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ON TENSOR PRODUCT DECOMPOSITION OF k-TRIDIAGONAL TOEPLITZ MATRICES

Asuka Ohashi[1 §], Tsuyoshi Sasaki Usuda[1],
Tomohiro Sogabe[2], Fatih Yılmaz[3]

[1]Graduate School of Information Science & Technology
Aichi Prefectural University
Aichi 480-1198, JAPAN

[2]Graduate School of Engineering
Nagoya University
Nagoya 464-8603, JAPAN

[3]Department of Mathematics
Polatlı Art and Science Faculty
Gazi University
06900, Ankara, TURKEY

Abstract: In the present paper, we provide a decomposition of a k-tridiagonal Toeplitz matrix via tensor product. By the decomposition, the required memory of the matrix is reduced and the matrix is easily analyzed since we can use properties of tensor product.

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1. Introduction

A tridiagonal Toeplitz matrix is one of tridiagonal matrices and has constant

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[§]Correspondence author

entries on each diagonal parallel to the main diagonal. It is widely known that the matrix arises in the finite difference discretization of the differential equation (cf. [8, §1.4]). For the recent developments, see, e.g., [1], [2], [4], [11], and [12].

A k-tridiagonal matrix is one of generalizations of a tridiagonal matrix and has received much attention in recent years (e.g., [3], [6], [9], and [10]). Here, let $T_n^{(k)}$ be an n-by-n k-tridiagonal matrix defined as

$$T_n^{(k)} := \begin{pmatrix} d_1 & 0 & \cdots & 0 & a_1 & 0 & \cdots & 0 \\ 0 & d_2 & & & & a_2 & & \vdots \\ \vdots & & \ddots & & & & \ddots & 0 \\ 0 & & & d_{n-k} & & & & a_{n-k} \\ b_{k+1} & & & & \ddots & & & 0 \\ 0 & b_{k+2} & & & & \ddots & & \vdots \\ \vdots & & \ddots & & & & d_{n-1} & 0 \\ 0 & \cdots & 0 & b_n & 0 & \cdots & 0 & d_n \end{pmatrix}. \tag{1}$$

If $d_i = d$ $(i = 1, 2, \ldots, n)$, $a_i = a$ $(i = 1, 2, \ldots, n-k)$, $b_i = b$ $(i = k+1, k+2, \ldots, n)$, where $1 \leq k < n$, $T_n^{(k)}$ is a k-tridiagonal Toeplitz matrix. Moreover, when $k = 1$, $T_n^{(1)}$ is an ordinary tridiagonal Toeplitz matrix. We consider a k-tridiagonal Toeplitz matrix $T_n^{(k)}$.

Hereafter, tensor product is briefly explained since it is used in a decomposition of k-tridiagonal Toeplitz matrices in the present paper. Tensor product is also referred to as the Kronecker product and represented by the symbol "$\otimes$". The definition of tensor product of matrices $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{p\times q}$ is

$$A \otimes B := \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix} \in \mathbb{C}^{mp\times nq}, \tag{2}$$

where a_{ij} is the (i,j) element of A. Let $\hat{\boldsymbol{a}}_i$ and $\boldsymbol{a}_j$ be the i-th row and the j-th column vectors in A, respectively. Similarly, let $\hat{\boldsymbol{b}}_i$ and $\boldsymbol{b}_j$ be the i-th row and the j-th column vectors in B, respectively. Then,

$$A \otimes B = \begin{pmatrix} \hat{\boldsymbol{a}}_1 \otimes \hat{\boldsymbol{b}}_1 \\ \hat{\boldsymbol{a}}_1 \otimes \hat{\boldsymbol{b}}_2 \\ \vdots \\ \hat{\boldsymbol{a}}_m \otimes \hat{\boldsymbol{b}}_p \end{pmatrix},$$

$$= (\boldsymbol{a}_1 \otimes \boldsymbol{b}_1, \boldsymbol{a}_1 \otimes \boldsymbol{b}_2, \dots, \boldsymbol{a}_n \otimes \boldsymbol{b}_q), \tag{3}$$

as the other expressions of $A \otimes B$.

The purpose of the present paper is to save the required memory of a k-tridiagonal Toeplitz matrix and to simplify analyses of that. We propose to decompose the k-tridiagonal Toeplitz matrix into a smaller matrix with the similar structure than the original one and an identity matrix via tensor product. Even if the number n in $T_n^{(k)}$ is very large, the matrix is decomposed into $T_2^{(1)}$ and the identity matrix under a certain condition. Then, one needs only analyses of $T_2^{(1)}$ in order to analyze $T_n^{(k)}$ by using properties of tensor product.

This paper is organized as follows. In Section 2, we give a theorem of the decomposition via tensor product and show two examples. In Section 3, the decomposition is applied in order to simplify the theorem and the proposition in [13] and to reduce a computational complexity.

2. Main Results

In this section, we present a theorem of a decomposition of a k-tridiagonal Toeplitz matrix $T_n^{(k)}$ and show two examples.

