a divisible open subgroup of G, H is determined.*

PROOF. 1. This follows from Proposition 1, taking into account that projections onto products of subfamilies are compact covering.

2. Observe that in this case $G \cong H \times G/H$, so we can apply item 1.

Remark 2. *Notice that a determined group G may contain an open subgroup H which is not determined ([9, Example 3.4(iii)]). Compare this with the following two results.*

Lemma 5. *Let G be a topological group, H an open subgroup of G and D a dense subgroup of G. Then D determines G if and only if $D \cap H$ determines H.*

PROOF. Suppose that D determines G. Fix a compact set $K \subseteq H$. Since D determines G, there exists a compact subset $R \subset D$ with $U_G(R) \subseteq U_G(K)$. Since the restriction map $D^\wedge \to (D \cap H)^\wedge$ is open ([6, 2.2(d)]), there is a compact subset $S \subseteq D \cap H$ with $U_G(R)|_H \supseteq U_H(S)$. Hence $U_H(S) \subseteq U_G(R)|_H \subseteq U_G(K)|_H \subseteq U_H(K)$.

Conversely, suppose that $D \cap H$ determines H. Write G as the disjoint union of cosets of H, say, $G = \bigcup_{i \in I}(g_i + H)$. Then $D = \bigcup_{i \in I}((g_i + H) \cap D)$ and by density and the fact that each coset is itself clopen we can assume that each $g_i \in D$. Doing so, it follows that $D = \bigcup_{i \in I}(g_i + (H \cap D))$. Let K be a compact subset of G. Set $K_i := K \cap (g_i + H)$. Then $K_i \neq \emptyset$ for only finitely many $i_1, \cdots, i_n$. Set $J := \{i_1, \cdots, i_n\}$. Then $K = \bigcup_{i \in J} K_i$. If we consider $i \in J$, then $K_i - g_i \subseteq H$. Since we are assuming that $H \cap D$ determines H, we can choose compact $C_i \subseteq H \cap D$ with $U_H(C_i, \frac{1}{8}) \subseteq U_H(K_i - g_i, \frac{1}{8})$. Hence $U_G(C_i, \frac{1}{8}) \subseteq U_G(K_i - g_i, \frac{1}{8})$, too. Set $C := \bigcup_{i \in J}(C_i \cup \{g_i\})$. Then C is a compact subset of D and we claim that $U_G(C, \frac{1}{8}) \subseteq U_G(K)$. If $x \in K$, then there is $i \in J$ with $x \in K_i$. Find $h \in H$ such that $x = g_i + h$. Then $h = x - g_i \in K_i - g_i$. Also, $C_i \subseteq C \Longrightarrow U_G(C, \frac{1}{8}) \subseteq U_G(C_i, \frac{1}{8}) \subseteq U_G(K_i - g_i, \frac{1}{8})$. Hence, if $\varphi \in U(C, \frac{1}{8})$, then $|\varphi(h)| \leq \frac{1}{8}$, and $|\varphi(g_i)| \leq \frac{1}{8}$. Therefore:

$$|\varphi(x)| = |\varphi(g_i) + \varphi(h)| \leq |\varphi(g_i)| + |\varphi(h)| \leq \frac{1}{8} + \frac{1}{8} = \frac{1}{4},$$

as required.

Theorem 6. *If G has an open subgroup which is determined, then G is itself determined.*

PROOF. This is an immediate consequence of Lemma 5.

Next we deal with products.

Lemma 7. *Let $\{G_i\}_{i\in I}$ be a family of topological abelian groups. If D_i determines G_i for every $i \in I$, then $\prod_{i\in I} D_i$ determines $\prod_{i\in I} G_i$.*

Proof. By [18], $(\prod_{i\in I} G_i)^\wedge$ is $\bigoplus_{i\in I} G_i^\wedge$ and $(\prod_{i\in I} D_i)^\wedge$ is $\bigoplus_{i\in I} D_i^\wedge$ via the natural isomorphisms. Since the restriction maps $\iota_i^\wedge : G_i^\wedge \to D_i^\wedge$ are topological isomorphisms, the associated map $\bigoplus_{i\in I} G_i^\wedge \to \bigoplus_{i\in I} D_i^\wedge$ is also a topological isomorphism. It is easy to see that this map coincides with the restriction map from $(\prod_{i\in I} G_i)^\wedge$ to $(\prod_{i\in I} D_i)^\wedge$.

