On the Intertranslatability of Autoepistemic, Default and Priority Logics, and Parallel Circumscription

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Abstract. This paper concentrates on comparing the relative expressive power of five non-monotonic logics that have appeared in the literature. The results on the computational complexity of these logics suggest that these logics have very similar expressive power that exceeds that of classical monotonic logic. A refined classification of non-monotonic logics by their expressive power can be obtained using translation functions that satisfy additional requirements such as faithfulness and modularity used by Gottlob. Basically, we adopt Gottlob's framework for our analysis, but propose a weaker notion of faithfulness. A surprising result is deduced in light of Gottlob's results: Moore's autoepistemic logic is less expressive than Reiter's default logic and Marek and Truszczyński's strong autoepistemic logic. The expressive power of priority logic by Wang et al. is also analyzed and shown to coincide with that of default logic. Finally, we present an exact classification of the non-monotonic logics under consideration in the framework proposed in the paper.

1 Introduction

A variety of *non-monotonic logics* have been proposed as formalizations of *non-monotonic reasoning* (NMR). Among these formalizations are *circumscription* by McCarthy [19], *default logic* by Reiter [24], *autoepistemic logic* by Moore [20], *strong autoepistemic logic* by Marek and Truszczyński [17] as well as *priority logic* by Wang, You and Yuan [28]. The main goal of this paper is to compare these five non-monotonic logics on the basis of their *expressive power*, i.e. their capability of representing various problems from the NMR domain.

A way of measuring the expressive power of a non-monotonic logic is to analyze the computational complexity of its decision problems, and to rank these decision problems in the *polynomial time hierarchy* (PH) [1]. In fact, complexity issues have received much attention in the NMR community recently, and the decision problems of default logic, (strong) autoepistemic logic and circumscription have been systematically analyzed [4, 5, 8, 18, 21, 26]. To summarize these results in the propositional case, the major decision problems of these four non-monotonic logics are complete problems on the second level of PH. These complexity results suggest that (i) the expressive powers of non-monotonic logics exceed that of classical monotonic logic and (ii) the non-monotonic logics mentioned are of equal expressive power – if measured by the levels of PH.

The expressibility issue can also addressed in terms of *translation functions* between theories of non-monotonic logics. For instance, a variety of translation functions have been proposed to transform a default theory into an autoepistemic one [2, 9, 12, 13, 17, 25, 27] and back [11, 18] such that sets of conclusions are preserved to a reasonable degree in the translation. Also, translation functions between various kinds of default theories have been considered [2, 6, 15]. In fact, complexity results are based on *polynomial transformations* between decision problems of non-monotonic logics and thus give rise to translation functions between non-monotonic theories, too. Unfortunately, the aim of such transformations is to preserve the yes/no-answers of decision problems and nothing more. This leaves room for translations that depend globally on the theory under translation so that a local modification to the theory changes the translation totally. However, it is possible to introduce further constraints for translation functions. A very promising requirement – *modularity* – is introduced by Imielinski [10] and then used by Gottlob [9] and Niemelä [23]. Roughly speaking, a modular translation function is in a sense *systematic*: local changes in a theory cause only local changes in its translation. Most importantly, it has been shown that a modular translation function between certain non-monotonic logics is not possible. This indicates that the non-monotonic logics involved differ in expressive power, although their decision problems are equally complex.

This paper takes the expressive power of five non-monotonic logics into reconsideration. A central concept – the notion of a polynomial, faithful and modular translation function – is adopted from Gottlob's work [9]. However, the notion of faithfulness is revised in an important way: it is assumed that sets of conclusions are preserved up to a fixed propositional language. This allows one to add, e.g., new propositional atoms in a translation if necessary and this is indeed the case with a number of translations functions addressed in this paper. Moreover, it is shown that polynomial, faithful and modular translations do not exist in certain cases in order to establish strict differences in expressive power. The comparisons made in the paper lead to an exact classification of non-monotonic logics. A particular novelty in this respect is that *Moore's autoepistemic logic is less expressive than Reiter's default logic*. Gottlob [9] employs a stronger notion of faithfulness and concludes the opposite. This demonstrates in an interesting way how the requirements imposed on translation functions affect the results on expressibility. Also, new light is shed on the interconnection of priority logic and default logic by showing that these logics are of equal expressive power.

The plan of this paper is as follows. In Section 2, we review the basic notions of non-monotonic logics mentioned. After this, the criteria for translation functions are set up in Section 3. In Section 4, actual translation functions are presented to rank non-monotonic logics by their expressive power. Some comparisons are also made with related work. Finally, the resulting classification of non-monotonic logics by their expressive power is illustrated and discussed in Section 5.

2 Logics of Interest

In this section, we review the basic definitions and notions of non-monotonic logics [14, 17, 20, 24, 28] that appear in the rest of this work. To allow a uniform treatment of these logics in sections to come, we have made the definitions similar as follows. (i) Only the propositional case is considered. Definitions are given relative to a propositional language $\mathcal{L}$ which is based on a finite or at most countable set of propositional atoms $\mathcal{A}$. (ii) A propositional subtheory $T \subseteq \mathcal{L}$ is distinguished for each *non-monotonic theory*, i.e. a theory of a non-monotonic logic. (iii) The sets of conclusions associated with non-monotonic theories are identified. Such sets are often called *extensions* or *expansions* and they determine the *semantics* of non-monotonic theories. Generally speaking, a non-monotonic theory may have a unique extension, several extensions, or sometimes even no extensions. We consider both *brave* and *cautious* reasoning strategies with extensions. In the former strategy, finding a single extension for a theory is of interest, while the intersection of extensions is considered in the latter.

2.1 Default Logic (DL)

A *default theory* [24] is a pair $\langle D, T\rangle$ where $T \subseteq \mathcal{L}$ and D is a set of *default rules* (or *defaults*) of the form $\alpha : \beta_1, \ldots, \beta_n/\gamma$ such that $n \geq 0$ and the *prerequisite* α, the *justifications* $\beta_1, \ldots, \beta_n$ and the *consequent* γ of the rule are sentences of $\mathcal{L}$. Marek and Truszczyński [18] reduce a set of defaults D with respect to a propositional theory $E \subseteq \mathcal{L}$ to a set of *inference rules* D_E which contains an inference rule α/γ whenever there is a default rule $\alpha : \beta_1, \ldots, \beta_n/\gamma \in D$ such that $E \cup \{\beta_i\}$ is consistent for all $0 < i \leq n$. We need also the closure of a theory $T \subseteq \mathcal{L}$ under a set of inference rules R, denoted by $\mathrm{Cn}^R(T)$, which is the least theory $E \subseteq \mathcal{L}$ satisfying (i) $T \subseteq E$, (ii) the set of propositional consequences $\mathrm{Cn}(E) = \{\phi \in \mathcal{L} \mid E \models \phi\} \subseteq E$ and (iii) $\{\gamma \mid \alpha/\gamma \in R \text{ and } \alpha \in E\} \subseteq E$.[1] The sets of conclusions associated with a default theory $\langle D, T\rangle$ are defined as follows.

