LONG CYCLES AND NEIGHBORHOOD UNION IN 1-TOUGH GRAPHS WITH LARGE DEGREE SUMS

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Abstract

For a 1-tough graph G we define \( \sigma_3(G) = \min \{d(u) + d(v) + d(w) : \{u, v, w\} \text{ is an independent set of vertices}\} \) and \( NC_{\sigma_3-n+5}(G) = \max \{\bigcup_{i=1}^{\sigma_3-n+5} N(v_i) : \{v_1, \ldots, v_{\sigma_3-n+5}\} \text{ is an independent set of vertices}\} \). We show that every 1-tough graph with \( \sigma_3(G) \geq n \) contains a cycle of length at least \( \min \{n, 2NC_{\sigma_3-n+5}(G) + 2\} \). This result implies some well-known results of Faßbender [2] and of Flan
drin, Jung & Li [6]. The main result of this paper also implies that \( c(G) \geq \min \{n, 2NC_2(G) + 2\} \) where \( NC_2(G) = \min \{|N(u) \cup N(v)| : d(u, v) = 2\} \). This strengthens a result that \( c(G) \geq \min \{n, 2NC_2(G)\} \) of Bauer, Fan and Veldman [3].

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Introduction

We consider only a finite undirected graph without loops and multiple edges. For undefined terms we refer to [3]. Let \( \omega(G) \) denote the number of components of a graph G. A graph G is 1-tough if for every nonempty proper subset S of the vertex set V(G) of G we have \( \omega(G - S) \leq |S| \). We use \( \alpha \) to denote the cardinality of a maximum independent set of vertices of G. A cycle C in G is called a dominating cycle if the vertices of the graph \( G - C \) are independent. The length \( \ell(C) \) of a longest cycle C of G is denoted by \( c(G) \). For \( k \leq \alpha \) we denote by \( \sigma_k \) the minimum value of the degree sum of any \( k \) pairwise nonadjacent vertices and by \( NC_k(G) \) the minimum cardinality of the neighborhood union of any \( k \) such vertices. For \( k > \alpha \) we set \( \sigma_k = k(n - \alpha(G)) \) and \( NC_k = n - \alpha(G) \). Instead of \( \sigma_1 \) and \( NC_1 \) we use the more common notation \( \delta(G) \). If no ambiguity can arise, we sometimes write \( \alpha \) instead of \( \alpha(G) \), etc.
A number of results have been established concerning long cycles in graphs with large degree sums. For details we refer to a survey [4] and [7]. Since, clearly, \( NC_t(G) \) is a non decreasing function of \( t \) and \( NC_t(G) \geq \frac{1}{t} \sigma_t(G) \), analogous results in terms of \( NC_t \) would extend well-known previous results [5].

Let \( d(u,v) \) denote the distance between \( u \) and \( v \). Our main result in the present paper is Lemma 9 and its consequence.

**Theorem 1.** If \( G \) is a 1-tough nonhamiltonian graph of order \( n \geq 3 \) with \( \sigma_3 \geq n \), then there exists in \( G \) an independent set of \( \sigma_3 - n + 5 \) vertices \( \{v_0, \ldots, v_{\sigma_3-n+4}\} \) such that \( d(v_0,v_i) = 2 \) (\( i \geq 1 \)) and \( c(G) \geq 2|\bigcup_{i=0}^{\sigma_3-n+4} N(v_i)| + 2 \).

Clearly, Theorem 1 strengthens the result of Bauer et al. (Theorem 26 in [5]) that under the same hypothesis \( c(G) \geq 2NC_2(G) \). Theorem 1 also implies the next result.

**Theorem 2.** If \( G \) is a 1-tough graph of order \( n \geq 3 \) with \( \sigma_3 \geq n \), then \( c(G) \geq \min\{n, 2NC_{3\sigma_3-n+5} + 2\} \).

Theorem 1 and Theorem 2 are strongly related to other results of Broersma, Van den Heuvel & Veldman [7] and in Van den Heuvel [8].

**Theorem 3** (Corollary 6 in [7]). If \( G \) is a 1-tough graph of order \( n \geq 3 \) with \( \sigma_3 \geq n \), then \( c(G) \geq \min\{n, 2NC_{3\delta-n+5}\} \), where \( \delta = \lceil \frac{2n}{3} \rceil \).

