

## How Connected is the Intuitionistic Continuum?<sup>1</sup>

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In the twenties Brouwer established the well-known continuity theorem “every real function is locally uniformly continuous”, [Brouwer 1924, Brouwer 1924a, Brouwer 1927]. From this theorem one immediately concludes that the continuum is indecomposable (*unzerlegbar*), i.e. if  $\mathbb{R} = A \cup B$  and  $A \cap B = \emptyset$  (denoted by  $\mathbb{R} = A + B$ ), then  $\mathbb{R} = A$  or  $\mathbb{R} = B$ .

Brouwer deduced the indecomposability directly from the fan theorem (cf. the 1927 Berline Lectures, [Brouwer 1992, p. 49]).

The theorem was published for the first time in [Brouwer 1928], it was used to refute the principle of the excluded middle:  $\neg \forall x \in \mathbb{R}(x \in \mathbb{Q} \vee \neg x \in \mathbb{Q})$ .

The indecomposability of  $\mathbb{R}$  is a peculiar feature of constructive universa, it shows that  $\mathbb{R}$  is much more closely knit in constructive mathematics, than in classical mathematics. The classical comparable fact is the topological connectedness of  $\mathbb{R}$ . In a way this characterizes the position of  $\mathbb{R}$ : the only (classically) connected subsets of  $\mathbb{R}$  are the various kinds of segments. In intuitionistic mathematics the situation is different; the continuum has, as it were, a syrupy nature, one cannot simply take away one point. In the classical continuum one can, thanks to the principle of the excluded third, do so. To put it picturesquely, the classical continuum is the frozen intuitionistic continuum. If one removes one point from the intuitionistic continuum, there still are all those points for which it is unknown whether or not they belong to the remaining part. This phenomenon is at the basis of the following considerations.

Suppose we remove one point from  $\mathbb{R}$ , i.e. consider  $\mathbb{R} - \{0\} = \{0\}^c = \{x \in \mathbb{R} | x \neq 0\}$ , we claim that this set is still indecomposable.

Let  $\mathbb{R} - \{0\} = A + B$ . Then  $\forall x \in \mathbb{R}(x \neq 0 \rightarrow x \in A \vee x \in B)$ .

We introduce  $\mathbb{R}^r = \{x \in \mathbb{R} | x \geq 0\}$ ,  $\mathbb{R}^l = \{x \in \mathbb{R} | x \leq 0\}$ , where by definition  $x \leq y \leftrightarrow \neg x > y$ . Let us assume that  $A + B$  is a proper decomposition.

Pick  $x \in \mathbb{R}^l$  with  $x < 0$ , then  $x \in A$  or  $x \in B$ , say  $x \in A$ . Now assume that  $y \in \mathbb{R}^l$  with  $y < 0$  and  $y \in B$ . Then  $(-\infty, 0) = (-\infty, 0) \cap A \cup (-\infty, 0) \cap B$ ,

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which yields an proper partitioning of  $(-\infty, 0)$ . This contradicts the indecomposability of  $\mathbb{R}$ .

So we get  $y < 0 \rightarrow \neg y \in B$ , and hence  $y \in B \rightarrow y \geq 0$ , i.e.  $B \subseteq \mathbb{R}^+$ .

Now let  $z > 0$  and suppose  $z \in A$ , then  $(0, \infty)$  has a proper decomposition, which conflicts with Brouwer's theorem. Hence  $z \in A \rightarrow z \leq 0$ , and thus  $A \subseteq \mathbb{R}^+$ .

Therefore we get

$$\forall x \in \mathbb{R}(x \neq 0 \rightarrow x \geq 0 \vee x \leq 0).$$

We now make use of Kripke's Schema, which says

$$(KS) \quad \exists \alpha \in \{0, 1\}^{\mathbb{N}}(\exists x(\alpha x = 1) \leftrightarrow \varphi).$$

i.e.  $\varphi$  is found to be true if and only if  $\alpha$  produces a 1.