Definition 1 (Marek and Truszczyński [18]). *A theory $E \subseteq \mathcal{L}$ is an extension of a default theory $\langle D, T\rangle$ if and only if $E = \mathrm{Cn}^{D_E}(T)$.*

2.2 Autoepistemic Logic (AEL)

An *autoepistemic language* $\mathcal{L}_\mathbf{B}$ is the unimodal extension of $\mathcal{L}$ with a modal operator $\mathbf{B}$ for beliefs whereas an *autoepistemic theory* $\Sigma \subseteq \mathcal{L}_\mathbf{B}$ [20]. Sentences of the form $\mathbf{B}\phi$ are known as *belief atoms* and the set of logical consequences $\mathrm{Cn}(\Sigma) \subseteq \mathcal{L}_\mathbf{B}$ is defined in the standard way by treating belief atoms as additional propositional atoms. In this paper, a pair of theories $\langle \Sigma, T\rangle$ where $T \subseteq \mathcal{L}$ is a propositional theory is also called an autoepistemic theory (then $\Sigma \cup T$ is a theory in Moore's sense). Moore's idea is to capture the sets of beliefs Δ of an *ideal and rational agent* which believes exactly the logical consequences of $\Sigma \subseteq \mathcal{L}_\mathbf{B}$ and its

[1] Marek and Truszczyński [18] propose a proof system to capture this closure.

beliefs $\mathbf{B}\Delta = \{\mathbf{B}\phi \mid \phi \in \Delta\}$ and disbeliefs $\neg\mathbf{B}\overline{\Delta} = \{\neg\mathbf{B}\phi \mid \phi \in \mathcal{L}_{\mathbf{B}} - \Delta\}$ obtained by *introspection*. Such sets of beliefs/conclusions are called *stable expansions*.

Definition 2 (Moore [20]). *A theory $\Delta \subseteq \mathcal{L}_{\mathbf{B}}$ is a stable expansion of an autoepistemic theory $\Sigma \subseteq \mathcal{L}_{\mathbf{B}}$ if and only if $\Delta = \mathrm{Cn}(\Sigma \cup \mathbf{B}\Delta \cup \neg\mathbf{B}\overline{\Delta})$.*

2.3 Strong Autoepistemic Logic (SAEL)

Theories of strong autoepistemic logic [17] are similar to those of AEL. Given $\Sigma \subseteq \mathcal{L}_{\mathbf{B}}$, we let $\mathrm{Cn}_{\mathbf{B}}(\Sigma)$ denote the closure of Σ under propositional inference and the standard *necessitation rule:* from ϕ infer $\mathbf{B}\phi$. More formally, $\mathrm{Cn}_{\mathbf{B}}(\Sigma)$ is the least theory $\Delta \subseteq \mathcal{L}_{\mathbf{B}}$ satisfying (i) $\Sigma \subseteq \Delta$, (ii) $\mathrm{Cn}(\Delta) \subseteq \Delta$ and (iii) $\mathbf{B}\Delta \subseteq \Delta$.[2] This leads to the definition of *iterative expansions* below. Iterative expansions of $\Sigma \subseteq \mathcal{L}_{\mathbf{B}}$ are also stable expansions of Σ, but not necessarily vice versa [17], and thus Σ is assigned a different semantics under iterative expansions.

Definition 3 (Marek and Truszczyński [17]). *A theory $\Delta \subseteq \mathcal{L}_{\mathbf{B}}$ is an iterative expansion of $\Sigma \subseteq \mathcal{L}_{\mathbf{B}}$ if and only if $\Delta = \mathrm{Cn}_{\mathbf{B}}(\Sigma \cup \neg\mathbf{B}\overline{\Delta})$.*

2.4 Parallel Circumscription (CIRC)

We present a generalization of McCarthy's approach [19], namely *parallel circumscription* by Lifschitz [14]. A *minimal model theory* is a triple $\langle P, F, T\rangle$ where $P \subseteq \mathcal{A}$ and $F \subseteq \mathcal{A}$ are mutually disjoint set of atoms and $T \subseteq \mathcal{L}$. The idea behind parallel circumscription [14] is to distinguish propositional models $\mathcal{M}$ that are minimal in the sense of Definition 4: as many atoms of P should be false in $\mathcal{M}$ as possible. Note that the atoms of F remain *fixed* in the minimization process while the atoms in $\mathcal{A} - (P \cup F)$ may *vary* freely.

Definition 4 (Lifschitz [14]). *A propositional model $\mathcal{M} \subseteq \mathcal{A}$ of $T \subseteq \mathcal{L}$ is $\langle P, F\rangle$-minimal if and only if there is no propositional model $\mathcal{M}' \subseteq \mathcal{A}$ of T such that $\mathcal{M}' \cap F = \mathcal{M} \cap F$ and $\mathcal{M}' \cap P \subset \mathcal{M} \cap P$.*

There is no explicit notion of extensions involved in parallel circumscription, but let us propose an implicit one. Given a propositional model $\mathcal{M}$, we let $\mathrm{True}(\mathcal{M})$ denote the theory $\{\phi \in \mathcal{L} \mid \mathcal{M} \models \phi\}$. A $\langle P, F\rangle$-minimal model $\mathcal{M}$ gives rise to a $\langle P, F\rangle$-extension $E \subseteq \mathcal{L}$ of T which is the intersection of the theories $\mathrm{True}(\mathcal{M}')$ for all $\langle P, F\rangle$-minimal models $\mathcal{M}'$ of T such that $\mathcal{M}' \cap (P \cup F) = \mathcal{M} \cap (P \cup F)$. In this setting, the $\langle P, F\rangle$-minimal models of T are divided into (equivalence) classes that give rise to $\langle P, F\rangle$-extensions. Obviously, there may be several models in one class, since the atoms in $\mathcal{A} - (P \cup F)$ may vary freely. What comes to the cautious reasoning strategy, the correspondence of $\langle P, F\rangle$-extensions and $\langle P, F\rangle$-minimal models given in Proposition 1 is straightforward to establish. The notion of $\langle P, F\rangle$-extensions proposed is also appropriate if the brave[3] reasoning strategy is used in conjunction with parallel circumscription.

[2] Gottlob's **B**-proofs [8] capture this closure.

[3] Note, e.g., that Eiter and Gottlob [5] consider the complexity of propositional circumscription according to the cautious strategy only.

Proposition 1. *Given a minimal model theory* $\langle P, F, T\rangle$*, the intersection of* $\langle P, F\rangle$*-extensions of the theory* $T \subseteq \mathcal{L}$ *coincide with the intersection of the theories* $\mathrm{True}(\mathcal{M})$ *for all* $\langle P, F\rangle$*-minimal models of* T*.*

2.5 Priority Logic (PL)

A theory of *priority logic* [28] is a triple $\langle R, P, T\rangle$ where R is a set of (monotonic) inference rules[4] in $\mathcal{L}$, $P \subseteq R \times R$ gives a priority relation among the rules of R and $T \subseteq \mathcal{L}$. The idea behind prioritisation of inference rules is that if two rules r_1 and r_2 from R are in the relation P (denoted by $r_1 \prec r_2$ in [28]), then the application of the rule r_2 *blocks* that of r_1 and the rule r_2 has a higher priority than r_1 in this sense. For a set of rules R and a theory $E \subseteq \mathcal{L}$, we write $\mathrm{App}(R, E)$ to denote the set $\{\alpha/\gamma \in R \mid E \models \alpha\}$ which contains the rules of R that are applicable given E. To interpret the priority relation P, we define $\mathrm{Nb}(R, P, R') \subseteq R$ as the set of rules $r \in R$ which are *not blocked* given that the rules of R' are applicable, i.e. there is no $r' \in R'$ such that $\langle r, r'\rangle \in P$. These notions suffice to define extensions for a priority theory $\langle R, P, T\rangle$. The set of rules R' in the definition is called a *stable argument* by Wang et al. [28].