In [16, 3.2] it was proved that if D_n determines G_n $(n < \omega)$, then $\bigoplus_{n<\omega} D_n$ determines $\prod_{n<\omega} G_n$. (Countability is essential, as in [16, 4.5] it is shown that $\bigoplus_{i\in I} \mathbb{R}$ does not determine $\mathbb{R}^I$ if I is uncountable.) It is clear that [16, 3.2] implies Lemma 7 for countable I.

The following two results contribute to a partial solution to the still unsolved problem of whether or not the product of two determined groups is determined.

Corollary 8. *Assume that G is a determined group and D is a discrete group. Then the product $G \times D$ is also a determined group.*

Proof. Follows easily from Theorem 6 and the fact that G is an open subgroup of $G \times D$.

Theorem 9. *Suppose M and N are topological torsion groups such that the orders of any two non-zero elements $g \in M$ and $h \in N$ are relatively prime. If M and N are determined, then $G := M \times N$ is determined.*

Proof. Any subgroup D of G is of the form $\pi_M(D) \times \pi_N(D)$; this is a direct consequence of the structure theorem for abelian torsion groups ([14, 8.4]). If D is dense, $\pi_M(D)$ and $\pi_N(D)$ are dense subgroups of M and N, respectively. Since the latter are determined, an application of Lemma 7 yields the result.

The next result is in the spirit of [16, 5.9].

Theorem 10. *No countable dense subgroup can determine $\mathbb{Z}^{\aleph_1}$.*

PROOF. Let N be a countable dense subgroup of $\mathbb{Z}^{\aleph_1}$. Consider the quotient map $\varphi : \mathbb{Z}^{\aleph_1} \longrightarrow \mathbb{Z}(2)^{\aleph_1}$. Then $\varphi(N)$ is a countable dense subgroup of $\mathbb{Z}(2)^{\aleph_1}$ and it therefore cannot determine $\mathbb{Z}(2)^{\aleph_1}$ [16, 5.9]. Set $K := \{0,1\}^{\aleph_1} \subseteq \mathbb{Z}^{\aleph_1}$. Since $\varphi(K) = \mathbb{Z}(2)^{\aleph_1}$ covers every compact subset of $\mathbb{Z}(2)^{\aleph_1}$, Proposition 1 implies our result.

Corollary 11. *$\mathbb{Z}^{\kappa}$ is determined if and only if $\kappa \leq \aleph_0$.*

PROOF. Taking into account that $\mathbb{Z}^{\aleph_1}$ does have countable dense subgroups [21, 16.4], we derive from Theorem 10 that $\mathbb{Z}^{\aleph_1}$ is not determined. Hence, if $\mathbb{Z}^{\kappa}$ is determined, we deduce $\kappa \leq \aleph_0$ using Corollary 4(1).

The converse is a consequence of the fact that metrizable abelian groups are determined ([3, 4.10]; [8]).

Remark 3. *Corollary 11 has also been proved as [16, 4.8], where the authors show that $\oplus_{\aleph_1}\mathbb{Z}$ cannot determine $\mathbb{Z}^{\aleph_1}$.*

Corollary 12. *Let $\prod_{i\in I} G_i$ be a product of non-trivial LCA compactly generated groups. Then the following statements are equivalent:*

1. *$\prod_{i\in I} G_i$ is determined.*
2. *$\prod_{i\in I} G_i$ is at most a countable product with each factor determined.*
3. *$\prod_{i\in I} G_i$ is metrizable.*

In particular, a LCA compactly generated group is determined if and only if it is metrizable.

