Distributed consensus control for linear multi-agent systems with discontinuous observations

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This article addresses the distributed consensus problem of linear multi-agent systems with discontinuous observations over a time-invariant undirected communication topology. Under the assumption that each agent can only intermittently share its outputs with the neighbours, a class of distributed observer-type protocols are designed and utilised to achieve consensus. By using appropriate matrix decomposition, it is shown that consensus in the closed-loop multi-agent systems under a connected topology can be converted to the simultaneous asymptotic stability of a set of switching systems whose dimensions are the same as each agent. From a multiple Lyapunov functions approach, it is proved that there exists a protocol to guarantee consensus if the communication time rate is larger than a threshold value. Furthermore, a distributed pinning control method is employed to solve the consensus problem on an arbitrary given topology which needs not be connected. Particularly, the questions of what kind of agents and at least how many agents should be pinned are addressed. The effectiveness of the analytical results is finally verified by numerical simulations.

Keywords: multi-agent system; consensus; pinning control; discontinuous observation; lyapunov function

1. Introduction

Recently, distributed cooperative control of multi-agent systems has received considerable attention from various scientific communities due to the growing interest in understanding the intriguing animal group behaviours, such as swarming (Gazi and Passino 2003) and flocking (Su, Wang, and Lin 2009; Wen, Duan, Li, and Chen 2012b), and their potential applications in formation control of satellites (Smith and Hadaegh 2005), teaming of multi-robotics (Ren and Sorensen 2008), and design of sensor networks (Yu, Chen, Wang, and Yang 2009b). Among the numerous research topics in distributed control of multi-agent systems, consensus problem is of particular importance, which refers to designing an appropriate protocol based only on the local relative information among neighbouring agents to guide all agents to reach an agreement (see survey papers Olfati-Saber, Fax, and Murray (2004) and the references therein).

Vicsek, Czirók, Ben-Jacob, Cohen, and Shochet (1995) introduced a simple yet effective discrete-time model for phase transition of a group of autonomous agents and numerically investigated the angle consensus behaviour of the multi-agent system. By using the tools from the algebraic graph theory, a theoretical explanation of the consensus behavior observed in Vicsek et al. (1995) was first given in Jadbabaie, Lin, and Morse (2003). A general framework of the consensus problem for a network of agents with single-integrator dynamics under a fixed or a switching topology and possible communication time delays was suggested and studied in Olfati-Saber and Murray (2004). The consensus conditions derived in Olfati-Saber and Murray (2004) were further relaxed in Ren and Beard (2005) by proving that consensus in multi-agent systems with single-integrator dynamics can be achieved if and only if the time-varying network topology contains a directed spanning tree frequently enough as the network evolves over time. Meanwhile, the consensus problem for multi-agent systems with second- and higher-order dynamics were addressed (Ren and Atkins 2007; Xie and Wang 2007; Hong, Chen, and Bushnell 2008; Wen, Duan, Yu, and Chen 2012d; Ren, Moore, and Chen 2007b; Zhang and Lewis 2012). Note that most of the above-mentioned works are concerned with the case where the agents are governed by integrator-type dynamics.
However, multi-agent systems with general linear node dynamics are more popular (Ma and Zhang 2010; Li, Duan, Chen, and Huang 2010; Zhang, Lewis, and Das 2011; Li, Duan, and Chen 2011), which include networks of agents with integrator-type of dynamics as special cases. In Ma and Zhang (2010), from a static output approach, some necessary and sufficient conditions were derived for consensus of multi-agent systems with general linear node dynamics under a fixed directed topology. Consensus in multi-agent systems with general linear node dynamics under a fixed directed topology was investigated with observer-type protocols appropriately designed in Li et al. (2010), Zhang et al. (2011), Li et al. (2011). Most of the above-mentioned results on the consensus problem in multi-agent systems with general linear dynamics are obtained based on the assumption that information is transmitted continuously among the agents, i.e. each agent has to share its state or output information with its neighbours all the time. However, this may not always be the case in reality. Sometimes, mobile agents can only communicate with their neighbours over some disconnected time intervals due to, for instance, temporary sonar equipment failures or the presence of communication obstacles, even if the distances among them are less than the communication radius. Yet, equipment failures may be recovered through repairing and communication obstacles may be bypassed as the system evolves in time. Motivated by these facts and based on the works reported in Wen, Duan, and Chen (2012a), Wen, Duan, Li, and Chen (2012c), the consensus problem for multi-agent systems with general linear node dynamics based on intermittent observations is studied in this article, where the ideal assumption that agents could transmit their output information to their neighbours at all times is removed. For convenience of analysis, the communication topology among the agents is assumed to be undirected. For achieving consensus, a new class of consensus protocols are proposed and analysed. By using tools from switching systems theory, it is theoretically shown that consensus in a closed-loop multi-agent system with a connected topology can be ensured if the communication rate is larger than a threshold value. The analytical expression of the threshold value is also explicitly given. By using a distributed pinning-based control method, the results are then extended to consensus in multi-agent systems with an arbitrary topology which needs not be connected. In particular, the questions of what kind of agents and at least how many agents should be pinned for achieving consensus are addressed and answered. Numerical examples are finally given to verify the theoretical analysis.

The rest of this article is organised as follows. Some preliminaries and the model formulation are presented in Section 2. Consensus problem for multi-agent systems with discontinuous observations under a time-invariant connected topology is studied in Section 3. Some extensions are given in Section 4. In Section 5, several numerical simulations are provided for illustration. Conclusions are finally drawn in Section 6.

Throughout this article, let $\mathbb{N}$ and $\mathbb{R}$ be the set of natural and real numbers, respectively, and $\mathbb{R}^{n \times n}$ be the sets of $n \times n$ real matrices. Let $I_n$ ($O_n$) be the $n \times n$ identity (zero) matrices, and $\mathbf{1}_n$ be the $n$-dimensional column vector with all entries equal to one (zero). Matrices, if not explicitly stated, are assumed to have compatible dimensions. The matrix inequality $A > B$ means that both $A$ and $B$ are square symmetric matrices and that $A - B$ is positive-definite. diag($a_1, a_2, \ldots, a_N$) represents a diagonal matrix with $a_i, i = 1, 2, \ldots, N$, being its diagonal elements. Notations $\otimes$ and $\| \cdot \|$ represent the Kronecker product and the Euclidian norm, respectively.