Definition 5. *A theory* $E \subseteq \mathcal{L}$ *is an extension of a priority theory* $\langle R, P, T\rangle$ *if and only if* $E = \mathrm{Cn}^{R'}(T)$ *for* $R' \subseteq R$ *satisfying* $R' = \mathrm{App}(\mathrm{Nb}(R, P, R'), E)$*.*

3 Requirements Imposed on Translation Functions

From now on, we restrict ourselves to **finite theories** of non-monotonic logics introduced in Section 2. In this section, we introduce the basic requirements for translation functions that map a theory of one non-monotonic logic to a theory of another. The requirements will be named as *polynomiality*, *faithfulness* and *modularity*. In the forthcoming mathematical formulations of these requirements, we let $\langle X, T\rangle$ stand for a non-monotonic theory where T is its propositional subtheory and X stands for any set(s) of syntactic elements which are specific to the non-monotonic logic in question (such as a set of defaults D in DL). The non-monotonic theories introduced in Section 2 are clearly of this form. Our first requirement involves the *length* of a non-monotonic theory $\langle X, T\rangle$, denoted by $||\langle X, T\rangle||$, which is the number of symbol occurrences needed to represent $\langle X, T\rangle$.

Definition 6 (Polynomiality). *A translation function* Tr *is polynomial, iff for all* $\langle X, T\rangle$*, the time required to compute* $\mathrm{Tr}(\langle X, T\rangle)$ *is polynomial in* $||\langle X, T\rangle||$*.*

To give an example of a such a function, we introduce a linear function that transforms a minimal model theory into an autoepistemic one.

Definition 7 (Niemelä [22]). *For all minimal model theories* $\langle P, F, T\rangle$*, let* $\mathrm{Tr}_{\mathrm{N}}(\langle P, F, T\rangle) = \langle\{\neg\mathbf{B}\mathrm{a} \rightarrow \neg\mathrm{a} \mid \mathrm{a} \in P \cup F\} \cup \{\neg\mathbf{B}\neg\mathrm{a} \rightarrow \mathrm{a} \mid \mathrm{a} \in F\}, T\rangle$*.*

[4] Wang et al. [28] use rules of the form $\gamma \leftarrow \alpha_1, \ldots, \alpha_n$ with multiple prerequisites, but such rules can be represented as $\alpha_1 \wedge \ldots \wedge \alpha_n/\gamma$ under propositional closure.

The next question is whether a translation function Tr preserves the semantics of a non-monotonic theory $\langle X, T\rangle$. We have used the following criteria to formulate our forthcoming definition. (i) Since the semantics of $\langle X, T\rangle$ is determined by its extensions and both brave and cautious reasoning strategies should be supported, a one-to-one correspondence of extensions is a natural solution. (ii) Only propositionally consistent extensions are taken into account in this one-to-one relationship, because we have in mind translation functions (such as Tr_1 in Definition 10) whose faithfulness depends on this restriction. (iii) Moreover, we are assuming that the language $\mathcal{L}$ of T is a fixed propositional language which is used for knowledge representation in a given domain. The propositional languages associated with $\langle X, T\rangle$ and $\mathrm{Tr}(\langle X, T\rangle)$ may extend $\mathcal{L}$, but we project the extensions of these theories with respect to $\mathcal{L}$. In particular, this means that a translation function can add new atoms, but within the bounds of our polynomiality requirement. This seems a crucial option in order to support different kinds of knowledge representation and reasoning techniques. For instance, the translation function Tr_{N} introduces belief atoms for these reasons.

Definition 8 (Faithfulness). *A translation function* Tr *is faithful, iff for all* $\langle X, T\rangle$*, the propositionally consistent extensions of* $\langle X, T\rangle$ *and* $\mathrm{Tr}(\langle X, T\rangle)$ *are in one-to-one correspondence and coincide up to the propositional language* $\mathcal{L}$ *of* T*.*

This definition ensures that given a faithful translation function Tr, any brave or cautious conclusion $\phi \in \mathcal{L}$ obtained from $\langle X, T\rangle$ can also be obtained from the translation $\mathrm{Tr}(\langle X, T\rangle)$, and vice versa. Note that this presumes that only propositionally consistent extensions are taken into account in the brave strategy. Let us yet point out that our notion is useful only if a notion of extensions is available for the (non-monotonic) logics involved. Fortunately, this is the case with logics addressed in this paper. Our notion of faithfulness is also closely related to the one by Gottlob [9]. The differences are that Gottlob does not allow new atoms to be introduced in a translation and he takes also the propositionally inconsistent extensions into account. Consequently, a translation function that is faithful in Gottlob's sense is also faithful in our sense. The converse does not hold in general – which is to be demonstrated in Theorem 4 and Example 1.

By Niemelä's results [22] and the notion of $\langle P, F\rangle$-extensions proposed in this paper, the translation function introduced in Definition 7 is faithful: the $\langle P, F\rangle$-extensions of T and the propositionally consistent stable expansions of $\mathrm{Tr}_{\mathrm{N}}(\langle P, F, T\rangle)$ are in a one-to-one correspondence and coincide up to the language $\mathcal{L}$ of T. An inconsistent stable expansion $\Delta = \mathcal{L}_{\mathbf{B}}$ appears only if T is inconsistent and there are no $\langle P, F\rangle$-extensions. Our last requirement follows.

Definition 9 (Modularity, Gottlob [9]). *A translation function* Tr *is modular, iff for all* $\langle X, T\rangle$, $\mathrm{Tr}(\langle X, T\rangle) = \langle X', T' \cup T\rangle$ *where* $\langle X', T'\rangle = \mathrm{Tr}(\langle X, \emptyset\rangle)$.

Our modularity requirement is a generalization of the one that Gottlob formulated for translations from DL into AEL [9]. In particular, a modular translation function provides a fixed translation for X (i.e. the non-monotonic theory $\mathrm{Tr}(\langle X, \emptyset\rangle)$) which is independent of T. Therefore, if T is updated, there is no need to recompute the fixed part $\mathrm{Tr}(\langle X, \emptyset \rangle)$ in order to compute $\mathrm{Tr}(\langle X, T \rangle)$. Note also that the translation function Tr_{N} of Definition 7 is modular in this sense, because P and F are translated into a fixed autoepistemic theory.

For the sake of brevity, we say that a **translation function is PFM** if it satisfies the three requirements set up in Definitions 6, 8 and 9. A fundamental property of PFM translations is pointed out in the following.

Proposition 2. *A composition of PFM translation functions is also PFM.*

PFM translation functions provide us the basis for analyzing the relative expressive power of non-monotonic logics. The motivation for this is that if there is a PFM translation function that maps theories of a non-monotonic logic L_1 to theories of a non-monotonic logic L_2, then we consider L_2 to be *at least as expressive as* L_1. This gives rise to a preorder among (non-monotonic) logics. For instance, AEL is at least as expressive as CIRC, because Tr_{N} is PFM. If – *in addition* – there are **no** PFM translation functions in the opposite direction, then we say that L_1 is *less expressive* than L_2. If there are PFM translation functions in both directions, then L_1 and L_2 are of equal expressive power. As concluded by Gottlob [9], this view identifies the expressive power of non-monotonic logics with their capability of representing different propositional closures in $\mathcal{L}$.

As a final issue in this section, we compare our approach with another by Gogic, Kautz, Papadimitriou and Selman [7]. They propose a framework for analyzing the *succinctness* of knowledge representation (i.e. the space required in knowledge representation) and thus also the expressive power of formalisms involved, but different kinds of translation functions are used. (i) Gogic et al. use a different polynomiality requirement: the length of the translation has to be polynomial in the length of the theory. This allows even exponential computations to obtain a translation as long as only a polynomial blow-up results in the translation. Our requirement restricts the translation time and thus also the translation space to be polynomial. (ii) Gogic et al. formulate their notion of faithfulness as a requirement that the propositional models of the theory under translation are preserved. If a translation function is faithful in our sense, then it is in their sense, too, provided that models of extensions are taken into account up to $\mathcal{L}$. The converse does not hold in general, since our notion of faithfulness presumes that a notion of extensions is available for the non-monotonic logics involved. (iii) Gogic et al. do not employ a modularity requirement.

