Introduction:

Wavelets, filter banks and multiresolution signal analysis, have been used independently in the fields of applied mathematics, computer vision and signal processing. It is interesting to note that they performed similar functions in different fields. It is recently, that they converged to form a single theory. In the paper, it is shown that the fundamental idea was the same in all the fields i.e. breaking down of the signal or functions into component signals at different resolutions.

The Wavelet transform are first compared to classical short-time Fourier Transform (STFT) and it is shown that the wavelet transform approach solves the shortcomings of STFT. Then, the relation between wavelets, filter banks and multiresolution analysis are explored.

The author explains how regular perfect reconstruction digital filter banks (PRFB) can be used to compute continuous wavelet transform and perform discrete wavelet transform. The set of conditions necessary to achieve PRFB are then derived in the paper. The author then discusses different design techniques for PRFB. Orthogonal filter banks are briefly discussed and design techniques for the same are mentioned in the paper.

Next, a comparison between perfect reconstruction conditions and Bezout identity is made. Using the Bezout identity, the author formulates a set of constraints, for a set of filters to achieve perfect reconstruction. Then, with an analogy to Diophantine equations, he shows that for a given low pass filter, how one can find all the complementary filters satisfying perfect reconstruction conditions. Also, an alternative approach of using the theory of continued fractions is discussed. The author specifically addresses the regularity condition and proposes new techniques to produce regular designs.

He finally uses the derived conditions and constraints to design highly regular filter banks and presents the results in the paper. In particular, biorhogonal compactly supported wavelets bases with symmetries using regular FIR are derived.

Main topics of the Paper:

The main topics/sections of the paper are :

- Relation between wavelets, multiresolution analysis and digital filter banks.
- General FIR perfect reconstruction filter banks and Biorthogonal Wavelets.
- Algebraic structure of FIR solutions.
1) Relation between wavelets, multiresolution analysis and digital filter banks.

Wavelet Transform as compared to STFT:

Fourier Transform (FT) cannot be used for analysis of non-stationary signals, because FT tells us which frequency components exist in the signal, but gives no time information about these frequency components. To solve this problem, STFT was developed. STFT used a window function to “view” a part of non-stationary signal as stationary and then perform analysis. However, STFT can have either good time resolution or good frequency resolution, but not both. This problem is related to Heisenberg’s uncertainty principle which states that we cannot have arbitrarily high resolution in both time and frequency domain.

The time-bandwidth product of basis functions is lower bounded by

\[ \Delta T \cdot \Delta \Omega \geq 4\pi \] … (1)

where \((\Delta T)^2\) & \((\Delta \Omega)^2\) are variances of absolute values of the function and Fourier Transform. Thus there was resolution problem in STFT. We can see from fig.1.a that we have equal time and frequency resolution at all frequencies in STFT.

To overcome this resolution problem, wavelet transform was developed. Wavelet analysis is performed using contracted and expanded versions of a single prototype function called a wavelet. We can achieve fine time resolution using contracted version of the wavelet, while fine frequency resolution can be achieved using expanded version as can be seen from fig.1.b.
The uncertainty principle applies to wavelet transform too. However, there is a tradeoff between time and frequency resolution in wavelet analysis (the tradeoff can be seen from fig 1.b i.e. there is good frequency resolutions at low frequencies and good time resolution at high frequencies) such that the bound given in equation (1) is met.

The wavelet transform is given by:

\[
X_w(a, b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} h^*(t - b) x(t) \, dt.
\]

… (2)

where \(a\) and \(b\) are wavelet function parameters and \(x(t)\) is the signal to be transformed.

The prototype wavelet function is given by

\[
h_{a,b}(t) = \frac{1}{\sqrt{a}} h \left( \frac{t - b}{a} \right)
\]

… (3)

The contraction and expansion parameters of the wavelet function given in eqn.(3) can be discretized. So, eqn.(3) then becomes

\[
h_{mn}(t) = a_0^{-m/2} \cdot h(a_0^{-m}t - nb_0),
\]

\[
m, n \in \mathbb{Z}, \quad a_0 > 1, \quad b_0 \neq 0
\]

… (4)

We can then construct wavelet functions which are orthonormal. Examples of prototype wavelet functions are Haar basis function, Daubechies basis functions, Shannon basis function, Mexican Hat function etc. These basis functions can be implemented using filters. Longer filters lead to smoother functions. Further in the paper, the author discusses techniques to design continuous wavelets, having additional properties like linear phase.
Multiresolution Signal Processing:

Wavelet can be thought of as a bandpass filter from signal processing point of view. In specific cases, wavelets can be considered as an octave band pass filter. So, we can use filter banks to implement wavelet transform. Thus, wavelet transform can be interpreted as constant-Q filtering with a set of octave band filters, followed by sampling at respective Nyquist frequencies. Also, by adding higher octave bands, we add resolution to the signal. The relation between filter banks and wavelets is explained in the next section.