2. Preliminaries and formulation of the model

In this section, some preliminaries and the model formulation for consensus in multi-agent systems with discontinuous observations are introduced.

2.1 Preliminaries

An undirected graph $G$ is a pair of $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \ldots, N\}$ is a node set and $\mathcal{E} \subseteq \{(i, j), i, j \in \mathcal{V}\}$ is an edge set in which an edge is represented by an unordered pair of distinct nodes. Two nodes $i, j$ are adjacent or neighboring, if $(i, j)$ is an edge of graph $G$, i.e. $(i, j) \in \mathcal{E}$. A path on $G$ from node $i_1$ to node $i_s$ is a sequence of ordered edges of the form $(i_k, i_{k+1})$, $k = 1, 2, \ldots, s - 1$. An undirected graph is connected if there exists a path between every pair of distinct nodes, otherwise is disconnected. Simple graphs are considered here, i.e. multiple edges and self-loops are forbidden in $G$. A connected subgraph of $G$, which is maximal, is called a component of $G$.

The adjacency matrix $A = [a_{ij}]_{N \times N}$ of a graph $G$ is defined by $a_{ij} = 0$ for $i = 1, 2, \ldots, N$, and $a_{ij} = a_{ji} > 0$ for $(i, j) \in \mathcal{E}$ but 0 otherwise. The Laplacian matrix $L = [l_{ij}]_{N \times N}$ is defined as $l_{ij} = -a_{ij}$, $i \neq j$, and $l_{ii} = \sum_{j=1}^{N} a_{ij}$ for $i = 1, 2, \ldots, N$. For an undirected graph $G$, both its adjacency matrix and Laplacian matrix are symmetric.

The following lemmas can be found in graph theory textbooks (e.g. Godsil and Royle 2001).
Lemma 1: Suppose that an undirected graph $G$ has $m$ components. Then, there exist a permutation matrix $W$ of order $N$, such that

$$ W^T \mathbf{L} W = \begin{bmatrix} \tilde{L}_{11} & O & \cdots & O \\ O & \tilde{L}_{22} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & \tilde{L}_{mm} \end{bmatrix}, $$

where $\tilde{L}_{11} \in \mathbb{R}^{q_1 \times q_1}$, $\tilde{L}_{22} \in \mathbb{R}^{q_2 \times q_2}, \ldots, \tilde{L}_{mm} \in \mathbb{R}^{q_m \times q_m}$ are symmetric matrices with zero row sums, with $\sum_{j=1}^{m} q_j = N$ and $1 \leq q_1 \leq \ldots \leq q_m \leq N$. Furthermore, $\mathcal{R}(L) = N - m$, where $\mathcal{R}(L)$ represents the rank of $L$.

Remark 1: Any undirected graph $G$ contains at least one component and at most $N$ components. Thus, according to Lemma 1, one has $0 \leq \mathcal{R}(L) \leq N - 1$.

Lemma 2: Consider a connected undirected graph $G$. Then, 0 is a simple eigenvalue of its Laplacian matrix $L$ and all the other eigenvalues of $L$ are positive real numbers.

2.2 Formulation of the model

Consider a network of identical agents with linear or linearised dynamics, described by

$$\begin{align*}
\dot{x}_i(t) &= Ax_i(t) + Bu_i(t), \\
y_i(t) &= Cx_i(t), \\
& \quad i = 1, 2, \ldots, N,
\end{align*}$$

where $x_i(t) \in \mathbb{R}^n$ is the state, $u_i(t) \in \mathbb{R}^m$ is the control input, $y_i(t) \in \mathbb{R}^p$ is the measured output, and $A$, $B$, and $C$ are constant real matrices with compatible dimensions.

The communication topology among the $N$ agents is represented by an undirected graph $G$ consisting of the node set $\mathcal{V} = \{1, 2, \ldots, N\}$ and the edge set $\mathcal{E} \subseteq \{(i, j), i, j \in \mathcal{V}\}$. An existing edge $(i, j)$ ($i \neq j$) means that agents $i$ and $j$ can obtain information from each other.

In some real situations, agents may only sense the outputs of their neighbours over some disconnected time intervals due to the unreliability of communication channels, failure of physical devices, etc. Motivated by this observation, the following distributed observer-type of consensus protocol with discontinuous dynamic output measurements is proposed for each agent $i$:

$$\begin{align*}
\dot{v}_i(t) &= Av_i(t) + Bu_i(t) + c F \sum_{j=1}^{N} a_{ij} [C(v_j(t) - y_j(t))] \\
\dot{y}_i(t) &= K y_i(t), \\
\dot{v}_j(t) &= Av_j(t), \\
\dot{u}_i(t) &= 0, \\
& \quad t \in [k \omega, k \omega + \delta),
\end{align*}$$

where $v_i(t) \in \mathbb{R}^p$ is the state of the observer embedded in agent $i$, $c > 0$ is the coupling strength, $F \in \mathbb{R}^{p \times p}$ and $K \in \mathbb{R}^{p \times n}$ are feedback matrices, $A \in \mathbb{R}^{n \times n}$ is adjacency matrix of graph $G$, and $\omega > \delta > 0$.

Let $\zeta(t) = (x_i(t))^T, \dot{\zeta}(t) = (x_i(t))^T$. Then, it follows from (2) and (3) that

$$\dot{\zeta}(t) = A_i \zeta(t) + c \sum_{j=1}^{N} a_{ij} H \zeta(t), \quad t \in [k \omega, k \omega + \delta),$$

where $A_1 = \begin{bmatrix} A & BK \\ O & A + BK \end{bmatrix}$, $A_2 = \begin{bmatrix} A & O \\ O & A \end{bmatrix}$, and $H = \begin{bmatrix} O & FC \\ -FC & FC \end{bmatrix}$.

and $L = [l_{ij}]_{N \times N}$ is the Laplacian matrix of graph $G$.