4 Classifying Non-monotonic Logics

Having set up the notion of a PFM translation function, such translation functions are exhibited in this section in order to classify non-monotonic logics by their expressive power. Moreover, counter-examples are provided to show that such translations are not possible in certain cases. Such non-equivalence proofs have already been devised for non-monotonic logics by Imielinski [10], Gottlob [9] and Niemelä [23]. In the forthcoming subsections, we perform a pairwise comparison of non-monotonic logics in the following order: CIRC, AEL, SAEL, DL and PL. But as a starter, we relate classical propositional logic (CL) with CIRC. This result is supported by other complexity and intranslatability results [3–5].

Theorem 1. *CL is less expressive than CIRC.*

Proof. A propositional theory $T \subseteq \mathcal{L}$ is translated into a minimal model theory $\mathrm{Tr}_0(T) = \langle \emptyset, \emptyset, T \rangle$ having the same propositional language $\mathcal{L}$. The only $\langle \emptyset, \emptyset \rangle$-extension of T is $\mathrm{Cn}(T)$, i.e. the natural "extension" of T in propositional logic. Thus it is easy to see that Tr_0 is PFM and CIRC is at least as expressive as CL. Let us then assume there is also a PFM translation function Tr in the other direction. Let $\mathcal{A} = \{\mathrm{a}, \mathrm{b}\}$ and $\mathcal{L}$ the corresponding propositional language. Then consider a minimal model theory $\langle P, F, T \rangle$ based on $\mathcal{L}$ where $T = \{\mathrm{a} \rightarrow \mathrm{b}\}$, $P = \{\mathrm{a}, \mathrm{b}\}$ and $F = \emptyset$. This has a unique $\langle P, F \rangle$-minimal model $\mathcal{M} = \emptyset$ so that a unique $\langle P, F \rangle$-extension $E = \mathrm{Cn}(\{\neg \mathrm{a}, \neg \mathrm{b}\})$ results. Thus the propositional translation $\mathrm{Tr}(\langle P, F, T \rangle)$ must entail $\neg$a and $\neg$b. However, if we update T to $T' = T \cup \{a\}$, there is a unique $\langle P, F \rangle$-extension $E' = \mathrm{Cn}(\{\mathrm{a}, \mathrm{b}\})$ of T'. By modularity, the translation $\mathrm{Tr}(\langle P, F, T' \rangle)$ has to be $\mathrm{Tr}(\langle P, F, T \rangle) \cup \{\mathrm{a}\}$ which is necessarily propositionally inconsistent. Thus Tr cannot be faithful, a contradiction.

4.1 Comparison of CIRC and AEL

The translation function Tr_{N} satisfies our requirements by Niemelä's results [22].

Theorem 2 (Niemelä [22]). *The translation function* Tr_{N} *is PFM.*

This indicates that reasoning corresponding to $\langle P, F \rangle$-minimal models is easily captured in terms of stable expansions of the translation and that AEL is at least as expressive as CIRC. In Theorem 3, we adopt a counter-example given by Niemelä [23] to show that there is no translation function meeting our criteria in the opposite direction. Thus CIRC is less expressive than AEL.

Theorem 3. *There is* **no** *PFM translation function from autoepistemic theories under stable expansions into minimal model theories.*

Proof. Let $\mathcal{A} = \{\mathrm{a}, \mathrm{b}\}$ and let $\mathcal{L}$ and $\mathcal{L}_{\mathbf{B}}$ be the respective propositional and autoepistemic languages. Let us make a hypothesis that there is a fixed polynomial translation of $\Sigma = \{\mathbf{B}\mathrm{a} \rightarrow \mathrm{b}\} \subseteq \mathcal{L}_{\mathbf{B}}$ into sets of atoms P and F and a propositional theory $\mathrm{Tr}(\Sigma)$ such that for all $T \subseteq \mathcal{L}$, the propositionally consistent stable expansions of $\langle \Sigma, T \rangle$ and $\langle P, F \rangle$-extensions of $\mathrm{Tr}(\Sigma) \cup T$ are in one-to-one correspondence and coincide up to $\mathcal{L}$. The language $\mathcal{L}'$ of $\mathrm{Tr}(\Sigma) \cup T$ is assumed to be based on $\mathcal{A}' \supseteq \mathcal{A}$ and the sets of atoms P and F are subsets of $\mathcal{A}'$.

For $T = \emptyset$, there is exactly one propositionally consistent stable expansion $\Delta = \{\neg \mathbf{B}\mathrm{a}, \neg \mathbf{B}\mathrm{b}, \ldots\}$ of $\langle \Sigma, T \rangle$ such that $\mathrm{a} \rightarrow \mathrm{b} \notin \Delta$. It follows by our hypothesis that there is a unique $\langle P, F \rangle$-extension E of $\mathrm{Tr}(\Sigma) \cup T$ such that $\Delta \cap \mathcal{L} = E \cap \mathcal{L}$. This implies that $\mathrm{a} \rightarrow \mathrm{b} \notin E$. So there is a $\langle P, F \rangle$-minimal model $\mathcal{M}$ of $\mathrm{Tr}(\Sigma) \cup T$ such that $\mathcal{M} \not\models \mathrm{a} \rightarrow \mathrm{b}$, i.e. $\mathcal{M} \models \mathrm{a}$ and $\mathcal{M} \not\models \mathrm{b}$. Then let $T' = \{\mathrm{a}\}$ so that also $\mathcal{M} \models \mathrm{Tr}(\Sigma) \cup T'$. It is easy to see that $\mathcal{M}$ is a $\langle P, F \rangle$-minimal model of

$\mathrm{Tr}(\Sigma) \cup T'$, since otherwise $\mathcal{M}$ would not be a $\langle P, F\rangle$-minimal model of $\mathrm{Tr}(\Sigma) \cup T$. It follows that b does not belong to the corresponding (propositionally consistent) $\langle P, F\rangle$-extension E' of $\mathrm{Tr}(\Sigma) \cup T'$. By our hypothesis, there is a propositionally consistent stable expansion Δ' of $\langle \Sigma, T'\rangle$ such that $\mathrm{b} \notin \Delta'$. But this is a contradiction, since the only stable expansion of $\langle \Sigma, T'\rangle$ is $\{\mathrm{a}, \mathbf{B}\mathrm{a}, \mathrm{b}, \mathbf{B}\mathrm{b} \ldots\}$. □

4.2 Comparison of AEL and SAEL

The author [11] presents a translation function that allows one to capture the stable expansions of an autoepistemic theory with the iterative expansions of the translation. We let $\mathrm{RBa}(\phi)$ denote the set of belief atoms that appear in an autoepistemic sentence ϕ recursively and define $\mathrm{RBa}(\Sigma) = \bigcup\{\mathrm{RBa}(\phi) \mid \phi \in \Sigma\}$ for sets of autoepistemic sentences Σ. To give a simple example, we note that $\mathrm{RBa}(\mathbf{B}(\mathbf{B}\mathrm{p} \wedge \mathbf{B}\mathrm{q})) = \{\mathbf{B}(\mathbf{B}\mathrm{p} \wedge \mathbf{B}\mathrm{q}), \mathbf{B}\mathrm{p}, \mathbf{B}\mathrm{q}\}$. The intuition behind the translation is that the positive introspection ($\mathbf{B}\Delta$) in the definition of stable expansions is realized using instances $\neg\mathbf{B}\neg\mathbf{B}\phi \rightarrow \mathbf{B}\phi$ of the axiom schema **5**.

Definition 10 (Janhunen [11]). *For all autoepistemic theories* $\langle \Sigma, T\rangle$, *the translation* $\mathrm{Tr}_1(\langle \Sigma, T\rangle) = \langle \Sigma \cup \{\neg\mathbf{B}\neg\mathbf{B}\phi \rightarrow \mathbf{B}\phi \mid \mathbf{B}\phi \in \mathrm{RBa}(\Sigma)\}, T\rangle$.