It is no restriction to assume that  $\alpha$  takes the value 1 at most once:  $\forall x \sum_{y=0}^x \alpha(y) \leq 1$ .

We will call such sequences *Kripke-sequences*. We will apply Kripke's Schema to the statements  $r \in \mathbb{Q}$  and  $r \notin \mathbb{Q}$ .

Let  $\alpha$  and  $\beta$  be Kripke sequences for  $r \in \mathbb{Q}$  and  $r \notin \mathbb{Q}$ , i.e.

$$\exists x(\alpha x = 1) \leftrightarrow r \in \mathbb{Q}, \quad \exists y(\beta y = 1) \leftrightarrow r \notin \mathbb{Q}.$$

$$\text{Put} \quad \begin{cases} \gamma(2x) & = & \alpha(x) \\ \gamma(2x+1) & = & \beta(x) \end{cases}$$

$$\text{and define } c_n = \begin{cases} (-2)^{-n} & \text{if } \forall k \leq n(\gamma k = 0) \\ (-2)^{-k} & \text{if } k \leq n \wedge \gamma k = 1. \end{cases}$$

$$c = (c_n)_n = 0 \leftrightarrow \forall k \gamma k = 0 \leftrightarrow \neg r \in \mathbb{Q} \wedge \neg r \in \mathbb{Q}. \text{ Contradiction. So } c \neq 0.$$

$$\text{Hence } c \geq 0 \vee c \leq 0.$$

By definition

$$c \geq 0 \leftrightarrow \neg \exists x \beta(x) = 1 \leftrightarrow \neg r \in \mathbb{Q}.$$

and

$$c \leq 0 \leftrightarrow \neg \exists x \alpha(x) = 1 \leftrightarrow \neg r \in \mathbb{Q}.$$

Since we have taken  $r$  to be arbitrary, we get

$\forall r \in \mathbb{R}(\neg r \in \mathbb{Q} \vee \neg r \in \mathbb{Q})$ , which conflicts with the indecomposability of  $\mathbb{R}$ .

Hence the decomposition  $A + B$  could not have been proper.

We therefore see that picking a hole in  $\mathbb{R}$  does not make it decomposable, let alone disconnected in the topological sense. Note, however, that it is not the case that  $A$  with  $\mathbb{R} - \{0\} \subseteq A \subseteq \mathbb{R}$  is likewise indecomposable, e.g.  $(\mathbb{R} - \{0\}) \cup \{0\}$  is decomposable (by definition of 'union' or 'disjunction'). Such an  $A$  is of course connected in the topological sense.

We can do still better than picking single holes in  $\mathbb{R}$ :

*Theorem.*  $\mathbb{Q}^c$  is undecomposable.

Proof. Put  $\mathbb{Q}^c = \{x \in \mathbb{R} \mid x \notin \mathbb{Q}\}$ ,  $\mathbb{Q}^\# = \{x \in \mathbb{R} \mid \forall y \in \mathbb{Q}(x \# y)\}$ .

$\mathbb{Q}^\#$  is called the set of *positive irrationals*, and we can apply the standard representation by continued fractions to  $\mathbb{Q}^\#$ :  $\mathbb{Q}^\#$  is homeomorphic to  $\mathbb{N}^{\mathbb{N}}$  (*Baire space*, or the spread  $U$  of all numerical choice sequences), cf. [Brouwer 1921].

Let  $\mathbb{Q}^c = A \cup A^c$ , then  $\forall x \in \mathbb{Q}^\#(x \in A \vee x \in A^c)$ .

By the bar theorem there is a bar  $B$  in the spread  $U$ , such that for each node  $n$  in the bar we have that all choice sequences through that node belong simultaneously to  $A$  or to  $A^c$ . By bar induction the bars are inductively defined, so we prove our theorem by induction.

- (i) The bar is the top node. Let us say that all positive irrationals are in  $A$ . We show that  $\forall x \in \mathbb{Q}^c(x \in A)$ . Suppose that there is an irrational in  $A^c$ , say  $a$ .