Remark 2: The consensus problem for multi-agent systems with general linear node dynamics based on intermittent relative state information has been studied recently in Wen et al. (2012a), using the state information of agents which are hard or impossible to obtain in practice. In contrast, the present protocol (3) depends only on the relative output information of neighboring agents.

Definition 1: The consensus problem of multi-agent system (2) is solved by protocol (3) if, for any initial conditions, the states of system (4) satisfy

$$\lim_{t \to \infty} \| \zeta(t) - \zeta_j(t) \| = 0, \quad \forall i, j = 1, 2, \ldots, N.$$  

Remark 3: From Definition 1, consensus in multi-agent system (2) is solved by protocol (3) if and only if both the states of agents and the protocols embedded in agents asymptotically approach the same values, respectively.

Finally, the following lemmas (referring to Boyd, Ghaoui, Ferion, and Balakrishnan 1994) are introduced.

Lemma 3: The maximum real parts of eigenvalues of a matrix $A \in \mathbb{R}^{n \times n}$ is less than $-\alpha$, where $\alpha > 0$, if and only if there exists a matrix $P = P^T > 0$ such that

$$A^T P + P A + 2 \alpha P < 0.$$  

Lemma 4: For matrices $A$, $B$, $C$, and $D$ of appropriate dimensions, one has

1. $(ya) \otimes B = A \otimes (ya) = y(A \otimes B), \forall y \in \mathbb{R}$;
2. $(A + B) \otimes C = A \otimes C + B \otimes C$;
3. $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.
3. Main results
In this section, the main results of this article are described and proved.

3.1 Consensus in multi-agent systems under a connected topology
In this subsection, distributed consensus in multi-agent system (2) under an undirected connected topology is addressed.

Assumption 1: The undirected communication topology \( \mathcal{G} \) is connected.

Under Assumption 1, it follows from Lemma 2 that \( \phi = (1/N)1_N \) is the left eigenvector of the Laplacian matrix \( \mathcal{L} \) associated with the zero eigenvalue. Let \( s(t) = (s_1(t)^T, s_2(t)^T, \ldots, s_N(t)^T)^T \), where \( s_i(t) = \xi_i(t) - \frac{1}{N} \sum_{j=1}^{N} \xi_j(t) \), \( i = 1, 2, \ldots, N \). Then,

\[
\dot{s}(t) = \left[(I_N - \frac{1}{N}1_N^T) \otimes I_n\right]\xi(t),
\]

where \( \xi(t) = (\xi_1(t)^T, \xi_2(t)^T, \ldots, \xi_N(t)^T)^T \). It is easy to verify that \( s(t) = 0 \) if and only if \( \xi_1(t) = \xi_2(t) = \cdots = \xi_N(t) \), for all \( t \geq 0 \). It then follows from (3) and (6) that

\[
\dot{s}(t) = (I_N \otimes A_1 + c\mathcal{L} \otimes H)s(t), \quad t \in [k\omega, k\omega + \delta),
\]

\[
\dot{s}(t) = (I_N \otimes A_2)s(t), \quad t \in [k\omega + \delta, (k + 1)\omega), \quad (7)
\]

where

\[
A_1 = \begin{pmatrix} A & BK \\ O & A + BK \end{pmatrix}, \quad A_2 = \begin{pmatrix} A & O \\ O & A \end{pmatrix},
\]

\[
H = \begin{pmatrix} O & -FC \\ FC & O \end{pmatrix},
\]

and \( \mathcal{L} = [L_{ij}]_{N \times N} \) is the Laplacian matrix of graph \( \mathcal{G} \). Let \( Y_1 \in \mathbb{R}^{N \times (N-1)}, \quad Y_2 \in \mathbb{R}^{(N-1) \times N}, \quad T \in \mathbb{R}^{N \times N} \) and a diagonal matrix \( \Delta \in \mathbb{R}^{(N-1) \times (N-1)} \) be such that

\[
T = (1_N, Y_1), \quad T^{-1} = \begin{pmatrix} \phi_1^T \\ \phi_2^T \end{pmatrix},
\]

\[
T^{-1}LT = \Delta = \begin{pmatrix} 0 & 0 \quad 0 \\ 0 & 0 \quad \Delta \end{pmatrix}, \quad (8)
\]

where the diagonal entries of \( \Delta \) are the non-zero eigenvalues of Laplacian matrix \( \mathcal{L} \). By using the linear transformation

\[
\mu(t) = (T^{-1} \otimes I_{2N})s(t)
\]

with \( \mu(t) = (\mu_1(t)^T, \mu_2(t)^T, \ldots, \mu_N(t)^T)^T \), it follows from Lemma 4 and (7) that

\[
\dot{\mu}(t) = (I_N \otimes A_1 + c\Delta \otimes H)\mu(t), \quad t \in [k\omega, k\omega + \delta),
\]

\[
\dot{\mu}(t) = (I_N \otimes A_2)\mu(t), \quad t \in [k\omega + \delta, (k + 1)\omega), \quad k \in \mathbb{N},
\]

where \( A_1, A_2, H \) are defined in (7) and \( \Lambda \) is given in (8). Note that \( \mu_i(t) = 0 \), for all \( t \geq 0 \). Thus, the consensus problem for multi-agent system (2) is solved by protocol (3) if and only if \( \mu_i(t), \quad i = 2, 3, \ldots, N \), converge asymptotically to zero, which is in turn equivalent to that the following \( N - 1 \) systems

\[
\dot{z}_i(t) = (I_N \otimes A_1 + c\lambda_i \otimes H)z_i(t), \quad t \in [k\omega, k\omega + \delta),
\]

\[
\dot{z}_i(t) = (I_N \otimes A_2)z_i(t), \quad t \in [k\omega + \delta, (k + 1)\omega), \quad k \in \mathbb{N},
\]

are simultaneously asymptotically stable, where \( \lambda_i, \quad i = 2, 3, \ldots, N \), are the non-zero eigenvalues of the Laplacian matrix \( \mathcal{L} \). Taking linear transformation

\[
\eta_i(t) = \begin{pmatrix} I & -I \\ 0 & I \end{pmatrix}z_i(t), \quad i = 2, 3, \ldots, N,
\]

and using (11), one obtains

\[
\dot{\eta}_i(t) = T_i\eta_i(t), \quad t \in [k\omega, k\omega + \delta),
\]

\[
\dot{\eta}_i(t) = A_2\eta_i(t), \quad t \in [k\omega + \delta, (k + 1)\omega), \quad k \in \mathbb{N},
\]

where

\[
T_i = \begin{pmatrix} A + c\lambda_iFC & O \\ -c\lambda_iFC & A + BK \end{pmatrix}, \quad A_2 = \begin{pmatrix} A & O \\ 0 & A \end{pmatrix},
\]

and \( \lambda_i, \quad i = 2, 3, \ldots, N \), are the non-zero eigenvalues of the Laplacian matrix \( \mathcal{L} \).