Theorem 4. *The translation function* Tr_1 *given in Definition 10 is PFM.*

Proof. The polynomiality and modularity of Tr_1 are easily seen from the definition. Faithfulness follows by the results of Marek et al. [16] and the author [11], namely the propositionally consistent stable expansions of $\langle \Sigma, T\rangle$ and the iterative expansions of $\mathrm{Tr}_1(\langle \Sigma, T\rangle)$ coincide. This implies the one-to-one correspondence of propositionally consistent expansions as required by Definition 8 so that the translation function Tr_1 is also faithful. Note that Tr_1 is not faithful in Gottlob's sense [9], since an autoepistemic theory $\langle \Sigma, \emptyset\rangle$ where $\Sigma = \{\mathbf{B}\mathrm{p} \rightarrow \mathrm{p} \wedge \mathrm{r},\ \mathbf{B}\mathrm{q} \rightarrow \mathrm{q} \wedge \neg\mathrm{r}\}$ [11] has a propositionally inconsistent stable expansion $\Delta = \mathcal{L}_{\mathbf{B}}$ which is not an iterative expansion of $\mathrm{Tr}_1(\langle \Sigma, \emptyset\rangle)$. □

Theorem 5 shows that a PFM translation in the opposite direction is not possible. The proof is obtained by modifying Gottlob's proof [9] which shows that a modular translation from DL into AEL cannot be realized (an analog of this result is considered later as Corollary 1). Theorems 4 and 5 signify together that AEL is *less expressive* than SAEL.

Theorem 5. *There is* **no** *PFM translation function from autoepistemic theories under iterative expansions into such theories under stable expansions.*

Proof. Let $\mathcal{A} = \{\mathrm{a}, \mathrm{b}\}$ be a set of atoms and let $\mathcal{L}$ and $\mathcal{L}_{\mathbf{B}}$ be the respective propositional and autoepistemic languages. Let us then make a hypothesis that there is a fixed polynomial translation of $\Sigma = \{\mathbf{B}\mathrm{a} \rightarrow \mathrm{b}, \mathbf{B}(\mathrm{a} \rightarrow \mathrm{b}) \rightarrow \mathrm{a}\} \subseteq \mathcal{L}_{\mathbf{B}}$ into $\Sigma' \subseteq \mathcal{L}'_{\mathbf{B}}$ and $T' \subseteq \mathcal{L}'$ such that for all $T \subseteq \mathcal{L}$ the propositionally consistent iterative expansions of $\langle \Sigma, T\rangle$ and the propositionally consistent stable expansions of $\langle \Sigma', T' \cup T\rangle$ are in one-to-one correspondence and coincide up to

$\mathcal{L}$. The languages $\mathcal{L}'$ and $\mathcal{L}'_{\mathbf{B}}$ are assumed to be based on a set of atoms $\mathcal{A}' \supseteq \mathcal{A}$. Note that we may assume a single translation $\mathrm{Tr}(\Sigma) = \Sigma' \cup T'$ without a loss of generality, since $\langle \Sigma', T' \cup T \rangle$ and $\langle \mathrm{Tr}(\Sigma), T \rangle$ are effectively the same in AEL. Let us then introduce propositional theories $T_0 = \emptyset$, $T_1 = \{\mathrm{a}\}$, $T_2 = \{\mathrm{a} \rightarrow \mathrm{b}\}$ and $T_3 = \{\mathrm{a}, \mathrm{a} \rightarrow \mathrm{b}\}$. For $i \in \{1,2,3\}$, the autoepistemic theory $\langle \Sigma, T_i \rangle$ has a unique, propositionally consistent iterative expansion $\Delta = \{\mathrm{a}, \mathbf{B}\mathrm{a}, \mathrm{b}, \mathbf{B}\mathrm{b}, \mathrm{a} \rightarrow \mathrm{b}, \mathbf{B}(\mathrm{a} \rightarrow \mathrm{b})\}$. As Tr is faithful, there are unique propositionally consistent stable expansions Δ'_i of $\langle \mathrm{Tr}(\Sigma), T_i \rangle$ that coincide with Δ up to $\mathcal{L}$.

Since $\mathrm{a} \rightarrow \mathrm{b} \in \Delta \cap \mathcal{L}$, it follows that $\mathrm{a} \rightarrow \mathrm{b} \in \Delta'_1$. Since Δ'_1 is a stable expansion of $\langle \mathrm{Tr}(\Sigma), T_1 \rangle$, it holds that $\Delta'_1 = \mathrm{Cn}(\mathrm{Tr}(\Sigma) \cup T_1 \cup \mathbf{B}\Delta'_1 \cup \neg\mathbf{B}\overline{\Delta'_1})$ and thus $\mathrm{Tr}(\Sigma) \cup T_1 \cup \mathbf{B}\Delta'_1 \cup \neg\mathbf{B}\overline{\Delta'_1} \models \mathrm{a} \rightarrow \mathrm{b}$. It follows that also $\Delta'_1 = \mathrm{Cn}(\mathrm{Tr}(\Sigma) \cup T_3 \cup \mathbf{B}\Delta'_1 \cup \neg\mathbf{B}\overline{\Delta'_1})$, i.e. Δ'_1 is a stable expansion of $\langle \mathrm{Tr}(\Sigma), T_3 \rangle$. Because $a \in \Delta \cap \mathcal{L}$, it follows similarly that Δ'_2 is a stable expansion of $\langle \mathrm{Tr}(\Sigma), T_3 \rangle$. Then $\Delta'_1 = \Delta'_2 = \Delta'_3$ is necessarily the case, as Δ'_3 is the unique stable expansion of $\langle \mathrm{Tr}(\Sigma), T_3 \rangle$. So let Δ' denote any of Δ'_i with $i \in \{1,2,3\}$. Since $\mathrm{b} \in \Delta \cap \mathcal{L}$, we know that $\mathrm{b} \in \Delta'$. Then it follows that $\mathrm{Tr}(\Sigma) \cup \{\mathrm{a}\} \cup \mathbf{B}\Delta' \cup \neg\mathbf{B}\overline{\Delta'} \models \mathrm{b}$ and the deduction theorem of propositional logic implies $\mathrm{Tr}(\Sigma) \cup \mathbf{B}\Delta' \cup \neg\mathbf{B}\overline{\Delta'} \models \mathrm{a} \rightarrow \mathrm{b}$. Thus $\Delta' = \mathrm{Cn}(\mathrm{Tr}(\Sigma) \cup \mathbf{B}\Delta' \cup \neg\mathbf{B}\overline{\Delta'})$, i.e. Δ' is a propositionally consistent stable expansion of $\langle \mathrm{Tr}(\Sigma), T_0 \rangle$. Then $\langle \Sigma, T_0 \rangle$ has a propositionally consistent iterative expansion which contains both a and b. But this is contradiction, since the only iterative expansion of $\langle \Sigma, T_0 \rangle$ is $\Delta'' = \{\neg\mathbf{B}\mathrm{a}, \neg\mathbf{B}\mathrm{b}, \neg\mathbf{B}(\mathrm{a} \rightarrow \mathrm{b}), \ldots\}$. □

4.3 Comparison of SAEL and DL

The author [11] has proposed an idea of representing autoepistemic introspection in terms of default rules. In this approach, default rules of the forms $\frac{\phi:}{\mathbf{B}\phi}$ and $\frac{:\neg\phi}{\neg\mathbf{B}\phi}$ capture the *positive* and the *negative* introspection of an autoepistemic sentence $\phi \in \mathcal{L}_{\mathbf{B}}$, respectively. A translation function is obtained as follows.