Discrete Time Case:

In discrete wavelet transform, the scale and resolution are varied, for detailed analysis of the signal. We can obtain signals of different resolution and scale, by passing them through various filters, followed by upsampling and downsampling processing.

Fig. 1.1. a) Haar Wavelet        Fig.1.1.b) Daubechies Wavelet

(From Wavelets and Filter banks: theory and design by M. Vetterli)

Fig.2) Scale and resolution change of the signal using filter banks.
Wavelets and Filter Banks: Theory and Design

In the above diagram, we can see how signals of different resolution and scales can be achieved by passing them through filter banks. Thus we can use digital filter banks to implement wavelet transform. So, the connection between Wavelet Transform, Multiresolution Signal Processing and Filter banks become clear.

![Filter Bank Diagram](image)

Fig.3.a) Structure 1 for Filter banks

![Filter Bank Diagram](image)

Fig.3.b) Structure 2 for Filter Banks

(From Informit.com)

We can see from fig. 3.a & 3.b that by using different structures for filter banks, we can obtain signals of different scales and resolution.

Also, the author states that we can use filter banks to generate orthogonal sets of wavelets. To generate these orthogonal sets of wavelets, the filters should exhibit orthogonality with respect to the even shifts and it should be regular (i.e. iterated filter should converge to a continuous function.)

**Image Pyramid:**

In computer vision, a successive approximation or multiresolution technique called an image pyramid is used. In this process, we derive a low resolution version of the signal, and then predict the original based on the coarse version. The difference between the original and the predicted version is calculated. At the reconstruction stage, the predicted part of the signal is added to the difference, which ensures perfect reconstruction. In the next section we see how perfect reconstruction can be achieved.

*2) General FIR perfect reconstruction filter banks and Biorthogonal Wavelets.*
Conditions for FIR Perfect Reconstruction Filter Banks (PRFB):

PRFB are used to construct wavelets. To achieve perfect reconstruction, the analysis and the synthesis filters should satisfy certain conditions which are mentioned below.

\[ \hat{X}(z) = \frac{1}{2} \begin{bmatrix} G_0(z) & G_1(z) \end{bmatrix} \begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix} \begin{bmatrix} X(z) \\ X(-z) \end{bmatrix}. \]  

In equation (5), \( H_0(z) \) and \( H_1(z) \) are the analysis filters, while \( G_0(z) \) and \( G_1(z) \) are the synthesis filters. In the fig. 4) synthesis filter \( \tilde{H}_0(z), \tilde{H}_1(z) \) are renamed as \( G_0(z) \) and \( G_1(z) \).

In order to cancel aliasing, the contribution due to \( X(-z) \) should be eliminated. In order to achieve the same, the analysis and the synthesis filter should satisfy the relation given in eqn 6.

\[ [G_0(z), G_1(z)] = C(z) [H_1(-z), -H_0(-z)]. \]  

Also, equation 5 can be further solved as:

\[ \det [H(z)] = H_0(z)H_1(-z) - H_0(-z)H_1(z) \]

\[ = P(z) - P(-z) \]  

where \( M \) denotes modulation of the filters.

For perfect reconstruction, with FIR synthesis filters after an FIR analysis, it is necessary and sufficient that equation 8 is satisfied. i.e.
\[
\det [\mathbf{H}_m(z)] = c \cdot z^{-2l-1}
\]  \quad \ldots (8)

By comparing eqn 7 & 8, we can see that \( P(z) \) can have a single nonzero odd indexed coefficient. A polynomial \( P(z) \) having a single nonzero odd indexed coefficient is termed as a valid polynomial. So, we can factorize \( P(z) \) into \( H_0(z) \) and \( H_1(-z) \) which would give a FIR PRFB. This is exactly similar to what we have studied in the class.

Then 2 design techniques for PRFB are discussed.

a) A valid polynomial satisfying equation 7 & 8 is found. Then this polynomial is factored into \( H_0(z) \) and \( H_1(-z) \).

b) In the second technique, a filter \( H_0(z) \) is taken and by solving a system of linear equations complementary filter \( H_1(z) \) is found. Having found \( H_0(z) \) & \( H_1(z) \), \( P(z) \) can be found.

Using the above techniques, we design the analysis filter. Once we have the analysis filters, then using equation (6), we can find the synthesis filters.