From the above analysis, one has converted the consensus problem for multi-agent system (2) with protocol (3) to the simultaneously asymptotically stability problem of \( N - 1 \) switching systems given by (13), which have the same low dimension as a single agent. One can see that the effects of the communication topology on the consensus are characterised by the non-zero eigenvalues of the Laplacian matrix \( \mathcal{L} \).

Next, an algorithm is presented to construct protocol (3).

Algorithm 1: Under Assumption 1 and the condition that \( (A, B, C) \) is both stabilisable and detectable, the consensus protocol (3) can be designed as follows.

(1) Choose \( \beta_0 > 0 \), and select feedback gain matrix \( K \) by Ackermann's formula, such that the real parts of the poles of \( A + BK \) are smaller than \( -\beta_0 \).

(2) Solve the following linear matrix inequality (LMI):

\[
A^TQ + QA - 2C^TC + 2\beta_0Q < 0,
\]

to obtain one solution \( Q > 0 \). Then, let the feedback gain matrix be \( F = -Q^{-1}C \).
Lemma 6: There exist some $W_i$ such that

\[
T_i = \begin{bmatrix}
    A + \gamma_i FC & O \\
    -\gamma_i FC & A + BK
\end{bmatrix},
\]

where $F = -Q^{-1}CT$, $Q$ is a solution of LMI (14), and $\lambda_i$, $i = 2, 3, \ldots, N$, are the non-zero eigenvalues of $L$.

Proof: Similarly to the proof of Proposition 1 in Li et al. (2011), this lemma can be proved. For the readers' convenience, a sketched proof is given as follows. Let $Q > 0$ be a solution of LMI (14). Take $F = -Q^{-1}CT$. Then, one gets

\[
(A + \gamma_i FC)Q^{-1} + Q^{-1}(A + \gamma_i FC)^T + 2\beta_0 Q^{-1}
\]

\[
= AQ^{-1} + Q^{-1}AT - 2\gamma_i Q^{-1}CTCQ^{-1} + 2\beta_0 Q^{-1}
\]

\[
\leq AQ^{-1} + Q^{-1}AT - 2Q^{-1}CTCQ^{-1} + 2\beta_0 Q^{-1},
\]

\[
(15)
\]

where the last inequality of (15) is derived by using the fact $c \geq 1/\lambda_2$, with $\lambda_2$ being the second smallest eigenvalue of the Laplacian matrix $L$.

Pre- and post-multiplying (15) by $Q$ and its transpose, according to (14), yields

\[
(A + \gamma_i FC)Q^{-1} + Q^{-1}(A + \gamma_i FC)^T + 2\beta_0 Q^{-1} < 0.
\]

\[
(16)
\]

It thus follows from Lemma 3 and (16) that the maximum real parts of the eigenvalues of matrices $T_i$ are less than $-\beta_0$, where

\[
T_i = \begin{bmatrix}
    A + \gamma_i FC & O \\
    -\gamma_i FC & A + BK
\end{bmatrix},
\]

Lemma 6: There exist some $W_i > 0$ such that

\[
T_i^TW_i + W_iT_i + 2\beta W_i < 0,
\]

where $\beta < \beta_0$.

Proof: It follows directly from Lemmas 3 and 5.

Lemma 7: There exists a $\gamma_0 > 0$, such that for all $\gamma > \gamma_0$, the following LMI

\[
A_4^TP + PA_4 - 2\gamma P < 0,
\]

\[
(18)
\]

where $A_4 = \begin{bmatrix} O & A \\ A & O \end{bmatrix}$, has a solution $P > 0$.

Proof: With $\gamma_0 = \max_{j=1,2,\ldots,n} \{\sum_{g=1}^{n} |a_{gj}|\}$, the lemma can be proved by using the Geršgorin disc theorem (Huang 1984).

The following theorem is one main result of this article.

Theorem 1: For the multi-agent system (2), under Assumption 1 and moreover (A, B, C) is both stabilisable and detectable, the protocol (3) constructed by Algorithm 1 solves the consensus problem if the communication time rate $\delta/\omega > \gamma/(\beta + \gamma) + \ln \rho[(\beta + \gamma)\omega]$, where $0 < \beta < \beta_0$, $\rho = \max_{i=2,3,\ldots,N} \{\lambda_{\max}(W_i)/\lambda_{\min}(P), \lambda_{\max}(P)/\lambda_{\min}(W_i)\}$, $\gamma > \gamma_0$, and matrices $W_i$ and $P$ are the positive-definite solutions of LMIs (17) and (18), respectively.

Proof: Construct the following multiple Lyapunov function candidate for the $i$th switching system of (13):}

\[
V(t) = \begin{cases}
    \eta_i(t)^TW_i\eta_i(t), & t \in [k\omega, k\omega + \delta), \\
    \eta_i(t)^TP\eta_i(t), & t \in [k\omega + \delta, (k + 1)\omega),
\end{cases}
\]

\[
V(t) = \begin{cases}
    \eta_i(t)^TW_i\eta_i(t), & t \in [k\omega, k\omega + \delta), \\
    \eta_i(t)^TP\eta_i(t), & t \in [k\omega + \delta, (k + 1)\omega),
\end{cases}
\]

\[
(19)
\]

where the positive-definite matrices $W_i$ and $P$ are the solutions of (17) and (18), respectively, $k \in \mathbb{N}, i = 2, 3, \ldots, N$.