Definition 11 (Janhunen [11]). *For all autoepistemic theories* $\langle \Sigma, T \rangle$, *let* $\mathrm{Tr}_2(\langle \Sigma, T \rangle) = \langle \{\frac{\phi:}{\mathbf{B}\phi} \mid \mathbf{B}\phi \in \mathrm{RBa}(\Sigma)\} \cup \{\frac{:\neg\phi}{\neg\mathbf{B}\phi} \mid \mathbf{B}\phi \in \mathrm{RBa}(\Sigma)\}, \Sigma \cup T \rangle$.

The propositional language $\mathcal{L}'$ of the translation $\mathrm{Tr}_2(\langle \Sigma, T \rangle)$ is assumed to contain atoms that correspond to the belief atoms in $\mathrm{RBa}(\Sigma)$ exactly.

Theorem 6. *The translation function* Tr_2 *given in Definition 11 is PFM.*

Proof. The translation function Tr_2 is clearly polynomial and modular. For the faithfulness of the translation, we refer to results shown by the author elsewhere [11, Theorem 13 and Proposition 16]. First of all, the author shows that the iterative expansions of an autoepistemic theory $\Sigma \subseteq \mathcal{L}_{\mathbf{B}}$ and the extensions of a translation $\langle \{\frac{\phi:}{\mathbf{B}\phi} \mid \phi \in \mathcal{L}_{\mathbf{B}}\} \cup \{\frac{:\neg\phi}{\neg\mathbf{B}\phi} \mid \phi \in \mathcal{L}_{\mathbf{B}}\}, \Sigma \rangle$ coincide (note that this is an extended and infinite translation). A translation obtained by Tr_2 is limited to belief atoms in $\mathrm{RBa}(\Sigma)$ and thus it captures essentially *full sets* [8] of Σ. This implies the one-to-one correspondence between the iterative expansions of $\langle \Sigma, T \rangle$ and the extensions of $\mathrm{Tr}_2(\langle \Sigma, T \rangle)$. In addition, the propositional parts of expansions and extensions in question coincide. □

A number of principles have been proposed to translate default theories into autoepistemic ones. Basically, the problem is to translate a default $\alpha : \beta_1, \ldots, \beta_n/\gamma$ into an autoepistemic sentence. Konolige [13] introduces a translation $\mathbf{B}\alpha \wedge \neg\mathbf{B}\neg\beta_1 \wedge \ldots \wedge \neg\mathbf{B}\neg\beta_n \rightarrow \gamma$ for a default. Unfortunately, the resulting translation function $\mathrm{Tr_K}$ for default theories is not faithful in general, as shown by Marek and Truszczyński [17]. As a response to this problem, they handle justifications β_i differently: $\neg\mathbf{B}\neg\beta_i$ is replaced by $\neg\mathbf{B}\mathbf{B}\neg\beta_i$ in their translation. Later, Truszczyński [27] ends up with a translation $\mathbf{B}\neg\mathbf{B}\neg\beta_i$ for justifications. This gives rise to a translation function for default theories as follows.

Definition 12 (Truszczyński [27]). *For all default theories $\langle D, T\rangle$, define* $\mathrm{Tr_T}(\langle D, T\rangle) = \langle\{\mathbf{B}\alpha \wedge \mathbf{B}\neg\mathbf{B}\neg\beta_1 \wedge \ldots \wedge \mathbf{B}\neg\mathbf{B}\neg\beta_n \rightarrow \gamma \mid \frac{\alpha:\beta_1,\ldots,\beta_n}{\gamma} \in D\}, T\rangle$.

Theorem 7 (Marek and Truszczyński [18]). *The translation function* $\mathrm{Tr_T}$ *given in Definition 12 is PFM.*

It is worth mentioning that the above result holds as long as the notion of faithfulness takes only the propositionally consistent expansions of the translation into account. Niemelä [22] demonstrates that for a set of defaults $D = \{\frac{\neg\mathrm{b}:}{\neg\mathrm{a}}, \frac{:\mathrm{b}}{\neg\mathrm{b}}\}$ and a theory $T = \{\mathrm{a}\}$ the translation $\mathrm{Tr_T}(\langle D, T\rangle)$ has an inconsistent iterative expansion while $\langle D, T\rangle$ has no extensions. However, Gottlob [9] proposes a variant of $\mathrm{Tr_T}$ that avoids such inconsistent iterative expansions.

Since PFM translations exist in both directions, we conclude that SAEL and DL have an equal expressive power according to the measure set up in Section 3. Note also that the theorems presented so far constitute an indirect proof of the following corollary. Therefore, Gottlob's intranslatability result [9] remains valid although a weaker notion of faithfulness is applied.

Corollary 1 (Gottlob [9]). *There is* **no** *PFM translation function from default theories into autoepistemic theories under stable expansions.*

Proof. Assume there is such a function. By Theorem 6 and Proposition 2, there is a PFM translation function that maps autoepistemic theories under iterative expansions to ones under stable expansions. But this contradicts Theorem 5. □

In spite of this result, Gottlob [9] sets up a non-modular translation function $\mathrm{Tr_G}$ to capture the extensions of a default theory $\langle D, T\rangle$ with the stable expansions of $\mathrm{Tr_G}(\langle D, T\rangle)$.[5] Then he provides a counter-example showing that faithful translations in the other direction are not possible and concludes then that DL is less expressive than AEL. This conclusion is in contrast with Theorems 4 and 6 and Proposition 2 which indicate that there is a PFM translation function (the composition of Tr_1 and Tr_2) for this purpose. The difference between the two views is due to the notions of faithfulness considered. Gottlob assumes that the language $\mathcal{L}$ of the default theory $\langle D, T\rangle$ obtained as a translation of an autoepistemic theory $\Sigma \subseteq \mathcal{L}_\mathbf{B}$ (under stable expansions) is the propositional sublanguage of $\mathcal{L}_\mathbf{B}$. However, we are ready to introduce new atoms to extend $\mathcal{L}$.

[5] Schwarz [25] proposes an alternative translation for this purpose.

To illustrate the effect of new atoms, we construct a default theory in order to capture the (propositionally consistent) stable expansions of an autoepistemic theory $\Sigma = \{\mathbf{B}\mathrm{p} \rightarrow \mathrm{p}\}$ used in Gottlob's counter-example [9].

Example 1. Let $\mathcal{A} = \{\mathrm{p}\}$ and let $\mathcal{L}$ and $\mathcal{L}_{\mathbf{B}}$ be the respective propositional and autoepistemic languages. Then let $\Sigma = \{\mathbf{B}\mathrm{p} \rightarrow \mathrm{p}\} \subseteq \mathcal{L}_{\mathbf{B}}$. The stable expansions of $\langle \Sigma, \emptyset \rangle$ are $\Delta_1 = \{\neg\mathbf{B}\mathrm{p}, \mathbf{B}\neg\mathbf{B}\mathrm{p}, \ldots\}$ and $\Delta_2 = \{\mathbf{B}\mathrm{p}, \mathrm{p}, \neg\mathbf{B}\neg\mathbf{B}\mathrm{p}, \ldots\}$ so that $\Delta_1 \cap \mathcal{L} = \mathrm{Cn}(\emptyset)$ and $\Delta_2 \cap \mathcal{L} = \mathrm{Cn}(\{\mathrm{p}\})$. Since $\mathrm{Cn}(\emptyset) \subseteq \mathrm{Cn}(\{\mathrm{p}\})$, Gottlob [9] concludes that there is no default theory with extensions $\mathrm{Cn}(\emptyset)$ and $\mathrm{Cn}(\{\mathrm{p}\})$, because the extensions of a default theory form an antichain [24].