PROOF. Notice that there are $(m_i, n_i) \in \omega^2$ and a compact group K_i such that $G_i = \mathbb{R}^{m_i} \times K_i \times \mathbb{Z}^{n_i}$.

(1$\Longrightarrow$2) By Corollary 4(1), for each $i \in I$, G_i must be determined. Applying Corollary 4(1) again, each K_i must be also determined, hence metric [16, 5.11]. Therefore each G_i must be metric, hence determined. Finally, since $\prod_{i\in I} G_i = \prod_{i\in I}(\mathbb{R}^{m_i} \times K_i \times \mathbb{Z}^{n_i}) = (\prod_{i\in I} \mathbb{R}^{m_i}) \times (\prod_{i\in I} K_i) \times (\prod_{i\in I} \mathbb{Z}^{n_i})$, an application of Corollary 4(1) again yields that each of the factors $(\prod_{i\in I} \mathbb{R}^{m_i}), (\prod_{i\in I} K_i)$ and $(\prod_{i\in I} \mathbb{Z}^{n_i})$ must be determined. If this is so, then [16, 5.11] together with [16, 4.8] or Corollary 11 would imply that each of these products is countable.

(2$\Longrightarrow$3) By Corollary 4(1), for each $k \in \omega$, K_k must be determined, hence metric [16, 5.11]. Therefore each G_k must be metric. Since the countable product of metric factors is itself metric, (3) follows.

(3$\Longrightarrow$1) follows from the fact that all metrizable groups are determined.

Determined LCA groups which are not compactly generated may be not metric as witnessed by examples 3.4 (ii) & (iii) in [9]. See also Example 15 below.

3. Non-determining pseudocompact groups

Recall that a topological abelian group is pseudocompact if and only if it is a G_δ-dense subgroup of a compact group. In this section we show that every nonmetrizable compact group has a dense pseudocompact subgroup which does not determine it. This answers negatively Question 5.12(iii) in [16].

The core of the proof is the following Lemma, which is based on the same arguments employed by Dikranjan and Tkachenko in [12, Theorem 5.5]. This lemma follows also from Propositions 3.3 and 3.4 of [15].

Recall that a subgroup D of a topological abelian group G is said to be h-embedded in G if every (not necessarily continuous) homomorphism from D to $\mathbb{T}$ admits an extension to a *continuous* homomorphism from G to $\mathbb{T}$.

Lemma 13. *Let H be either $\mathbb{T}$ or $\mathbb{Z}(p)$ for a prime p. Then $H^{\mathfrak{c}}$ admits a dense pseudocompact subgroup G such that all countable subgroups of G are h-embedded.*

Proof. Put $\Sigma = \bigcup_{K\subset\mathfrak{c},\,|K|\leq\omega} H^K$.

Enumerate Σ as $\Sigma = \{x_\gamma : \gamma < \mathfrak{c}\}$. For every $\gamma < \mathfrak{c}$ there exists $K_\gamma \subset \mathfrak{c}$, $|K_\gamma| \leq \omega$ such that $x_\gamma \in H^{K_\gamma}$.

Denote by $A(\mathfrak{c})$ the free abelian group over $\mathfrak{c}$. For every $l \in A(\mathfrak{c})$ we denote by $\mathrm{supp}(l)$ the minimal finite subset F of $\mathfrak{c}$ such that $l \in \langle F\rangle$. Enumerate the countable subsets of $\mathfrak{c}$ as C_ν, $\nu < \mathfrak{c}$. We denote by $A(C_\nu)$ the corresponding free abelian group over C_ν, and write $L_\nu := \bigcup_{\gamma\in C_\nu} K_\gamma$. Note that L_ν is a countable set.