For $t \in [k\omega, k\omega + \delta)$ and an arbitrarily given $k \in \mathbb{N}$, taking the time derivative of $V(t)$ along the trajectories of system (13) gives

\[
\dot{V}(t) < -2\beta V(t), \quad t \in [k\omega, k\omega + \delta),
\]

\[
(20)
\]

where $\beta$ is defined in Lemma 6.

Similarly, one has

\[
\dot{V}(t) < 2\gamma V(t), \quad t \in [k\omega + \delta, (k + 1)\omega).
\]

\[
(21)
\]

Note that the switching systems (13) switches at $t = k\omega$ and $t = k\omega + \delta$, $k \in \mathbb{N}$. Based on the above analysis, one obtains

\[
V(\omega) < \rho e^{2\gamma(\omega - \delta)}V(\delta)
\]

\[
< \rho^2 e^{-2\beta\delta + 2\gamma(\omega - \delta)}V(0)
\]

\[
= e^{-\tau}V(0),
\]

where $\tau = 2\beta\delta - 2\gamma(\omega - \delta) - 2\ln \rho$. From the condition $\delta/\omega > \gamma/(\beta + \gamma) + \ln \rho[(\beta + \gamma)\omega]$, one has $\tau > 0$. By recursion, for any positive integer $k$, one gets

\[
V(k\omega) < V(0)e^{-k\tau}.
\]

\[
(22)
\]

For any $t > 0$, there exists an $r \in \mathbb{N}$ such that $r\omega \leq t < (r + 1)\omega$. Then, one has

\[
\dot{V}(t) < V(r\omega)e^{2\gamma t}
\]

\[
< V(0)e^{-rr\omega + 2\gamma t}
\]

\[
< e^{2\gamma t}V(0)e^{-\frac{\tau}{t}(\omega + 1)^{t}},
\]
i.e.
\[ V(t) < \Omega_0 e^{-\theta t}, \quad \text{for all } t \geq 0, \]
where \( \Omega_0 = e^{\gamma_0 V(0)} \) and \( \theta = \tau/(\omega + 1) \). This indicates that the states of agents exponentially converge to the same.

**Remark 4:** In Theorem 1, it is assumed that the agents can obtain the intermittent relative outputs of their neighbours periodically. However, one may extend the results to the consensus in linear multi-agent systems with aperiodically intermittent output communications by using the present approach. Similarly to the proof of Theorem 1, some corresponding theoretical results can be derived, which are omitted here for brevity.

**Remark 5:** To successfully construct protocol (3) according to Algorithm 1, it should be firstly shown that both steps (1) and (2) of Algorithm 1 are feasible for some \( \beta_0 > 0 \). Note that there always exist some positive scalar \( \beta_0 \) such that steps (1) and (2) are feasible if \((A, B, C)\) is stabilisable and detectable. It is also worth noting that both steps (1) and (2) are feasible for any given \( \beta_0 > 0 \) if the matrix triple \((A, B, C)\) is controllable and observable.

**Remark 6:** In the context of multi-agent systems with intermittent observations, one interesting and important issue is what the minimum admissible communication rate is for achieving consensus for a given topology \( \mathcal{G} \) with fixed parameters \( \beta \) and \( \gamma \). However, LMIs (17) and (18) are solved independently, which may introduce conservativeness in determining the admissible communication rate to satisfy the consensus condition. From the proof of Theorem 1, one can see that the minimum admissible communication rate, under a given communication topology with fixed parameters \( \beta \) and \( \gamma \), can be obtained by minimising \( \rho \) in Theorem 1. Actually, the minimum \( \rho \) can be obtained by solving the following optimisation problem:

1. Minimise \( \rho_i \) subject to: \( W_i > 0, P > W_i, P < \rho_i W_i, T_i^T W_i + W_i T_i + 2\beta W_i < 0 \), \( A_3^T P + P A_4 - 2\gamma P < 0 \), where \( T_i = \begin{bmatrix} A + c_3 FC & O \\ -c_3 FC & A + BK \end{bmatrix}, \quad A_4 = \begin{bmatrix} A & O \\ O & A \end{bmatrix}, \)

and \( \lambda_i, \quad i = 2, 3, \ldots, N, \) are the non-zero eigenvalues of Laplacian matrix \( \mathcal{L} \).

2. Take \( \rho_{\min} = \max_{i=2, 3, \ldots, N} \{\rho_i\} \).

Then, \( \rho_{\min} \) is the minimum value of \( \rho \).

**Remark 7:** In Xia and Cao (2009), Cai, Liu, Xu, and Sun (2009) and Wang, Hao, and Zuo (2010), some interesting state-based intermittent feedback methods are proposed and used to analyse the synchronisation behaviours of coupled complex dynamical networks, which is a closely related topic with consensus for multi-agent systems. However, from the perspective of control theory, the states of a dynamical system are internal information which is difficult or impossible to simultaneously obtain. In contrast, protocol (3) is designed based only on the relative outputs of neighboring agents, which is more practical.

### 3.2 Extensions

In the last subsection, consensus in multi-agent systems with a connected communication topology is studied. Notice that the communication topology may be disconnected in real multi-agent systems due to external disturbances and/or sensing range limitations. It is thus interesting and important to further investigate consensus in multi-agent systems with an arbitrarily given communication topology which may not be connected. Motivated by the works reported in Li, Wang, and Chen (2004), Xiang, Liu, Chen, Chen, and Yuan (2007) and Yu, Chen, and Lü (2009a) and based on the analysis given in the last subsection, a pinning-based distributed control method is utilised here to guarantee consensus in the multi-agent system (2) under an arbitrarily given communication topology with discontinuous observations.

Suppose that the communication topology of the multi-agent system (2) is given by \( \mathcal{G} \), with node set \( \{1, 2, \ldots, N\} \). Lemma 1 implies that changing the order of the node indexes of \( \mathcal{G} \) will yield a new graph \( \mathcal{G} \) with the Laplacian matrix \( \mathcal{L} \) in the form of (1). To realise this transformation, an algorithm is given below.