As the first step, we apply Tr_1 and add an instance of the schema **5** to Σ and obtain $\Sigma' = \{\mathbf{B}\mathrm{p} \rightarrow \mathrm{p}, \neg\mathbf{B}\neg\mathbf{B}\mathrm{p} \rightarrow \mathbf{B}\mathrm{p}\}$. It is easy to see that Δ_1 and Δ_2 are the iterative expansions of $\langle \Sigma', \emptyset \rangle$. In particular, note that $\neg\mathbf{B}\mathrm{p} \notin \Delta_2$ implies that $\neg\mathbf{B}\neg\mathbf{B}\mathrm{p} \in \neg\mathbf{B}\overline{\Delta_2}$ so that $\mathbf{B}\mathrm{p}$ and p are $\mathbf{B}$-provable from $\Sigma' \cup \neg\mathbf{B}\overline{\Delta_2}$, since Σ' contains the critical instance $\neg\mathbf{B}\neg\mathbf{B}\mathrm{p} \rightarrow \mathbf{B}\mathrm{p}$ of **5**.

The next step is to apply Tr_2. An extended propositional language $\mathcal{L}'$ based on a set of atoms $\mathcal{A}' = \mathcal{A} \cup \{\mathbf{B}\mathrm{p}, \mathbf{B}\neg\mathbf{B}\mathrm{p}\}$ is introduced. The set of defaults introduced by Tr_2 is $D = \{\frac{\mathrm{p}:}{\mathbf{B}\mathrm{p}}, \frac{:\neg\mathrm{p}}{\neg\mathbf{B}\mathrm{p}}, \frac{\neg\mathbf{B}\mathrm{p}:}{\mathbf{B}\neg\mathbf{B}\mathrm{p}}, \frac{:\neg\neg\mathbf{B}\mathrm{p}}{\neg\mathbf{B}\neg\mathbf{B}\mathrm{p}}\}$. Consequently, the extensions of the resulting default theory $\langle D, \Sigma' \rangle$ are $E_1 = \mathrm{Cn}(\Sigma' \cup \{\neg\mathbf{B}\mathrm{p}, \mathbf{B}\neg\mathbf{B}\mathrm{p}\})$ and $E_2 = \mathrm{Cn}(\Sigma' \cup \{\mathbf{B}\mathrm{p}, \neg\mathbf{B}\neg\mathbf{B}\mathrm{p}\})$, because the reductions of D are $D_{E_1} = \{\mathrm{p}/\mathbf{B}\mathrm{p}, \top/\neg\mathbf{B}\mathrm{p}, \neg\mathbf{B}\mathrm{p}/\mathbf{B}\neg\mathbf{B}\mathrm{p}\}$ and $D_{E_2} = \{\mathrm{p}/\mathbf{B}\mathrm{p}, \neg\mathbf{B}\mathrm{p}/\mathbf{B}\neg\mathbf{B}\mathrm{p}, \top/\neg\mathbf{B}\neg\mathbf{B}\mathrm{p}\}$. It follows that $E_1 \cap \mathcal{L} = \mathrm{Cn}(\emptyset)$ and $E_2 \cap \mathcal{L} = \mathrm{Cn}(\{\mathrm{p}\})$. Thus the stable expansions of $\langle \Sigma, \emptyset \rangle$ and the extensions of $\langle D, \Sigma' \rangle$ coincide up to $\mathcal{L}$. In particular, the extended language $\mathcal{L}'$ allows the relationship $E_1 \cap \mathcal{L} \subseteq E_2 \cap \mathcal{L}$, although $E_1 \not\subseteq E_2$.

Bonatti and Eiter [2] analyze the expressive power of non-monotonic logics as query languages for disjunctive databases. A comparison with our results is possible in the propositional case, if a restriction to empty databases is made. Then Theorems 6.3 and 7.3 in [2] speak about the intertranslatability of non-monotonic theories. These theorems involve two translations functions, $\mathrm{Tr}_{\mathrm{BE}}$ and Tr_{K}. The latter function Tr_{K} is due to Konolige [13], and it allows one to capture the extensions of a prerequisite-free[6] default theory $\langle D, T \rangle$ in terms of stable expansions of the translation $\mathrm{Tr}_{\mathrm{K}}(\langle D, T \rangle)$. The results by Marek and Truszczyński [18, Section 12.5] and Gottlob [9] suggest that this translation is PFM in our sense, implying that AEL is at least as expressive as prerequisite-free default logic (PDL). It follows by Corollary 1 and compositionality that there is **no** PFM translation from default theories into prerequisite-free ones. Interestingly, the translation function $\mathrm{Tr}_{\mathrm{BE}}$ is proposed to remove prerequisites from a default theory. However, such a translation cannot be PFM by our remarks above. It seems that $\mathrm{Tr}_{\mathrm{BE}}$ is polynomial and modular so that $\mathrm{Tr}_{\mathrm{BE}}$ cannot be faithful in our sense. Indeed, the idea behind $\mathrm{Tr}_{\mathrm{BE}}$ is to simulate the defaults of the original default theory $\langle D, T \rangle$ without actually applying them and consequently the extensions produced for the translation $\mathrm{Tr}_{\mathrm{BE}}(\langle D, T \rangle)$ do not coincide with the extensions of $\langle D, T \rangle$ up to $\mathcal{L}$ (i.e. the language of $\langle D, T \rangle$). Moreover, Theorem 6.3 in [2] does not establish a one-to-one correspondence of extensions.

[6] A default $\alpha : \beta_1, \ldots, \beta_n / \gamma$ is called prerequisite-free, if $\alpha = \top$.

Marek et al. [15] and Engelfriet et al. [6] propose mappings to translate a default theory $\langle D, T\rangle$ into a prerequisite-free one such that extensions are preserved. However, these translations introduce a new default for each *quasi-proof* which is a sequence of defaults from D. Consequently, these translations are not polynomial in general and no contradiction arises with Corollary 1 as discussed above. Let us also note that the latter approach [6] deals with *infinitary* defaults whereas only finite defaults and default theories are considered here.

4.4 Comparison of DL and PL

Wang, You and Yuan [28] propose a translation of a default theory $\langle D, T\rangle$ into a priority theory $\langle R, P, T\rangle$. The idea is to break a default $\alpha : \beta_1, \ldots, \beta_n/\gamma \in D$ to inference rules α/γ, $\neg\beta_1/\neg\beta_1, \ldots, \neg\beta_n/\neg\beta_n$ to be included in R. The priority relation P is chosen such that the rule α/γ has a lower priority than the rules $\neg\beta_1/\neg\beta_1, \ldots, \neg\beta_n/\neg\beta_n$. As reported by Wang et al., this translation is faithful only if *dissimilar* sets of defaults D are considered, i.e. sets of defaults D which do not contain two defaults with exactly same prerequisite α and consequent γ. Wang et al. argue that this restriction is not significant, since it is possible to differentiate the prerequisites of defaults without changing their semantics. An unrestricted translation function introduces a new atom p_d for each $d \in D$.

Definition 13. *For all default theories $\langle D, T\rangle$, the translation* $\mathrm{Tr}_{\mathrm{W}}(\langle D, T\rangle) = \langle R, P, T\rangle$ *where R and P are such that for each $d = \frac{\alpha:\beta_1,\ldots,\beta_n}{\gamma} \in D$, (i) the rules $\alpha \wedge (\mathrm{p}_d \vee \neg\mathrm{p}_d)/\gamma$ and $\neg\beta_1/\neg\beta_1, \ldots, \neg\beta_n/\neg\beta_n$ belong to R and (ii) the rule $\alpha \wedge (\mathrm{p}_d \vee \neg\mathrm{p}_d)/\gamma$ is in the relation P with the rules $\neg\beta_1/\neg\beta_1, \ldots, \neg\beta_n/\neg\beta_n$.*

Theorem 8. *The translation function* Tr_{W} *given in Definition 13 is PFM.*

Proof. It is clear that Tr_{W} is polynomial and modular. To establish the faithfulness of Tr_{W}, we note that adding the tautology $\mathrm{p}_d \vee \neg\mathrm{p}_d$ to the prerequisite of a default $d \in D$ does not affect the applicability of the default d by any means. Let D' denote D modified in this way. It is clear that the extensions of $\langle D, T\rangle$ and $\langle D', T\rangle$ coincide up to the language $\mathcal{L}$ of $\langle D, T\rangle$. Since D' is definitely dissimilar, the translation function Tr_{W} is faithful by the one-to-one correspondence of extensions established by Wang et al. [28, Theorem 8]. □

A translation function in the other direction can also be obtained and it seems that one cannot do without new atoms in this case. The idea is that an atom a_r is introduced to denote that a rule r of a priority theory $\langle R, P, T\rangle$ is applied. Then the priority relation P of the priority theory is easily representable in terms of the justifications of defaults. Note that finiteness of R is essential in this translation: a single default is sufficient to represent a rule $r \in R$.