For every $\nu < \mathfrak{c}$ denote by $\mathcal{H}_\nu$ the family of all homomorphisms from $A(C_\nu)$ to H. The family $\mathcal{H} = \bigcup_{\nu<\mathfrak{c}} \mathcal{H}_\nu$ has cardinality $\mathfrak{c}$ and we can enumerate it in this way: $\mathcal{H} = \{h_\alpha : \alpha < \mathfrak{c},\ \alpha \text{ not a limit ordinal}\}$. (This is possible because the set of nonlimit ordinals $\alpha < \mathfrak{c}$ has cardinality $\mathfrak{c}$.) For every nonlimit $\alpha < \mathfrak{c}$ there exists $\nu(\alpha) < \mathfrak{c}$ such that $h_\alpha \in \mathcal{H}_{\nu(\alpha)}$.

For every $K \subset L \subset \mathfrak{c}$, let $\pi^L_K : H^L \to H^K$ be the natural projection.

For every $\alpha < \mathfrak{c}$ we are going to define, by transfinite induction, subsets A_α of $\mathfrak{c}$, satisfying

(1) $A_\beta \subset A_\alpha$ for every $\beta < \alpha < \mathfrak{c}$

(2) $\alpha \subset A_\alpha$ for every $\alpha < \mathfrak{c}$

(3) $|A_\alpha| \le |\alpha| \cdot \omega$

and taking for every $\alpha < \mathfrak{c}$ and $\gamma < \mathfrak{c}$, $B_{\gamma,\alpha} = A_\alpha \cup K_\gamma$, we will define $y_{\gamma,\alpha} \in H^{B_{\gamma,\alpha}}$ satisfying

(4) $\pi^{B_{\gamma,\alpha}}_{K_\gamma}(y_{\gamma,\alpha}) = x_\gamma$ for every $\gamma, \alpha < \mathfrak{c}$

(5) $\pi^{B_{\gamma,\alpha}}_{B_{\gamma,\beta}}(y_{\gamma,\alpha}) = y_{\gamma,\beta}$ for every $\beta < \alpha < \mathfrak{c}$ and $\gamma < \mathfrak{c}$

Zero case. Put $A_0 = \emptyset$, $y_{\gamma,0} = x_\gamma$ for every $\gamma < \mathfrak{c}$.

Sucessor case. Suppose $\alpha = \beta + 1$. Recall that $\nu(\alpha) < \mathfrak{c}$ satisfies $h_\alpha \in \mathcal{H}_{\nu(\alpha)}$. Consider the homomorphism $h_\alpha : A(C_{\nu(\alpha)}) \to H$.

We put $A_\alpha = A_\beta \cup L_{\nu(\alpha)} \cup \{\beta, \delta(\alpha)\}$ where $\delta(\alpha) \in \mathfrak{c} \setminus (A_\beta \cup L_{\nu(\alpha)} \cup \{\beta\})$ is arbitrary. Now for every $\gamma < \mathfrak{c}$, we define $y_{\gamma,\alpha}$ in the following way.

For every $\lambda \in B_{\gamma,\alpha}$

$$y_{\gamma,\alpha}(\lambda) = \begin{cases} h_\alpha(\gamma) & \text{if } \lambda = \delta(\alpha) \text{ and } \gamma \in C_{\nu(\alpha)} \\ y_{\gamma,\beta}(\lambda) & \text{if } \lambda \in B_{\gamma,\beta} \\ 0 & \text{in other case} \end{cases}$$

This definition is consistent because $\gamma \in C_{\nu(\alpha)}$ and $\delta(\alpha) \in B_{\gamma,\beta}$ cannot hold simultaneously: note that $\delta(\alpha) \notin A_\beta$ and, on the other hand, $[\gamma \in C_{\nu(\alpha)} \Rightarrow K_\gamma \subset L_{\nu(\alpha)} \Rightarrow \delta(\alpha) \notin K_\gamma]$. Moreover, $y_{\gamma,\alpha}$ clearly satisfies (4) and (5) for every $\gamma < \mathfrak{c}$.