We can find a sequence  $(a_n)$  in  $\mathbb{Q}^\#$  with  $\lim a_n = a$ . Now consider a Kripke sequence  $\alpha$  for  $r = 0$  (where  $r \in \mathbb{R}$ ):

$$\exists x(\alpha x = 1) \leftrightarrow r = 0.$$

$$\text{Define } b_n = \begin{cases} a_n & \text{if } \forall k \leq n \ \alpha k = 0 \\ a_k & \text{if } k \leq n \wedge \alpha k = 1 \end{cases}$$

If  $b = (b_n)_n \in \mathbb{Q}$ , then we would have  $\neg \exists x \alpha x = 1$ , hence  $\forall x \alpha x = 0$ . Therefore  $b = a \in \mathbb{Q}^c$ . Contradiction: So  $b \in \mathbb{Q}^c$ , and hence  $b \in A \vee b \in A^c$ .

If  $b \in A$  then  $b \neq a$ , so by definition  $\neg \forall x \ \alpha x = 0$ , i.e.  $\neg \neg r = 0$ , so  $r = 0$ .

If  $b \in A^c$ , then  $\forall k(b \neq a_k)$ , hence  $\forall x(\alpha x = 0)$ , hence  $r \neq 0$ .

So  $r = 0 \vee r \neq 0$ . Again we conclude  $\forall r \in \mathbb{R}(r = 0 \vee r \neq 0)$ , which

contradicts the undecomposability of  $\mathbb{R}$ . Thus  $a \in A$ , i.e.  $\mathbb{Q}^c = A$ .

- (ii) The bar  $B$  is the sum of denumerably many (sub-) bars  $B_i$  and by induction hypothesis each of the corresponding open segments  $I_i$  (under the standard homeomorphism) contain only irrationals in  $A$  or only irrationals in  $A^c$ . Suppose  $I_i$  and  $I_j$  determine different parts of the decomposition, e.g.

$$I_i \cap \mathbb{Q}^c \subseteq A \quad \wedge \quad I_j \cap \mathbb{Q}^c \subseteq A^c \quad (i < j).$$

Since for each irrational it is decidable whether it is in  $A$  or  $A^c$ , we can determine adjacent segments  $I_k$  and  $I_{k+1}$  corresponding respectively to  $A$  and  $A^c$ . Now consider the rational number that separates  $I_k$  and  $I_{k+1}$ . By repeating the argument used for the undecomposability of  $\mathbb{R} - \{0\}$  above, we obtain a contradiction. Hence  $\mathbb{Q}^c = A$  or  $\mathbb{Q}^c = A^c$ .

This proves the theorem.

Note that the proof uses the continuity principle, bar induction and Kripke's schema ( $SC + KS$ , cf. [Troelstra-van Dalen 1988]). It would be interesting to see which alternative principles yield the undecomposability of the irrationals. The theorem is not true for recursive mathematics with Markov's Principle, this principle (in its general form) is actually equivalent to  $\mathbb{R} - \{0\}$  being disconnected. Note that in recursive mathematics the continuum is, of course, unzerlegbar, but whether the irrationals are undecomposable under Church's Thesis without Markov's Principle is an open problem.

In a forthcoming paper we will establish an even stronger result, i.e. all dense negative subsets of the continuum are "unzerlegbar", assuming Kripke's schema.

Note that the above result rather influences intuitionistic dimension theory, e.g. classically one gets the one-dimensional continuum as the sum of the two obvious zero-dimensional subsets, the rationals and the irrationals. But intuitionistically the irrationals are themselves already one-dimensional. On the basis of the above mentioned stronger undecomposability result, we can even assert that the complement of a dense zero dimensional set is one dimensional.

Brouwer considered in his [Brouwer 1926] dimension only for located compact sets; it is quite possible that he had realised the strange behaviour of sets such as the irrationals.

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