**Theorem 4** (Theorem 11 in [7]). If \( G \) is a 1-tough graph of order \( n \geq 3 \) with \( \sigma_3 \geq n + r \geq n \) and \( n \geq 8t - 6t - 17 \), then \( c(G) \geq \min\{n, 2NC_t\} \).

**Theorem 5** (Corollary 7.12 in [8]). If \( G \) is a 1-tough graph on \( n \geq 3 \) vertices, then \( c(G) \geq \min\{n, 2NC_\frac{1}{2}(4\delta-n+3)\} \).

Theorem 2 is in a sense best possible. This can be seen from the construction by Bauer et al. [3] of a 1-tough graph \( G_n \) for odd \( n \geq 15 \). The graph \( G_n \) is obtained from \( K_{(n-1)/2} \cup K_3 \cup K_{(n-5)/2} \) by joining every vertex of \( K_{(n-5)/2} \) to all vertices in \( K_{(n-1)/2} \cup K_3 \) and by adding a matching between the vertices of \( K_3 \) and three vertices in \( K_{(n-1)/2} \). A variation of the graph \( G_n \), with \( K_{(n-5)/2} \) replaced by \( K_{(n-5)/2} \), has already appeared in [1].

But we do not know of the existence of 1-tough graphs \( G \) on \( n \geq 3 \) vertices with \( \sigma_3 \geq n \) and \( c(G) < n - 1 \) for which Theorem 2 is best possible. Moreover, we cannot conclude Theorem 2 from Theorem 3, Theorem 4
and Theorem 5. Let $G_{(n,p)}$ denote the graph $(F_p \cup K_{(n-1)/2-(2p+1)}) + K_{(n+1)/2-(2p+1)}$ for odd $n \geq 12p + 3 \geq 27$, where $F_p$ denotes the unique graph with a degree sequence $(d_1 = 1, d_2 = 1, ..., d_{2p+1} = 1, d_{2p+2} = 2p + 1, ..., d_{4p+2} = 2p + 1)$. Then $G_{(n,p)}$ is a 1-tough graph on $n \geq 27$ vertices with $\sigma_3 \geq n$. By Theorem 2, $c(G_{(n,p)}) \geq n + 1 - 4p$ which cannot be deduced from Theorem 3, Theorem 4 and Theorem 5.

Theorem 2 immediately implies a result of Flandrin, Jung & Li [6].

Corollary 6. If $G$ is a 1-tough graph of order $n \geq 13$ with $\sigma_3 \geq \frac{3n-14}{2}$, then $G$ is hamiltonian.

Proof. Clearly, $\sigma_3 \geq n$ for $n \geq 13$ and $\sigma_3 - n + 5 \geq \frac{n-4}{2}$ if $\sigma_3 \geq \frac{3n-14}{2}$. Since $G$ is a 1-tough graph, $NC_{\lceil \frac{n-4}{2} \rceil} \geq \frac{n-2}{2}$. Hence, $2NC_{\sigma_3 - n + 5} + 2 \geq 2n - 2 + 2 = n$. By Theorem 2, $c(G) \geq \min\{n, 2NC_{\sigma_3 - n + 5} + 2\} = n$. Thus, $G$ is hamiltonian.

Theorem 2 immediately implies a result of Flandrin, Jung & Li [6].

Corollary 7. If $G$ is a 2-connected graph of order $n$ such that $d(u) + d(v) + d(w) \geq n + |N(u) \cap N(v) \cap N(w)|$ for every independent set $\{u,v,w\}$, then $G$ is hamiltonian.

Proof. Let $G$ satisfy the stated conditions. Then $G$ is 1-tough [4] and $n \leq 2NC_{3}$ [7]. The proof is completed by applying Theorem 2 (note that $NC_{\sigma_3 - n + 5} \geq NC_{3}$).