Limit case. Define $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ and for every $\gamma < \mathfrak{c}$, define $y_{\gamma,\alpha}$ satisfying (4) and (5) for every $\beta < \alpha$.

This finishes the construction, thus defining for every $\gamma < \mathfrak{c}$ an element $y_\gamma \in H^\mathfrak{c}$ in such a way that $\pi^\mathfrak{c}_{B_{\gamma,\alpha}}(y_\gamma) = y_{\gamma,\alpha}$ for every $\gamma, \alpha < \mathfrak{c}$. In particular we have $\pi^\mathfrak{c}_{K_\gamma}(y_\gamma) = x_\gamma$ for every $\gamma < \mathfrak{c}$, and $y_\gamma(\delta(\alpha)) = h_\alpha(\gamma)$ for every $\gamma < \mathfrak{c}$ and every nonlimit $\alpha < \mathfrak{c}$ with $\gamma \in C_{\nu(\alpha)}$.

We next define the homomorphism $\varphi : A(\mathfrak{c}) \to H^\mathfrak{c}$ as follows:

$$\varphi(\gamma) = y_\gamma, \quad \gamma < \mathfrak{c}.$$

Let us see that the group $G = \varphi(A(\mathfrak{c})) = \langle \{y_\gamma : \gamma < \mathfrak{c}\} \rangle$ satisfies the required properties.

All countable subgroups of G are h-embedded in G: Fix any countable subgroup $S \subseteq G$ countable and a homomorphism $h : S \to \mathbb{T}$. Let S_0 be a countable subgroup of $A(\mathfrak{c})$ such that $\varphi(S_0) = S$. Since S_0 is countable, there exists some $\nu < \mathfrak{c}$ such that $C_\nu = \bigcup_{l \in S_0} \operatorname{supp}(l)$; in particular $S_0 \leq A(C_\nu)$. We extend h to some homomorphism $\tilde{h} : \varphi(A(C_\nu)) \to \mathbb{T}$. In the case $H = \mathbb{Z}(p)$, it is clear that $\tilde{h}(\varphi(A(C_\nu)))$ is a subgroup of $\mathbb{Z}(p)$. Hence in any case there exists a nonlimit $\alpha < \mathfrak{c}$ such that $\nu = \nu(\alpha)$ and $h_\alpha = \tilde{h} \circ \varphi$. In particular we have

$$y_\gamma(\delta(\alpha)) = h_\alpha(\gamma) = \tilde{h}(\varphi(\gamma)) = \tilde{h}(y_\gamma)$$

for every $\gamma \in C_\nu$. This implies that the restriction to $G = \langle \{y_\gamma : \gamma < \mathfrak{c}\} \rangle$ of the projection on the coordinate $\delta(\alpha)$ extends $\tilde{h}$ and thus, h.

G is pseudocompact and dense in $H^{\mathfrak{c}}$: It is enough to show that for every countable $K \subseteq \mathfrak{c}$ and every $x \in H^K$ there exists $y \in G$ with $\pi^{\mathfrak{c}}_K(y) = x$. Fix K and x. Then $x = x_\gamma$ and $K = K_\gamma$ for some $\gamma < \mathfrak{c}$. Hence $y = y_\gamma$ satisfies $\pi^{\mathfrak{c}}_K(y) = x$, by construction.

Theorem 14. *Every compact abelian group G with $w(G) \geq \mathfrak{c}$ has a dense pseudocompact subgroup which does not determine G.*

Proof. There is a continuous epimorphism $\varphi : G \to H^{\mathfrak{c}}$ where $H = \mathbb{T}$ or $H = \mathbb{Z}(p)$ for some prime p (see [9, Theorem 4.15] for details). Note that φ is actually a quotient map. By Lemma 13 there is a dense pseudocompact subgroup D of $H^{\mathfrak{c}}$ all whose countable subgroups are h-embedded. Pseudocompact groups all whose countable subgroups are h-embedded contain no infinite compact subsets and hence are reflexive; this fact has been proved independently in [1, Theorem 2.11] and in [15, Lemma 2.3 and Theorem 6.1]. In particular D does not determine $H^{\mathfrak{c}}$. The subgroup $\varphi^{-1}(D)$ of G is pseudocompact and dense in G and by Proposition 1, it does not determine G either.