Definition 14. *For all priority theories $\langle R, P, T\rangle$, the translation* $\mathrm{Tr}_3(\langle R, P, T\rangle)$ *is $\langle D, T\rangle$ where D contains for each rule $r = \alpha/\gamma \in R$ a default $\frac{\alpha:\neg\mathrm{a}_{r_1},\ldots,\neg\mathrm{a}_{r_n}}{\gamma\wedge\mathrm{a}_r}$ where $r_1, \ldots, r_n$ are all the rules of R such that $\langle r, r_1\rangle \in P, \ldots, \langle r, r_n\rangle \in P$.*

Theorem 9. *The translation function* Tr_3 *given in Definition 14 is PFM.*

Proof sketch. It is obvious that Tr_3 is polynomial and modular. Let us then sketch how Tr_3 is proved faithful. Consider a priority theory $\langle R, P, T\rangle$ and the set of defaults D introduced by Tr_3. Let $\mathcal{L}$ be the language of $\langle R, P, T\rangle$ based on a set of atoms $\mathcal{A}$. Thus the language $\mathcal{L}'$ of the resulting default theory $\langle D, T\rangle$ is based on a set of atoms $\mathcal{A}' = \mathcal{A} \cup \{\mathrm{a}_r \mid r \in R\}$.

(i) Consider an extension $E \subseteq \mathcal{L}$ of $\langle R, P, T\rangle$ based on a set of rules $R' \subseteq R$ satisfying the stability condition of Definition 5. Define $E' = \mathrm{Cn}(E \cup A') \subseteq \mathcal{L}'$ where A' is the set of atoms $\{\mathrm{a}_r \mid r \in R'\}$. Then the reduct $D_{E'}$ contains the inference rule $\alpha/\gamma \wedge \mathrm{a}_r$ if and only if $r = \alpha/\gamma$ belongs to $\mathrm{Nb}(R, P, R')$. Consequently, it can be shown that E' is the unique extension of $\langle D, T\rangle$ with the property $\{r \in R \mid \mathrm{a}_r \in E'\} = R'$. (ii) Then assume that there is an extension $E' \subseteq \mathcal{L}'$ of $\langle D, T\rangle$ and let $R' = \{r \in R \mid \mathrm{a}_r \in E'\}$. It follows that R' satisfies the stability condition. It follows that $E = \mathrm{Cn}^{R'}(T) = E' \cap \mathcal{L}$ is an extension of $\langle R, P, T\rangle$. The steps (i) and (ii) above establish a one-to-one correspondence of extensions. Moreover, these extensions coincide up to the language $\mathcal{L}$. □

The results of Theorems 8 and 9 entitle us to conclude that default logic and priority logic are of equal expressive power. It is also worth pointing out that the translations presented lead to straightforward reductions between the decision problems of DL and PL corresponding to brave and cautious strategies. Thus our results and the complexity results on DL [8] have the following corollary.

Corollary 2. *The decision problems of PL corresponding brave and cautious reasoning strategies are* $\mathbf{\Sigma}_2^{\mathrm{p}}$*-complete and* $\mathbf{\Pi}_2^{\mathrm{p}}$*-complete problems, respectively.*

5 Conclusions

A framework of polynomial, faithful and modular (PFM) translation functions is proposed in this paper to classify non-monotonic logics by their expressive power. If there is a PFM translation function that maps theories of one non-monotonic logic L_1 into theories of another L_2, the sets of conclusions induced by a theory of L_1 – which determine the semantics of the theory – are effectively captured by the sets of conclusions induced by a theory of L_2. This is interpreted to indicate that the non-monotonic logic L_2 at least as expressive as L_1. A number of translation functions are considered, and three novel translation functions Tr_1, Tr_2 and Tr_3 are proposed in the paper for classification purposes. The first two are merely obtained by modifying existing translation functions while the last is completely new. It is established that these translation functions are PFM. Two impossibility proofs are also provided to establish strict relationships in the expressive power of non-monotonic logics under consideration.

To conclude, the comparisons made in the paper give rise to a classification illustrated in Figure 1. Classical propositional logic CL is also included in the figure to complete our view. Solid arrows denote PFM translation functions from one non-monotonic logic to another that are considered in this work. Dotted arrows in the figure denote translation functions obtained as compositions of others. Such compositions are not necessarily optimal for a particular purpose. Note, for instance, that many unnecessary atoms and sentences would be introduced if a minimal model theory $\langle P, F, T\rangle$ were translated into a default theory using $\mathrm{Tr_N}$, $\mathrm{Tr_1}$ and $\mathrm{Tr_2}$ that involve the intermediate representations of $\langle P, F, T\rangle$ as an autoepistemic theory. Nevertheless, the resulting translation function $\mathrm{Tr_N} \circ \mathrm{Tr_1} \circ \mathrm{Tr_2}$ is still PFM by compositionality.

The *non-monotonic* logics under consideration are divided in three equivalence classes by their expressive power. The strongest class contains SAEL, DL and PL. The class below this contains AEL which is *less expressive* than SAEL, DL and PL. The third and the least expressive class contains CIRC which is less expressive than AEL. Below these three classes, there is the fourth class containing CL which is less expressive than any of the non-monotonic logics considered. The relationships depicted in Figure 1 refine the classification of non-monotonic logics based on earlier results [2, 7, 9, 10, 23] on the expressive power of non-monotonic logics. Finally, we want to emphasize that the ranking of non-monotonic logics by their expressive power is very sensitive to the requirements imposed on translation functions. It is demonstrated in this paper how a slight change in the notion of faithfulness changes the relative ordering of AEL and DL to the opposite compared to Gottlob's results [9].

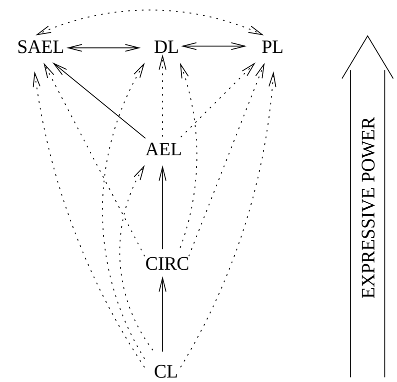

Fig. 1: Non-monotonic Logics Ordered by Their Expressive Power

Future Work. The notion of modularity considered in the paper is rather weak, i.e. only changes in the propositional subtheory are tolerated. We expect that the translations considered are also modular in a stronger sense and thus a stronger notion of modularity can be introduced such that the classification depicted in Figure 1 remains intact. For instance, changes in the defaults of a default theory $\langle D, T\rangle$ cause only local changes in the respective sentences of the translation $\mathrm{Tr_T}(\langle D, T\rangle)$. Moreover, there are also other non-monotonic logics as well as variants of those considered in the paper. These logics should be analysed in terms of PFM translation functions in order to classify them in the hierarchy. For instance, PDL is interesting in this respect on the basis of Section 4.3.

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