Proofs

Let $C$ be a cycle in $G$ with an assigned orientation. If $x$ and $y$ are two vertices of $C$ then $x \overrightarrow{C} y$ denotes the path on $C$ from $x$ to $y$, inclusively $x$ and $y$, following the assigned orientation. The same vertices in a reverse order are given by $y \overleftarrow{C} x$. We will consider $x \overrightarrow{C} y$ and $y \overleftarrow{C} x$ both as a path and as a vertex set. If $c$ is a vertex on $C$, then $c^+$ and $c^-$ are its successor and predecessor on $C$, respectively, according to the assigned orientation. If $X$ is a set of vertices on $C$ let $X^+ := \{x^+ : x \in X\}$ and $X^- := \{x^- : x \in X\}$. If $v \in V(G)$ and $H \subseteq V(G)$ then $N_H(v)$ is the set of all vertices in $H$ adjacent to $v$. We denote $|N_H(v)|$ by $d_H(v)$. If $G$ is a nonhamiltonian graph, we set $\mu(C) = \max\{d(v) : v \in V(G) - V(C)\}$ and $\mu(G) = \max\{\mu(C) : C$ is a longest cycle in $G\}$.

The following lemmas are already proved in [3].
Lemma 1 (Theorem 5 [3]). Let $G$ be a 1-tough graph with $\sigma_3 \geq n$. Then every longest cycle in $G$ is a dominating cycle.

Lemma 2 (see proof of Theorem 9 [3]). Let $G$ be a 1-tough graph with $\sigma_3 \geq n$. If $G$ is nonhamiltonian, then $G$ has a longest cycle $C$ such that $C$ avoids a vertex $v_0$ with $d(v_0) \geq \frac{\sigma_3}{3}$ in $G$.

Lemma 3 (Lemma 8 [3]). Let $G$ be a 1-tough graph with $\sigma_3 \geq n$. Suppose $C$ is a longest cycle in $G$. If $v_0 \in V(G) - V(C)$ and $A = N(v_0)$, then $(V(G) - V(C)) \cup A^+$ is an independent set of vertices.

Assume $G$ is nonhamiltonian. Let $C$ be a cycle in $G$ with an assigned orientation, $v \in V(G) - V(C)$ and $v_1, ..., v_k$ be the elements of $N(v)$, occurring on $C$ in a consecutive order. For $i = 1, 2, ..., k$ set $u_i = v_i^+$ and $w_i = v_{i+1}$ (indices modulo $k$). We set, for convenience, $3 = \{ i : \text{there exists some } j \neq i \text{ such that } u_iw_j \in E(G) \}$.

The set $u_i \rightarrow w_i$ will be called a segment; $u_i \rightarrow w_i$ is a $p$-segment if $|u_i \rightarrow w_i| = p$. Let $S$ denote the set of 1-segments. The following lemma is observation (1) in the proof of Theorem 4 in Broersma et al. [7].

Lemma 4. $(V(G) - V(C)) \cup N(v)^+ \cup N(S)^+$ is an independent set of vertices.

If $d(v) = \mu(G)$ then $d(v) \geq n/3$ because of Lemma 2 and therefore $S \neq \emptyset$. Let $u_{i_1}, u_{i_2}, ..., u_{i_s}$ be the vertices of the 1-segments and assume, without loss of generality, that $i_1 = 1$ and $d(u_1) \geq d(u_{i_2}) \geq ... \geq d(u_{i_s})$. Since $C' : v v_2 \rightarrow v_1 v$ is a longest cycle, $\mu(G) \geq d(u_1)$.

Lemma 5. If $\mu(G) = d(v) \leq \frac{\sigma_3 + 2}{3}$, then $d(v) = d(u_1)$.

Proof. Suppose to the contrary that $d(u_1) \leq d(v) - 1$. Let $t_C(v) = |V(C) - (N(v) \cup N(v)^+ \cup N(v)^-)|$. By $n - 1 \geq \ell(C) = 3d(v) - s + t_C(v)$, $n - 1 + s - 3d(v) \geq t_C(v)$ (*). We distinguish 3 cases:

Case 1. $s = 1$.

By (*) and by Lemma 2, in fact, $\ell(C) = n - 1$, $d(v) = \frac{n}{3}$ and $t_C(v) = 0$. Since $G$ is a 1-tough graph, $G - N(v)$ contains at most $d(v)$ components. Hence, there is $i_0 \neq j_0$ and some edge joining $u_{i_0}$ with $w_{j_0}$. Now, $C' : v v_{j_0 + 1} \rightarrow u_{i_0} w_{j_0} \rightarrow u_{i_0 + 1} v$ is also a longest cycle which avoids $w_{i_0}$. Thus, $d(w_{i_0}) \leq d(v)$ by the maximality of $d(v)$, and therefore $d(u_1) + d(w_{i_0}) + d(v) \leq 3d(v) - 1 = n - 1$, a contradiction.
Case 2. \( s = 2 \).