**Algorithm 2:**

1. Set \( \text{NS} = \{1, 2, \ldots, N\} \) and \( m = 0 \).
2. Arbitrarily select a node \( i \) from \( \text{NS} \), and use the depth-first search algorithm Tarjan (1972) to find the component \( \overline{\text{CC}}(i) \) of graph \( \mathcal{G} \) containing node \( i \). Let \( m = m + 1 \) and \( \text{NS} = \text{NS} \cap \text{NSCC}(i) \), where \( \text{NSCC}(i) \) denotes the node set of \( \overline{\text{CC}}(i) \), \( \text{NSCC}(i) \cup \text{NSCC}(i) = \{1, 2, \ldots, N\} \) and \( \text{NSCC}(i) \cap \text{NSCC}(i) = \emptyset \).
3. Check the condition \( \text{NS} = \emptyset \); if not, re-perform step (2); if so, go to step (4).
4. Arrange the \( m \) components of \( \mathcal{G} \) in a size-descending order. Then, relabel the nodes from the first component to the last one so as to obtain the graph \( \mathcal{G} \).
Note that $\mathcal{G}$ and $\mathcal{G}$ are isomorphic to each other. And, under a given protocol (3), consensus in the closed-loop multi-agent system (2) with topology $\mathcal{G}$ can be achieved if and only if consensus in the closed-loop multi-agent system (2) with topology $\mathcal{G}$ can be achieved. It is easy to check that the Laplacian matrix $L$ of $\mathcal{G}$ is in the form of (1).

In the following, consensus in multi-agent system (2) with topology $\mathcal{G}$ is discussed. Obviously, consensus cannot be achieved if $\mathcal{G}$ is disconnected. To guarantee consensus in system (2) with an arbitrarily given topology, a virtual leader labeled $N+1$ is introduced whose dynamics are given as follows:

\[ \begin{align*}
    \dot{x}_{N+1}(t) &= Ax_{N+1}(t) + Bu_{N+1}(t), \\
    y_{N+1}(t) &=Cx_{N+1}(t),
\end{align*} \]

where $x_{N+1}(t) \in \mathbb{R}^n$ is the state, $u_{N+1}(t) \in \mathbb{R}^m$ is the control input, and $y_{N+1}(t) \in \mathbb{R}^p$ is the measured output. Here, the virtual leader plays the role of a command generator providing a reference state for the followers to track. Thus, it is assumed that $u_{N+1}(t) \equiv 0$ and $y_{N+1}(t) \equiv 0$, i.e., the states of the virtual leader evolves without being affected by the followers, and the virtual leader is no need to observe the stats or outputs of any followers. It is furthermore assumed that only a subset of agents, called pinned agents, have access to the outputs of the virtual leader, but intermittently. For notational convenience, let $\mathcal{P}$ be the set of pinned agents and $\mathcal{G}$ be the augmented graph with adjacency matrix $\tilde{A} = [a_{ij}]_{N \times (N+1)}$, where $a_{ij(N+1)} = 0$, $a_{i(N+1)} \geq 0$, and $a_{(N+1)} > 0$ if and only if $i \not\in \mathcal{P}$. The objective here is to find an appropriate control protocol for system (2) to achieve consensus in the sense of $\|x_i(t) - x_{N+1}(t)\| = 0, \forall i = 1, 2, \ldots, N$, where $\xi_{N+1}(t) = (x_{N+1}(t)^T, y_{N+1}(t)^T)^T$. To do so, a pinning-based distributed protocol based on (3) is presented for each follower $i$ as follows:

\[ \begin{align*}
    \dot{v}_i(t) &= A v_i(t) + B u_i(t) + cF \sum_{j=1}^{N+1} a_{ij} [C(v_i(t) - v_j(t)) - (y_i(t) - y_j(t))], \\
    u_i(t) &= K y_i(t), \quad t \in [k\omega, k\omega + \delta), \\
    \dot{v}_i(t) &= A v_i(t), \\
    u_i(t) &= 0, \quad t \in [k\omega + \delta, (k+1)\omega), \quad k \in \mathbb{N},
\end{align*} \]

where $v_i(t) \in \mathbb{R}^p$ is the state of the observer embedded in agent $i$, $c > 0$ is the coupling strength, and $F \in \mathbb{R}^{d \times p}$ and $K \in \mathbb{R}^{m \times n}$ are the feedback matrices.

Before moving forward, the following lemma is introduced. Let $\mathcal{V}(i)$ be the node set of the $i$th $(1 \leq i \leq m)$ connected component of $\mathcal{G}$.

**Lemma 8:** Suppose that for each $i \in \{1, 2, \ldots, m\}$, there exists at least one node $j_i$ such that $j_i \in \mathcal{V}(i) \cap \mathcal{P}$. Then, matrix $\tilde{L} > 0$, where $\tilde{L} = L + \Theta$, $L$ is the Laplacian matrix of $\mathcal{G}$, and $\Theta = \text{diag}\{a_{i(N+1)}, a_{2(N+1)}, \ldots, a_{N(N+1)}\} \in \mathbb{R}^{N \times N}$.

**Proof:** Since, for each $i \in \{1, 2, \ldots, m\}$, there exists at least one node $j_i$ such that $j_i \in \mathcal{V}(i) \cap \mathcal{P}$, it thus follows from Corollaries 1 and 2 in Ren, Beard, and McLain (2005) that $\tilde{L}$ has a simple zero eigenvalue and all the other eigenvalues have positive real parts, where $\tilde{L}$ is the Laplacian matrix of the augmented graph $\mathcal{G}$. Taking $M = (a_{i(N+1)}, a_{2(N+1)}, \ldots, a_{N(N+1)})^T \in \mathbb{R}^N$, one has

\[ \tilde{L} = \begin{bmatrix} \tilde{L} & M \\ 0^T & 0 \end{bmatrix}. \]

where $\tilde{L} = L + \Theta$. Thus, all the eigenvalues of $\tilde{L}$ have positive real parts. Since $\tilde{L}$ is symmetric, $\tilde{L}$ is positive-definite.