Recently Bruguera and Tkachenko proved ([7, Theorem 4.6]) that every compact abelian group G with $w(G) \geq \mathfrak{c}$ contains a proper dense reflexive pseudocompact subgroup H. It is clear that such a subgroup H, being reflexive, cannot determine G. Hence Theorem 14 can be deduced from their result, whose proof uses the existence of the subgroup resulting from the construction in Lemma 13, too.

In connection with these results, Dikranjan and Shakhmatov [10] have shown that no countable dense subgroup, if any, of a non-metrizable compact group is determining. A related result by the same authors [11] is that under CH, a compact abelian group with all its dense, countably compact subgroups being determining is necessarily metrizable.

4. Examples

As noticed in [9, 3.4], topological groups without proper dense subgroups are "trivially determined". We now present further examples of LCA, determined topological groups that are not metric and which have proper dense subgroups. Examples 3.4 (ii) & (iii) in [9] show the existence of determined LCA groups without proper dense subgroups that are not metric (such examples were first noticed in [20]). As far as we know, the following is the first example of a nonmetric LCA group *with proper dense subgroups* that is determined.

Example 15. *A determined LCA group with proper dense subgroups that is not metric.*

PROOF. Let p be a prime number. We denote by $\mathbb{Z}(p^\infty)$ the quasicyclic p-group. Consider the subgroup G of $(\mathbb{Z}(p^\infty))^{\aleph_1}$ generated by $K := \mathbb{Z}(p)^{\aleph_1}$, and $M := \bigoplus_{\aleph_1} \mathbb{Z}(p^\infty)$. Equip K with the product topology and make it open in G, i. e. U is a neighborhood of 0 in G if and only if $U \cap K$ is a neighborhood of 0 in K. Clearly G is a LCA torsion group that is not metric.
Claim 1. If D is a dense subgroup of G, then $\bigoplus_{\aleph_1} \mathbb{Z}(p) \subseteq D \cap K$. For, let $x \in \bigoplus_{\aleph_1} \mathbb{Z}(p)$. There exists $y \in M$ with $py = x$. Since $y + K$ is open in G, we can choose $z \in (y + K) \cap D$, say $z = y + k \in D$ wih $k \in K$. Then $x = py = py + pk = pz \in D$, as required.

By [9, 3.12] , $\bigoplus_{\aleph_1} \mathbb{Z}(p)$ determines K. Hence by last claim and Lemma 5, it follows that G is determined.
Claim 2. M is a proper dense subgroup of G. For, assume U is open in G, and consider $g \in U$ with $g = k+m$ with $k \in K$ and $m \in M$. Then $k \in (U-m) \cap K$ and since the latter is open in K there is $y \in \bigoplus_{\aleph_1} \mathbb{Z}(p) \cap ((U - m) \cap K)$, say $y = u - m$ with $u \in U$. But then $u = y + m \in U \cap M$, as required.

Examples [9] (3.4 (ii) & (iii)) show the existence of determined locally bounded but not complete groups without proper dense subgroups that are not metric. Noncomplete determined locally bounded groups with proper dense subgroups that are not metric exist: consider $\mathbb{R}$ equipped with its Bohr topology [9, 2.3]. An application of Theorem 9 yields further product-like examples:

Example 16. *A determined locally bounded but not complete group with proper dense subgroups that is not metric.*

Proof. Take $N := H \times G$ with G as above, and $H := \mathbb{Z}(q^\infty) \subset \mathbb{T}$ with $p \neq q$.