By (\( * \)), \( \frac{(n+1)}{3} \geq d(v) \) and therefore \( d(u_1) + d(u_2) + d(v) \leq 3d(v) - 2 \leq n - 1 \), a contradiction.

Case 3. \( s \geq 3 \).

In this case we have \( d(u_1) + d(u_2) + d(u_3) \leq 3d(v) - 3 \leq \sigma_3 - 1 \), a contradiction. Thus, Lemma 5 is true.

\begin{lemma}
If \( C \) contains only \( p \)-segments with \( p \leq 3 \), then \( \exists \neq \emptyset \).
\end{lemma}

\begin{proof}
Suppose to the contrary that \( \exists = \emptyset \). We consider \( G - (N(v) \cup \{ u_i^+: u_i \in C \} \) is a 3-segment and \( u_iw_i \notin E(G) \}). Since \( G \) is a 1-tough graph there exists \( i \neq j \) and some arc \( B \) joining a vertex \( p \in u_i \) with a vertex \( q \) in \( u_j \). By Lemma 3 and since \( \exists = \emptyset \), \( p = u_i^+ = w_i \) or \( q = u_j^+ = w_j \), say \( p = u_i^+ = w_i \) and therefore \( u_iw_i \in E(G) \). We distinguish two cases:

1. \( q = u_j \) (similar for the case \( q = w_j \)).

In this case \( C' : vv_jw_iu_ipBu_jw_i \) would be a cycle longer than \( C \), a contradiction.

2. \( q = u_j^+ = w_j \).

In this case \( C' : vv_jw_iu_ipBu_jw_i \) would be a cycle longer than \( C \), a contradiction. Thus Lemma 6 is true.
\end{proof}

\begin{lemma}
Suppose that \( \exists \neq \emptyset \). Let \( i_0 = \max \exists \) and \( j_0 \neq i_0 \) such that \( u_iw_j \in E(G) \). Suppose that \( v_iu_1 \in E(G) \) or \( \{ u_1v_{j_0+1}, u_1v_{j_0} \} \subseteq E(G) \). Then \( d(u_{j_0}) + 2d(v) \leq \ell(C) + x \), where \( x \) is the number of vertices \( u_i = w_i \) such that \( v_iu_i \notin E(G) \) and \( \{ u_iu_{j_0+1}, u_iu_{j_0} \} \subseteq E(G) \).
\end{lemma}

\begin{proof}
To prove this lemma we start with a trivial observation.

(\( * \)) If \( u_iw_i \in E(G) \) or \( u_{i+1}w_i \in E(G) \) then \( u_i \in v_{j_0}u_{j_0} \).

For \( i = 1, 2, \ldots, k \) we set \( L_i := u_i \) for \( i \neq j_0 \). Then \( d_L(u_{j_0}) \leq |L_i| - 1 \) because of \( u_iu_{j_0} \notin E(G) \) by Lemma 3. Since \( d(u_{j_0}) = \sum_{i=1}^{k} d(u_i) \) it suffices to show that \( d_L(u_{j_0}) \leq |L_i| - 2 \) (i.e. there exists on \( L_i \) some \( z \neq u_i \) such that \( zv_{j_0} \notin E(G) \) for \( u_i \neq w_i \) and for \( u_i = w_i \) with \( v_iu_i \in E(G) \) or \( \{ u_iu_{j_0+1}, u_iu_{j_0} \} \subseteq E(G) \).