Based on the above analysis, an algorithm is given here to construct protocol (24).

**Algorithm 3:** Suppose that the communication topology $\mathcal{G}$ has $m$ component and that $(A, B, C)$ is both stabilisable and detectable. Then, the consensus protocol (24) can be designed as follows.

1. Using Algorithm 2 to relabel the node indexes of $\mathcal{G}$ so as to get a graph $\mathcal{G}$ whose Laplacian matrix $L$ has the form of (1). Then, pin $m$ different nodes $j_1, j_2, \ldots, j_m$, where $j_i \in \mathcal{V}(i)$, and $\mathcal{V}(i)$ is the node set of the $i$th component of $\mathcal{G}$, $i = 1, 2, \ldots, m$.

2. Choose $\varsigma_0 > 0$, and select a feedback gain matrix $K$ by Ackermann’s formula, such that the real parts of the poles of $A + BK$ are smaller than $-\varsigma_0$.

3. Solve the LMI

\[ A^TQ + QA - 2C^TC + 2\varsigma_0Q < 0 \]

(26) to obtain solution $Q > 0$. Then, take the feedback gain matrix $F = -Q^{-1}C^T$.

4. Take the coupling strength $c \geq 1/\lambda_0$, where $\lambda_0$ is the smallest eigenvalue of matrix $\tilde{L}$.

To derive another main result, the following lemmas are introduced for which the proofs are simple therefore omitted.

**Lemma 9:** The maximum real parts of the eigenvalues of matrices $T_i$, are less than $-\varsigma_0$, where

\[ T_i = \begin{bmatrix} A + c\lambda_i FC & O \\ -c\lambda_i FC & A + BK \end{bmatrix}. \]

$F = -Q^{-1}C^T$, $Q$ is a solution of LMI (26), and $\lambda_i$, $i = 1, 2, \ldots, N$, are the eigenvalues of $\tilde{L}$. 

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Lemma 10: There exist some $W_i > 0$ such that
\[ T_i^T W_i + W_i T_i + 2 \xi W_i < 0, \quad (27) \]
where $\xi < \xi_0$.

\[ T_i = \begin{bmatrix} A + c \lambda_i FC & O \\ -c \lambda_i FC & A + BK \end{bmatrix}, \]
and $\lambda_i, i = 1, 2, \ldots, N$, are the eigenvalues of $\mathcal{L}$.

Now, it is to present another main result of this article.

Theorem 2: For the multi-agent system (2) with a communication topology $\mathcal{G}'$, the protocol (24) constructed by Algorithm 3 solves the consensus problem if the communication rate $\delta/\omega > \gamma/(\xi + \gamma) + \ln \|Q\|/[(\xi + \gamma)\omega]$, where $0 < \xi < \xi_0, \quad 0 < \gamma < \gamma_0, \quad \mathcal{G} = \max_{i=1,2,\ldots,N} \{\lambda_{\max}(W_i)/\lambda_{\min}(P), \lambda_{\max}(P)/\lambda_{\min}(W_i)\}, \quad \gamma > \gamma_0, \quad \gamma_0$ is given in Lemma 7, and matrices $P$ and $W_i$ are positive-definite solutions of LMI (19) and (27), respectively.

Proof: Take $\hat{\xi}_i(t) = x_i(t) - x_{\mathcal{N}+i}(t)$ and $\hat{v}_i(t) = v_i(t) - v_{\mathcal{N}+i}(t)$, where $i = 1, 2, \ldots, N$. Obviously, consensus in (2) can be achieved if and only if $\|\hat{\xi}_i(t)\| \to 0$ and $\|\hat{v}_i(t)\| \to 0$, for all $i = 1, 2, \ldots, N$. Based on the above analysis and according to (24), one gets
\[ \hat{\xi}_i(t) = A \hat{\xi}_i(t) + B u_i(t) + c \sum_{j=1}^{N} \hat{\xi}_j C (\hat{v}_i(t) - \hat{\xi}_i(t)), \]
\[ u_i(t) = K \hat{\xi}_i(t), \quad t \in [k \omega, k \omega + \delta), \]
\[ \hat{v}_i(t) = A \hat{v}_i(t), \]
\[ u_i(t) = 0, \quad t \in [k \omega + \delta, (k + 1) \omega), \quad k \in \mathbb{N}, \]
where $\mathcal{L} = \begin{bmatrix} \hat{\xi}_i \end{bmatrix}_{N \times N}$.

Let $\hat{\zeta}_i(t) = (\hat{\xi}_i(t)^T, \hat{v}_i(t)^T)^T$, $i = 1, 2, \ldots, N$. Then, it follows from (2) and (3) that
\[ \hat{\zeta}_i(t) = A \hat{\zeta}_i(t) + c \sum_{j=1}^{N} \hat{\xi}_j H \hat{\zeta}_i(t), \quad t \in [k \omega, k \omega + \delta), \]
\[ \hat{\zeta}_i(t) = A \hat{\zeta}_i(t), \quad t \in [k \omega + \delta, (k + 1) \omega), \quad k \in \mathbb{N}, \]
where
\[ A_1 = \begin{bmatrix} A & BK \\ O & A + BK \end{bmatrix}, \quad A_2 = \begin{bmatrix} A & O \\ O & A \end{bmatrix}, \]
\[ H = \begin{bmatrix} O & -FC \\ FC & A + BK \end{bmatrix}. \]

Similarly to the proof of Theorem 1, the rest of the proof can be completed.

Remark 8: The minimum admissible communication rate for achieving consensus under a given communication topology with fixed parameters $\xi$ and $\gamma$ can be obtained by minimising the parameter $\varrho$ in Theorem 2. And, the minimum $\varrho$ can be obtained by solving the following optimisation problem:

(1) Minimise $\varrho_i$
subject to: $W_i > 0, \quad P > \varrho_i W_i, \quad T_i^T W_i + W_i T_i + 2 \xi W_i < 0, \quad A_2^T P + PA_2 - 2 \gamma P < 0,$
where $T_i = \begin{bmatrix} A + c \lambda_i FC & O \\ -c \lambda_i FC & A + BK \end{bmatrix}, \quad A_2 = \begin{bmatrix} A & O \\ O & A \end{bmatrix},$
and $\lambda_i, i = 1, 2, \ldots, N$, are the eigenvalues of $\mathcal{L}$.