As far as we know, the following is the first example of its kind:

Example 17. *A non determined, countable group.*

Proof. For a topological abelian group G and a subgroup H of G let us denote by $\mathcal{A}(G^\wedge, H)$ the annihilator of H in $G^\wedge$, that is, the subgroup of $G^\wedge$ formed by those characters whose restriction to H is identically zero.

Let $\mathbb{T}_d$ denote the group $\mathbb{T}$ endowed with the discrete topology. Its Pontryagin dual $(\mathbb{T}_d)^\wedge$ can be identified with $b\mathbb{Z}$, the Bohr compactification of $\mathbb{Z}$. We denote by $\mathbb{Z}^\#$ the subgroup $\mathbb{Z}$ of $b\mathbb{Z}$ endowed with the induced topology.

Let A and B be proper subgroups of $\mathbb{T}$ such that A is dense with respect to the usual topology, B is countably infinite, and $\mathbb{T} = A \times B$ algebraically [17, (A.8)]. Put $N = \mathcal{A}(b\mathbb{Z}, A \times \{0_B\})$. N is a closed (hence compact) subgroup of $b\mathbb{Z}$. By [17, 23.25] and Pontryagin duality theorem, we have $N^\wedge \cong \mathbb{T}_d/(A \times \{0_B\})$ canonically. Since this last group is isomorphic with $\{0_A\} \times B$, we deduce that N has a countable dual group and thus it is metrizable [17, 24.15]. In particular N is determined and separable. Let D be a countable dense subgroup of N, and set $G = \langle \mathbb{Z}^\# \cup D \rangle_{b\mathbb{Z}}$. It is clear that G is countable and $\mathbb{Z}^\#$ is one of its dense subgroups.

We claim that $\mathbb{Z}^\#$ does not determine G. To show this, we must find a compact subset C of G such that $U_{\mathbb{Z}^\#}(K) \not\subseteq U_G(C)$ for every compact subset K in $\mathbb{Z}^\#$. Since compact subsets of $\mathbb{Z}^\#$ are finite by Glicksberg's theorem, it is enough to show that $U_G(C)$ cannot contain an open interval around zero in $\mathbb{T}$. Let us see that we can choose C in such a way that $U_G(C)$ is actually contained in the proper dense subgroup A of $\mathbb{T}$.

Choose a compact $C \subseteq D$ such that $U_N(C) = \{0\}$; this is possible because N is determined and compact and D is dense in N. Every element ψ in $U_G(C)$, being an element of $\mathbb{T}$, can be written as $\psi = \psi_1 + \psi_2$, with $\psi_1 \in A$ and $\psi_2 \in B$, and every element $c \in C \subseteq N = \mathcal{A}(b\mathbb{Z}, A \times \{0_B\})$ satisfies $\psi(C) = \psi_2(c)$. Thus $\psi \in U_G(C)$ implies $\psi_2 \in U_N(C) = \{0\}$; it follows that $U_G(C) \subseteq A$, as required.

5. Questions and Acknowledgements

Question 1. *With N as in Example 16, is $G := \mathbb{T} \times N$ determined?*

We next recall two questions from Sections 1 and 2:

Question 2. *Is there a group G with dense subgroup H such that $G^\wedge$ and $H^\wedge$ are topologically isomorphic, but the restriction map from $G^\wedge$ to $H^\wedge$ is not a topological isomorphism?*

Question 3. *According to Proposition 1, if $\varphi : G \to H$ is a continuous compact-covering homomorphism between the topological groups G and H and N determines G, then $\varphi(N)$ determines H. Can one replace "compact-covering" with "onto" in this result?*

The following natural question has been posed by the Referee.

Question 4. *Is the restriction on compactness of H necessary for the result of Proposition 3 to still hold?*

Acknowledgements. The authors are indebted to S. Ardanza-Trevijano, L. Aussenhofer, W. Comfort and S. Hernández for helpful discussions regarding the contents of this article. We have also benefitted from an unusually constructive Referee's report and we are grateful for his/her comments and suggestions.

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