Note that \( j_0 > i_0 \) and \( v_{j_0+1} \neq v_{i_0} \) by (\( * \)) (for \( i = 1 \)). Thus \( w_iu_{j_0} \notin E(G) \) if \( w_i \neq u_i \) and \( i \neq j_0 \) because of the maximality of \( i_0 \). If \( i = j_0 \), then \( v_{j_0+1}w_{j_0} \notin E(G) \) by (\( * \)). If \( u_i = w_i \) with \( v_iu_i \in E(G) \) or \( \{ u_iu_{j_0+1}, u_iu_{j_0} \} \subseteq E(G) \) then \( u_i \in w_{j_0}u_{j_0} \) by (\( * \)) and therefore \( v_{i+1}u_{j_0} \notin E(G) \). Otherwise, \( C' : vv_{j_0+1}u_{j_0}u_{j_0}u_{j_0+1}u_{j_0}u_{j_0}u_{j_0}v \), when \( u_iu_{j_0} \in E(G) \),
and $C' : v_{i_o} \overrightarrow{w_{j_0}} u_i, v_{j_0} \overrightarrow{w_{j_0}} u_{i_o} \overrightarrow{v_{i_o}}$, when $\{u_i v_{j_0+1}, u_i v_{j_0}\} \subseteq E(G)$ would be a cycle longer than $C$, a contradiction. Thus Lemma 7 is true.

Theorem 1 is obviously established by the next two lemmas.

**Lemma 8.** Let $X = N(v) \cup \{N(u_i) : u_i \in S\}$. Then $f(C) \geq 2|X| + 2$.

**Proof.** Let $x_1, ..., x_y$ be the vertices of $X$, occurring on $C$ in a consecutive order. By Lemma 4, $X \cap X^+ = \emptyset$. Since $G$ is a 1-tough graph, there exist some $i \neq j$ and some arc joining a vertex $y$ on $x_i^+ \overrightarrow{x_{i+1}}$ and a vertex $z$ on $x_j^+ \overrightarrow{x_{j+1}}$. Without loss of generality, assume that $|x_i^+ \overrightarrow{x_{i+1}}| \leq |x_j^+ \overrightarrow{x_{j+1}}|$. Then by Lemma 4, $z \notin \{x_j^+, x_{j+1}\}$ if $x_i^+ = x_{i+1}$. Thus, $f(C) \geq 2|X| + 2$.

Following Broersma et al. [7], we say that a property $P$ holds by the longest cycle argument, denoted by $P(C')$, if the contrary implies the existence of a cycle $C'$ longer than $C$.

Now, we give and prove a lower bound of so called 1-segments. Theorem 1 is established by the last lemma.

**Lemma 9.** Let $G$ be a 1-tough nonhamiltonian graph on $n \geq 3$ vertices with $\sigma_3 \geq n$. Then $G$ contains a longest cycle $C$ avoiding a vertex $v$ with $d(v) = \mu(G)$ and $s \geq \sigma_3 - n + 4$.

**Proof.** Assume to the contrary that $s \leq \sigma_3 - n + 3$ for any longest cycle $C$ avoiding a vertex $v$ with $d(v) = \mu(G)$. Let $t_C(v) = |V(C) - (N(v) \cup N(v^+) \cup N(v^-))|$.

**Claim 1.** If $C$ is a longest cycle in $G$ avoiding a vertex $v$ with $d(v) = \mu(G)$, then $d(v) \leq \frac{\sigma_3 + 2}{3}$ and $t_C(v) \leq 2$ with strict inequality if $\mu(G) \neq \frac{\sigma_3}{3}$ or $\ell(C) \neq n - 1$.

**Proof.** Counting the vertices on $C$ we get $n - 1 \geq \ell(C) = 3d(v) - s + t_C(v)$. Thus, $\sigma_3 + 2 - t_C(v) \geq 3d(v)$ and $\sigma_3 - 3d(v) + 2 \geq t_C(v)$, establishing Claim 1.