**Orthogonal Filter Banks:**

In the paper, the author discusses on how unitary operators can be constructed from filters which are orthogonal to their even translates i.e. the even terms of the autocorrelation function of the filter are all zero, with the exception of the central one \((H_i(z)H_i(z^{-1}) = 1)\). Now the central coefficient equals to unity for normalized filters. So we have the equation

\[
H_i(z)H_i(z^{-1}) + H_i(-z)H_i(-z^{-1}) = 2 \quad \text{for } i \in \{0,1\} \quad \ldots (9)
\]

Also the 2 analysis filters i.e. \( H_0(z) \) and \( H_1(z) \) are orthogonal to each other. So the even terms of cross correlation are all zero. Thus the equation

\[
H_0(z)H_1(z^{-1}) + H_0(-z)H_1(-z^{-1}) = 0 \quad \ldots (10)
\]

Thus if the two filters satisfy equation (9) and (10), then they form an orthogonal basis for the space of square summable sequences.

Now, based on the above equations, it can be stated as:

Fact 1.1 : Consider an FIR perfect reconstruction filter bank such that \( H_0(z) \) satisfies equation (9). Then for \( H_1(z) \) and \( H_0(z) \) to form an orthogonal basis, (i.e. for \( H_1(z) \) to satisfy equation (9) and equation (10)) it is necessary and sufficient that

\[
H_1(z) = z^{-2k-1}H_0(z) \quad \text{where } z^{-2k-1} \text{ represents the delay} \quad \ldots (11)
\]
Proof for Fact 1.1:

Sufficiency: The length \( L \) of \( H_0(z) \) filter must be *even* for it to form an orthonormal basis. If length \( L \) of \( H_0(z) \) filter is *odd*, then either \( h_0(0) \) or \( h_0(L-1) \) should be zero for \( H_0(z) \) to form an orthonormal basis. By substituting eqn. 11 in eqn. 9 and eqn. 10, we can see that the conditions are satisfied.

Necessity: The necessity part is proved by contradiction. From equation (10), we can see that \( H_0(z) H_1(z^{-1}) \) is a polynomial with only odd coefficients. i.e.

\[
H_0(z) H_1(z^{-1}) = z^{-2n-1} Q(z^2)
\]  

\[\text{...(12)}\]

From equation (12), we can see that the zeros for \( H_0(z) \) appear in pairs at \((\alpha, -\alpha)\). Now, this is contradictory to the conditions to achieve perfect reconstruction, because to achieve perfect reconstruction, for every zero at \( z = \alpha \) in \( H_0(z) \), there must be a corresponding zero at \( z = -\alpha \) in \( H_1(z^{-1}) \). Thus, it is proved that \( H_1(z) \) should have the form given in eqn. (11).

The design techniques for orthogonal filter banks are discussed in the paper as stated below:

a) In the first technique, we need to find an autocorrelation function that has only a single even-indexed coefficient other than zero. Also, this single even indexed coefficient should be symmetric. Then once a function is found, we can factorize it into \( H_0(z) \) and \( H_1(z^{-1}) \). This method was used by Smith and Barnwell; the design technique which we studied in the class.

b) In the second design technique, we use lattice structures to synthesize paraunitary matrices, for which complete factorizations tables are available.

*General Perfect Reconstructions Filter Banks:*

Two channel orthogonal system cannot have linear phase except in simple structures (sum and difference of 2 delays). So, in order to obtain filter banks with linear phase, orthogonality must be sacrificed, which leads to biorthogonal linear phase filter banks. Thus, by sacrificing orthogonality, we have greater degree of freedom. So, we can use arbitrary length linear phase filters.

Now, linear phase perfect reconstruction real FIR filter banks using \( H_0(z) \) and \( H_1(z^{-1}) \) should have one of the following forms:

a) Both the filters are symmetric and of odd lengths, differing by an odd multiple of 2.
b) One filter is symmetric and other is antisymmetric; both lengths are even and are equal or differ by even multiple of 2.

c) One filter is of odd length, the other is even; both have all zeros on the unit circle. Either both filters are symmetric or one is symmetric and other is antisymmetric.

Also, the author mentions that in FIR case it is necessary to give up orthogonality to get non trivial linear phase solutions but in IIR case we can have orthogonality and linear phase filters at the same time.

It is shown in the paper that how by using infinitely iterated biorthogonal perfect reconstruction filter banks; biorthogonal sets of functions can be generated.

**Filter Design:**

The author then discusses conditions by which we can get *maximally regular filters*. The maximally regular filters should be of the form

\[ H(z) = \left[ \left(1 + z^{-1}\right) / 2 \right]^N F(z) \]  

where N corresponds to the number of zeros at \( \Pi \). So, to get maximally regular filters, we should maximize number of zeros at \( \Pi \) and simultaneously minimize \( F(e^{j\omega}) \). However, we can easily control only \( N \) i.e. the number of zeros at \( \Pi \).