First, the theorem is as follows:

Theorem 1. *Let $T_n^{(k)}$ be an n-by-n k-tridiagonal Toeplitz matrix. If there exist natural numbers n', k', and m such that $n = mn'$ and $k = mk'$, where $m > 1$, $T_n^{(k)}$ is decomposed into the form:*

$$T_n^{(k)} = T_{n'}^{(k')} \otimes I_m, \tag{4}$$

where I_m represents the identity matrix of order m.

Proof. First, let T and S be Toeplitz matrices of the same size. Then, T is equal to S if and only if both of the following equations are satisfied: $(T)_{1:} = (S)_{1:}$ for the first row vectors in T and S; $(T)_{:1} = (S)_{:1}$ for the first column vectors in those. Here, $(T)_{i:}$ and $(T)_{:j}$ denote the i-th row and the j-th column vectors in T, respectively.

Since $T_{n'}^{(k')}$ in (4) has Toeplitz structure, $T_{n'}^{(k')} \otimes I_m$ also has Toeplitz structure. Hence, the two matrices are the same if both of the first column and row vectors in $T_n^{(k)}$ and in $T_{n'}^{(k')} \otimes I_m$ are equal. From (3), the first row and column vectors are obtained as follows:

$$(T_{n'}^{(k')} \otimes I_m)_{1:} = (T_{n'}^{(k')})_{1:} \otimes (I_m)_{1:}$$

$$
\begin{aligned}
&= (d_1, 0, \ldots, 0, a_1, 0, \ldots, 0) \otimes \boldsymbol{e}_1^{\mathrm{T}} \\
&= \left(d_1 \boldsymbol{e}_1^{\mathrm{T}}, \mathbf{0}_m^{\mathrm{T}}, \ldots, \mathbf{0}_m^{\mathrm{T}}, a_1 \boldsymbol{e}_1^{\mathrm{T}}, \mathbf{0}_m^{\mathrm{T}}, \ldots, \mathbf{0}_m^{\mathrm{T}}\right),
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
(T_{n'}^{(k')} \otimes I_m)_{:1} &= (T_{n'}^{(k')})_{:1} \otimes (I_m)_{:1} \\
&= \begin{pmatrix} d_1 \\ 0 \\ \vdots \\ 0 \\ b_{k'+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \otimes \boldsymbol{e}_1 = \begin{pmatrix} d_1 \boldsymbol{e}_1 \\ \mathbf{0}_m \\ \vdots \\ \mathbf{0}_m \\ b_{k'+1} \boldsymbol{e}_1 \\ \mathbf{0}_m \\ \vdots \\ \mathbf{0}_m \end{pmatrix},
\end{aligned} \tag{6}
$$

where $\boldsymbol{e}_1$ and $\mathbf{0}_m$ represent the m-dimensional first canonical vector and the m-dimensional zero vector, respectively. From (5) and (6), we have $(T_{n'}^{(k')} \otimes I_m)_{1:} = (T_n^{(k)})_{1:}$ and $(T_{n'}^{(k')} \otimes I_m)_{:1} = (T_n^{(k)})_{:1}$. Thus,

$$
T_{n'}^{(k')} \otimes I_m = T_n^{(k)}.
$$

This completes the proof. $\square$

Theorem 1 provides three notes: first, the required memory of $T_n^{(k)}$ is the lowest in all the values m when $m = \gcd(n, k)$; second, the k'-tridiagonal Toeplitz matrix $T_{n'}^{(k')}$ is the tridiagonal Toeplitz matrix $T_{n'}^{(1)}$ under the condition that $n = mk$; third, the determinant, the eigenvalues, integer powers, and the inversion of $T_n^{(k)}$ are easily computed from those of $T_{n'}^{(k')}$. As for the condition in the second note, the original matrix $T_n^{(k)}$ is decomposed into the tridiagonal Toeplitz matrix of order 2 and the identity matrix of order k, i.e., $T_n^{(k)} = T_2^{(1)} \otimes I_k$, under the condition that $n = 2k$.

Next, two examples under the condition that $n = mk$ are shown.

Example 2. *Let $n = 8$ and $k = 2$. Setting $m = 2$, then the 2-tridiagonal Toeplitz matrix $T_8^{(2)}$ is decomposed such as $T_8^{(2)} = T_4^{(1)} \otimes I_2$. The matrices are* *specifically denoted as follows:*

$$T_8^{(2)} = \left(\begin{array}{cc|cc|cc|cc} d & 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & d & 0 & a & 0 & 0 & 0 & 0 \\ \hline b & 0 & d & 0 & a & 0 & 0 & 0 \\ 0 & b & 0 & d & 0 & a & 0 & 0 \\ \hline 0 & 0 & b & 0 & d & 0 & a & 0 \\ 0 & 0 & 0 & b & 0 & d & 0 & a \\ \hline 0 & 0 & 0 & 0 & b & 0 & d & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 & d \end{array}\right) = \begin{pmatrix} d & a & 0 & 0 \\ b & d & a & 0 \\ 0 & b & d & a \\ 0 & 0 & b & d \end{pmatrix} \otimes I_2.$$