**Claim 2.** If $C$ is a longest cycle avoiding a vertex $v$ with $d(v) = \mu(G)$, then $\delta = 0$.

**Proof.** Supposing that $\delta \neq \emptyset$, we determine $i_0 = \max \delta$ and $j_0 \neq i_0$ such that $u_{i_0} w_{j_0} \in E(G)$. First note that if $u_i = w_i$ and $d(u_i) = d(v)$, then by $P(C')$ $u_i^+ \overrightarrow{w_{j_0}} u_i^+ \overrightarrow{w_{j_0}} u_{i_0}^+ v_{j_0+1}$ when $u_i \in u_{i_0} \overrightarrow{w_{j_0}}$ and $C' : v_{i_0+1} \overrightarrow{w_{j_0}} u_{i_0}^+ w_{j_0} u_{i_0}^+ v_{j_0+1}$ when $u_i \in w_{j_0} \overrightarrow{u_{i_0}}$. Similarly,
Lemma 5 and by Claim 1, otherwise, $t_{C'}(u_i) \geq 3$ where $C' : vv_i \overrightarrow{C} v_{i+1}v$, which contradicts Claim 1. By Lemma 5 and by Claim 1, $d(u_i) = d(v)$. Now using Lemma 7 we have $d(u_i) + d(v) + d(u_{j_0}) = 2d(v) + d(u_{j_0}) \leq \ell(C) + x$, where $x$ is the number of vertices $u_i = w_i$ such that $d(u_i) \leq d(v) - 1$. By $\sigma_3 \geq n$, $x \geq 1$. Hence, $d(v) + d(u_i) + d(u_{j_0}) \leq \ell(C) + x - 1$ and, by similar argument, $x \geq 2$. Note that by $\frac{\sigma_3 + 2}{3} \geq d(v)$, $x \leq 2$ by $d(u_i) + d(u_{i+1}) + d(u_{i+2}) \geq \sigma_3$ and, by $x \geq 2$, in fact, $x = 2$. Now we get $d(u_{i+1}) + d(u_i) + d(u_{j_0}) \leq \ell(C) < n$, a contradiction.

The next claim is obviously established by Lemma 6, Claim 2 and Claim 1.

**Claim 3.** If $C$ is a longest cycle and $v \in V(G) - V(C)$ such that $d(v) = \mu(G)$, then $t_C(v) = 2$ and $C$ contains a 4-segment.

By Claim 1, we get $\ell(C) = n - 1$ and $d(v) = \sigma_3/3$. Using the inequality $n - 1 \geq 3d(v) - s + t_C(v)$ and $t_C(v) = 2$ by Claim 3, we get $s \geq \sigma_3 - n + 3 \geq 3$. By $d(u_i) + d(u_i) + d(v) \geq \sigma_3$, we easily get:

**Claim 4.** If $C$ is a longest cycle avoiding a vertex $v$ with $d(v) = \mu(G)$, then $d(u_i) \geq d(v) = \frac{\sigma_3}{3}$ and $d(w_i) \geq d(v)$ with equality if $u_i = w_i$.

**Claim 5.** If $C$ is a longest cycle avoiding a vertex $v$ with $d(v) = \mu(G)$, then $N(u_i) = N(v)$ for any $u_i = w_i$.

**Proof.** Suppose that there exists some $u_i = v_i$ such that $N(u_i) \neq N(v)$. By Claim 4, either $u_i^+ w_t^+ u_i \in E(G)$ or $w_t^- u_i \in E(G)$, say $w_t^- u_i \in E(G)$.

Note that $u_i w_t^+ \notin E(G)$ ($C' : vv_i u_i^- \overrightarrow{C} u_i u_i^+ v_{i+1}v$) and $u_i w_t^- \notin E(G)$ ($C' : vv_i u_i^- \overrightarrow{C} u_i u_i^+ v_{i+1}v$).

Therefore there exists some $j$ such that either $w_i^- w_j \in E(G)$ or $w_i^- w_j \notin E(G)$ since $\omega(G - N(v) - \{u_i^+\}) \leq d(v) + 1$ by the toughness of $G$ and by Claim 2. But $w_i^- u_j \notin E(G)$ ($C' : vv_{i+1} u_i^+ v_{i+1}u_i^- \overrightarrow{C} u_i w_{j}^+ v_{i+1}v$) and therefore $w_i^- w_j \notin E(G)$.