Also, it should be ensured that both the filter’s i.e. \( H_0(z) \) and \( H_1(z) \) are regular. The author uses the conditions mentioned in this report to design the filter.

**3) Algebraic Structure of FIR solutions:**

**Bezout’s Identity**

In this section author shows how the perfect reconstruction conditions are related to Bezout’s identity and how we can use the identity to examine constraints for the analysis low pass and high pass filters.

The perfect reconstruction conditions for the analysis and the synthesis filters can be stated as :

\[ H_0(z)H_1(-z) - H_0(-z)H_1(z) = 2z^{-2l-1}. \]  

\[ H_{00}(z)H_{11}(z) - H_{01}(z)H_{10}(z) = z^{-l} \]  

where equation (15) is written in the polyphase notation. Each of the conditions implies the other and we can use either of them to satisfy perfect reconstruction.
Now eqn (14) and eqn (15) are special forms of Bezout identity. This identity arises in Euclidean algorithm and it calculates the greatest common divisor of the two polynomials \( a_0(x) \) and \( a_1(x) \):

\[
a_r(x) = \alpha \gcd (a_0(x), a_1(x)) \quad \text{...(16)}
\]

where \( a_r(x) \) is the last divisor of the algorithm and \( \alpha \) is a constant.

Also we can write

\[
a_r(x) = p_0(x)a_0(x) + p_1(x)a_1(x) \quad \text{for some } p_0(x), p_1(x). \quad \text{...(17)}
\]

If \( a_0(x) \) and \( a_1(x) \) are coprime, then \( a_r(x) \) has zeros at 0 or \( \infty \) only. Thus, from equations (14), (15), (16) & (17), we can see the relation between perfect reconstruction conditions and Bezout identity.

Thus for \( H_0(z) \) and \( H_1(z) \) to form a perfect reconstruction pair, it is necessary that they be coprime. i.e. the only zero common to both of them could be at \( z = \infty \). If there is another common zero other than at \( z = \infty \), then some frequencies would not pass through the filter and we might not get perfect reconstruction at the output.

Based on the above seen relations between Bezout identity and PRFB, author states some propositions, for which proofs are given in the paper.

Proposition 1) Assume that the filters \( H_0(z) \) and \( H_1(z) \) are FIR and causal. Then given one of the pairs \([H_{00}(z) \ H_{01}(z)], [H_{10}(z) \ H_{11}(z)], [H_{00}(z) \ H_{10}(z)] \) or \([H_{01}(z) \ H_{11}(z)]\) in order to calculate the other necessary pair, it is necessary and sufficient that the given pair be coprime.

Proposition 2) A filter \( H_0(z) \) has a complementary filter if and only if it has no zeros in pairs at \( z = \alpha \) and \( z = -\alpha \).

Proposition 3) There is always a complementary filter to the binomial filter. This can be written as:

\[
H_0(z) = (1 + z^{-1})^k = H_{00}(z^2) + z^{-1}H_{01}(z^2). \quad \text{...(18)}
\]

**Diophantine Equations**

For a given filter \( H_0(z) \), we can calculate its complement \( H_1(z) \). However one important thing to note is that the filter \( H_0(z) \) can have many complement filters. We can find all the complementary filters for \( H_0(z) \) by a simple mechanism.

In number theory, equations with integer coefficients for which integer solutions are sought are known as basic Diophantine equations. The most basic Diophantine equation is given by
where all quantities are integers and the solution to \((x,y)\) is found out.

So, in the Diophantine equation, we can replace integers by polynomials and equation 19 becomes similar to equation 14. Thus using this similarity, we can find all the filters which are complementary for a given filter \(H_0(z)\).

**Design Results:**

The author now designs a filter using the conditions detailed in this report till now. He designs a linear phase filter which satisfies perfect reconstruction conditions has maximum number of zero’s at \(z = -1\). Also, if \(P(z)\) has a single zero at \(z = -1\), it cannot have any zero at \(z = 1\) and vice-versa. The results obtained are then discussed in the paper.

**Conclusion:**

In the paper, relationships between wavelets, filter banks and multiresolution analysis are developed and shown the similarity in their fundamental ideas. It is shown as to how digital filter banks can be used to perform wavelet analysis. The author finally derives biorthogonal compactly supported wavelet bases using regular FIR PRFB’s.

References:

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4) Linear Phase wavelets : Theory and Design by M. Vetterli and C. Herley
6) Tutorial on Wavelets by Robi Polikar.