Example 3. *Let $n = 8$ and $k = 4$. Setting $m = 4$, then the 4-tridiagonal Toeplitz matrix $T_8^{(4)}$ is decomposed such as $T_8^{(4)} = T_2^{(1)} \otimes I_4$. The matrices are specifically denoted as follows:*

$$T_8^{(4)} = \left(\begin{array}{cccc|cccc} d & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & d & 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & d & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & d & 0 & 0 & 0 & a \\ \hline b & 0 & 0 & 0 & d & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & d & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & b & 0 & 0 & 0 & d \end{array}\right) = \begin{pmatrix} d & a \\ b & d \end{pmatrix} \otimes I_4.$$

From Examples 2 and 3, we can confirm the first and second notes. Particularly, we can see that the number of nonzero elements of matrices is reduced by one m-th.

3. Applications

In Subsection 3.1, we present some corollaries, which are obtained from Theorem 1. Some of the corollaries imply the theorem and the proposition that were proved in [13], however the corollaries in the present paper have simpler and more general expressions than in [13]. In Subsection 3.2, we show that Theorem 1 is used in order to reduce a computational complexity.

3.1. The Symmetric k-Tridiagonal Toeplitz Matrices

We consider a specialized form of symmetric k-tridiagonal Toeplitz matrices, i.e.,

$$(S_n^{(k)})_{i,j} = \begin{cases} a, & |i-j| = k, \\ d, & i = j, \\ 0, & \text{otherwise}, \end{cases} \tag{7}$$

where $i, j = 1, 2, \ldots, n$, and parameters n and k are natural numbers such that $n = mk$. Applying Theorem 1 to $S_n^{(k)}$ in (7), we obtain Corollary 4.

Corollary 4. *Let $S_n^{(k)}$ be the matrix as in (7). Then*

$$S_n^{(k)} = S_m^{(1)} \otimes I_k.$$

Proof. By the definition of $S_n^{(k)}$ and Theorem 1, the result can be obtained. □

Using Corollary 4, the determinant, the eigenvalues, and arbitrary integer powers of $S_n^{(k)}$ are easily computed as below.

Corollary 5. *Let $S_n^{(k)}$ be the matrix as in (7). Then*

$$\det(S_n^{(k)}) = [\det(S_m^{(1)})]^k.$$

Proof. By Corollary 4 and the property of the determinant of tensor product, we have

$$\det(S_n^{(k)}) = \det(S_m^{(1)} \otimes I_k) = [\det(S_m^{(1)})]^k.$$

□

Corollary 6. *Let $S_n^{(k)}$ be the matrix as in (7). Then, the eigenvalues λ_j of $S_n^{(k)}$ are represented by*

$$\lambda_j = d + 2a \cos\left(\frac{j\pi}{n+1}\right),$$

where $j = 1, 2, \ldots, m$.

Proof. The result is obtained by Corollary 4, the analytical forms of the eigenvalues of the tridiagonal Toeplitz matrix (cf. [8, Example 7.2.5]), and the eigenvalues of tensor product. □

Note that the eigenvalues are particular forms of those in [7].

Corollary 7. *Let $S_n^{(k)}$ be the matrix as in (7). Then*

$$(S_n^{(k)})^r = (S_m^{(1)})^r \otimes I_k,$$

where r is an arbitrary integer.

Proof. By Corollary 4 and the property of tensor product, the result is obvious. □

Let $a = 1$, $d = 0$, and $m = 4$ in (7). Then Corollary 6 corresponds to [13, Proposition 2]. As for Corollary 7, we have

$$(S_n^{(k)})^r = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}^r \otimes I_k,$$

which corresponds to [13, Theorem 5].

3.2. A Reduction of a Computational Complexity by the Decomposition

Theorem 1 can be used in order to reduce the computational complexity. We here focus on the computation of the inversion of the k-tridiagonal Toeplitz matrix $T_n^{(k)}$ using [5, Theorem 2.1]. An algorithm to compute the inversion with the decomposition is shown below.

First, the k-tridiagonal Toeplitz matrix $T_n^{(k)}$ is decomposed by Theorem 1. Then, the inversion of $T_n^{(k)}$ is computed by $(T_n^{(k)})^{-1} = (T_{n'}^{(k')})^{-1} \otimes I_m$. Here, the algorithm in [5, Theorem 2.1] computes the inversion element-wise. Therefore, the fewer $T_n^{(k)}$ has elements, the lower the computational complexity of the algorithm is. Since the number of nonzero elements of $T_{n'}^{(k')}$ is reduced by one m-th, the computational complexity is also reduced by one m-th.

4. Conclusion

In the present paper, we gave a decomposition of a k-tridiagonal Toeplitz matrix via tensor product. As applications of the decomposition, we have shown that the determinant, the eigenvalues, and arbitrary integer powers of the matrix are easily computed and that the inversion of the matrix is computed with lower computational complexity than that without the decomposition.

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