Moreover, $w_j \in u_i^- w_i^-$ ($C' : vv_{i+1} u_i^+ v_{i+1}u_i^- \overrightarrow{C} u_i w_{j}^+ v_{i+1}v$) and $w_j \in u_i^- w_i^-$ ($C' : vv_{i+1} u_i^+ v_{i+1}u_i^- \overrightarrow{C} u_i w_{j}^+ v_{i+1}v$). By Claim 2, $d(w_i) \leq d(v) - 1$ since $w_i w_{i+1} \notin E(G)$ ($C' : vv_{i+1} u_i^+ v_{i+1}u_i^- \overrightarrow{C} u_i w_{j}^+ v_{i+1}v$) and $w_i w_{i+1} \notin E(G)$ ($C' : vv_{i+1} u_i^+ v_{i+1}u_i^- \overrightarrow{C} u_i w_{j}^+ v_{i+1}v$) and $w_i w_{i+1} \notin E(G)$ ($C' : vv_{i+1} u_i^+ v_{i+1}u_i^- \overrightarrow{C} u_i w_{j}^+ v_{i+1}v$) (note that $w_i w_j \notin E(G)$), which contradicts Claim 4.

Thus Claim 5 is true.

Now, a longest cycle $C$ and a vertex $v_0 \in V(G) - V(C)$ with $d(v_0) = \mu(G)$ are fixed. Then there exists one $t$ such that $|u_i^- w_t^-| = 4$ and $|u_i^+ w_t^-| \leq 2$ for any $i \neq t$. 

Since $G$ is a 1-tough graph, $\omega(G - N(v_0)) \leq d(v_0)$ and therefore there exist $i \neq j$ and some $y \in u_i \overline{C} w_i, z \in u_j \overline{C} w_j$ such that $yz \in E(G)$. Since $S = \emptyset$ by Claim 2, either $i = t$ or $j = t$, say $j = t$ and assume, without loss of generality, that $y = u_i$. We distinguish two cases.

Case 1. $u_i u_t^+ \in E(G)$. We consider the pair $u_t$ and $C': v_0 v_t \overline{C} u_i^+ u_i \overline{C} v_t v_0$. By the maximality of $d(v_0)$, Claim 4 for $v_0$ and $C$ yields $\mu(C') = d(u_t) = \mu(G)$. Now, Claim 3, 4 and 5 can be applied to $u_i$ and $C'$. If $v_i u_t \notin E(G)$, then $v_i^+ v_t^+ v_i v_0$ is the 4-segment of $u_t$ and $C'$, consequently $u_{i-1} \neq v_{i-1}$. If $v_j u_t \notin E(G)$ for some $v_j \neq v_i, v_t$ then $v_j^+ v_{j+1} v_j v_{j+1}$ is the 4-segment of $u_t$ and $C'$, therefore either $u_i \overline{C} w_i$ or $u_{i-1} \overline{C} w_{i-1}$ is a 2-segment of $v_t$ and $C$. It follows by $s \geq 3$ that some 1-segment of $v_0$ and $C$ is also a 1-segment of $u_t$ and $C'$. But this contradicts $N(u_t) \neq N(v)$ and Claim 5 (applied to both pairs $v_0, C$ and $u_t, C'$). This rejects Case 1.

Case 2. $u_i w_i^+ \in E(G)$.

In this Case $N(u_i^+) \cap N(v_0)^+ = \{u_t\}$ by Case 1. Since $G$ is 1-tough and $u_i w_i \notin E(G)$ ($C': v_0 v_t \overline{C} u_i^+ u_i \overline{C} u_i w_t \overline{C} v_t v_0$) it follows that $u_i^+$ has a neighbor $w_j$. Clearly $w_j$ is on $u_i \overline{C} v_i$ ($C': v_0 v_j+1 \overline{C} u_i^+ u_i \overline{C} u_i w_t \overline{C} v_t v_0$). Now consider the pair $w_i$ and $C': v_0 v_{i+1} \overline{C} u_i^+ u_i \overline{C} w_j \overline{C} u_i \overline{C} v_{j+1} v_0$ to obtain a contradiction as in Case 1.

Conjecture

The lower bound on the number of so called 1-segments on a longest cycle in Lemma 9 is best possible only for $c(G) = n - 1$.

**Conjecture.** Let $G$ be a 1-tough nonhamiltonian graph on $n \geq 3$ vertices with $\sigma_3 \geq n$. Then $G$ contains a longest cycle $C$ (with an assigned orientation) avoiding a vertex $v$ with $d(v) = \mu(G)$ and $|N_C(v)^+ \cap N_C(v)^-| \geq \sigma_3 - n + 3\omega(G - C) + 1$.

The graphs $G_{(n, p)}$ show that our Conjecture, if true, is best possible, also in case $c(G) < n - 1$.

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References


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