(2) Take $\varrho_{\min} = \max_{i=1,2,\ldots,N} \{\varrho_i\}$.
Then, $\varrho_{\min}$ is the minimum value of $\varrho$.

4. Simulation examples
In this section, two simulation examples are provided to verify the theoretical analysis.

Example 1: Take each agent in a multi-agent system to be a two-mass-spring system with a single force input (Zhang, Lewis, and Qu 2012), whose dynamics are described by (2) with
\[ x_i(t) = \begin{bmatrix} x_{i1}(t) \\ x_{i2}(t) \\ x_{i3}(t) \\ x_{i4}(t) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_1-k_2 & 0 & k_2 & 0 \\ 0 & 0 & 0 & 1 \\ k_2 & 0 & -k_2 & m_2 \end{bmatrix}, \]
\[ B = \begin{bmatrix} 0 \\ \frac{1}{m_1} \\ 0 \\ 0 \end{bmatrix}, \quad \text{ where } m_1 \text{ and } m_2 \text{ are two masses, and } k_1 \text{ and } k_2 \text{ are spring constants. Furthermore, take the output matrix }\]
\[ C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \]
\[ m_1 = 1.2 \text{ kg}, \quad m_2 = 1.0 \text{ kg}, \quad k_1 = 1.4 \text{ N/m}, \quad \text{ and } k_2 = 1.0 \text{ N/m}. \]

Some simple calculations show that $(A, B, C)$ is controllable and observable.

Consider a group of four agents with undirected communication topology $\mathcal{G}_1$ as shown in Figure 1, where the weights are indicated on the edges. Obviously, $\mathcal{G}_1$ is connected. An observer-type of consensus protocol in the form of (3) is designed according to Algorithm 1. Take $\omega = 4$ and $\rho_0 = 2$. 

Then, some calculations give the feedback gain matrices in (3) as

\[ K = \begin{pmatrix} -3.2838 & 0.6105 & -0.0594 \\ 0.6105 & -3.9534 & -0.6675 \\ -0.0594 & -0.6675 & -8.1926 \\ -0.1739 & -0.0167 & -20.7378 \end{pmatrix}, \]

(31)

According to Lemmas 6 and 7, one may take \( \beta = 1.15 \) and \( \gamma = 1.5 \). Letting \( c = 2.75 \) and solving the optimisation problem in Remark 6 gives \( \rho_{\text{min}} = 11 \). Thus, the minimum communication rate is 79.22\%. Take \( \delta = 3.20 \), which means that the communication rate is 80\%, see Figure 2 for illustration. According to Theorem 1, the states of all agents in system (2) will converge to the same value. The state trajectories of the agents are shown in Figures 3–6, respectively. Use

\[ E_1(t) = \sqrt{\sum_{i=1}^{3} \| x_i(t) - x_4(t) \|^2} \]

to denote the consensus errors of system (2) under protocol (3). Figure 7 demonstrates that consensus is indeed achieved.

**Example 2:** Consider a group of seven agents with a communication topology \( G_2 \) as shown in Figure 8, where the agents 1, 2, 3 belong to the first component and agents 4, 5, 6, 7 belong to the second component. The agent dynamics and the parameters are the same as those in Example 1. To achieve consensus, a virtual leader labeled by 8 is introduced. Then, an observer-type of consensus protocol in the form of (24) is designed according to Algorithm 3. According to
step (1) of Algorithm 3, agents 1 and 7 are chosen as the pinned agents, i.e. $P = \{1, 7\}$. Simple calculations yield that \( C_{21}^0 = 0.5493 \), where \( C_{21}^0 \) is the smallest eigenvalue of \( bL \) given in (25). Take \( \delta = 1.15 \), \( \gamma = 1.5 \) and \( \omega = 4 \). Selecting \( c = 2.75 \) and solving the optimization problem in Remark 8 gives \( \varrho_{\text{min}} = 10.70 \). According to Theorem 2, the minimum communication rate is 78.96%. Take \( \delta = 3.20 \), which means that the communication rate is 80%, see Figure 2 for illustration. The feedback matrices \( F \) and \( K \) are taken to be the same as those in Example 1. Then, according to Theorem 2, the states of the agents in system (2) will approach those of the virtual leader. The state trajectories of agents are shown in Figures 9–12, respectively. Use

\[
E_2(t) = \sqrt{\sum_{i=1}^{7} \|x_i(t) - x_8(t)\|^2}
\]

to denote the consensus errors of system (2) under protocol (24). Figure 13 demonstrates that consensus is indeed achieved.

5. Conclusions

In this article, the consensus problem for multi-agent systems with general linear node dynamics and

Figure 6. Consensus of trajectories of \( x_{i4}(t) \), \( i = 1, 2, 3, 4 \), in Example 1.

Figure 7. Trajectories of consensus errors \( E_1(t) \), in Example 1.

Figure 8. Communication topology \( G_2 \), in Example 2.

Figure 9. Consensus of trajectories of \( x_i(t) \), \( i = 1, 2, \ldots, 8 \), in Example 2.

Figure 10. Consensus of trajectories of \( x_{i2}(t) \), \( i = 1, 2, \ldots, 8 \), in Example 2.
discontinuous output measurements has been studied. To achieve consensus, a class of observer-type of protocols have been proposed. By using tools from switching systems theory and matrix analysis, it has been proved that consensus in the closed-loop multi-agent systems can be guaranteed under a fixed connected undirected topology if the communication rate is larger than a threshold value. Furthermore, consensus for multi-agent systems with a disconnected communication topology has been studied from a pinning-based distributed control approach. The effectiveness of the theoretical analysis has been verified by numerical simulations. Future work will focus on consensus for multi-agent systems with higher-order nonlinear dynamics and discontinuous output measurements.

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