Absorbing boundary conditions: a spectral collocation approach

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SUMMARY

We introduce a spectral collocation method for the discretisation of the shallow water equations on a one-dimensional semi-infinite domain, employing scaled Laguerre basis functions to obtain an accurate description of the solutions on finite regions of arbitrary size. The time discretisation is based on a semi-implicit, semi-Lagrangian approach that handles the highly inhomogeneous node distribution without loss of efficiency. The method is first validated on standard test cases and then applied to the implementation of absorbing open boundary conditions by coupling the semi-infinite domain to a finite size domain on which the same equations are discretised by standard finite volume methods. Numerical experiments show that the proposed approach does not produce significant spurious reflections at the interface between the finite and infinite domain, thus providing a reliable tool for absorbing boundary conditions. Copyright © 2013 John Wiley & Sons, Ltd.

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1. INTRODUCTION

In this paper, we introduce a pseudo-spectral method based on scaled Laguerre functions expansions for the approximation of wave propagation problems on semi-infinite domains. Our development is motivated by the fact that, in many environmental applications, it is of interest to consider computational domains that span very different length scales. In climate modelling, for example, increasing attention has been devoted to stratospheric phenomena, for example, [1,2], which requires to extend the typical computational domain of standard climate models by a substantial amount in the vertical direction. In an early attempt to use Laguerre basis functions for the vertical discretisation of the equations of atmospheric flow, Francis [3] reported that they ‘entail a computational penalty that more than compensates for the advantages gained by using analytic vertical representation’. It is therefore of interest to assess whether, with the theoretical and computational resources currently at our disposal, spectral or pseudo-spectral approaches can be used to devise accurate and efficient discretisations for domains with arbitrarily large length scales.

If an unbounded domain is to be modelled, for computational reasons it is necessary to restrict the simulation to a bounded domain of interest and to apply some kind of open boundary condition to let waves propagate out of the computational domain. However, as this artificial boundary has no physical counterpart, it must be transparent to the passage of waves propagating in either direction, and no spurious phenomena such as reflection of outgoing waves into the finite computational domain should occur. Referring for example to atmospheric modelling, outgoing waves

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reflected by the top boundary can spread to the region of interest and corrupt the numerical solution, for example, [4]. Therefore, an accurate analytical and numerical design of the boundary conditions on the artificial boundary has to be developed. Critical issues in this context involve stability, accuracy, and ease of implementation of the numerical method approximating the problem in the unbounded domain. Extensive reviews of the classical approaches to these issues can be found in [5] and [6], for a more recent overview see [7]. Early suggestions for boundary conditions at the artificial ‘ceiling’ in atmospheric models include setting the vertical velocity to zero at some finite height or, in a pressure coordinate, at $p = 0$ (infinite height). However, both solutions entail total energy reflection or lead to ill-posed formulations.

Presently, the two most common attempts to solve this problem are radiation boundary conditions and absorbing/sponge layers. The basic idea of the radiation or characteristic boundary conditions [8, 9] is that disturbances propagate across the artificial boundary as waves that can be described by a simplification of the full model dynamics [10]. Developments of this approach are also present in recent work on Higdon-type conditions in the framework of linearised Euler and shallow water models [11–13].

On the other hand, relaxation methods extend the computational domain to let disturbances leave and penetrate into a limited region (the absorbing layer) where they can be damped by viscous or reactive terms towards the external solution [14–16]. If the external solution is zero, this approach amounts to the introduction of a damping term active only in the absorbing layer [17]. In this context, the choice of the relaxation coefficient is a trade-off between the aim of damping efficiently outgoing perturbations on the one hand, and limited spatial resolution, which does not allow for high spatial variations of wave amplitude, on the other. Various functional forms for the parameters have been tested but, as pointed out in [18], the choice of these coefficients and the thickness of the absorbing layer are mostly dependent on empirical or ad hoc criteria. Moreover, a major drawback of this approach is its relatively high computational cost, as considerable effort must be put into the solution of modified equations outside the area of interest [7]. To the best of our knowledge, at the present time, most of the operational atmospheric models need to set aside at least a fifth of the total vertical DOFs only for damping purposes. Advanced versions of sponge layer formulations have also been developed e.g. [19], leading to perfectly matched layers, mainly to simulate electromagnetic phenomena [20]. Most relevant to our work are the contributions in [21–23], which adapt the extension and strength of the damping, tuning it according to the form of the linear shape functions.

Approaches similar to the one presented here have also been proposed in [24], which explored mapping techniques to extend spectral methods to unbounded domains, [25], in which the use of unscaled Laguerre functions is analysed (see also the related work [26] for an application to eigenvalue problems) and [27]. Yet another approach to the problem is given by infinite elements, widely employed in numerical models of wave propagation phenomena from scattering or radiating objects, for example, [28, 29]. Similar to finite element, infinite element formulations are local, but their shape functions are chosen to mimic the asymptotic behaviour of the solution at infinity. A clear advantage of such discretisations is the sparseness of the coefficient matrices, whereas stability and ill-conditioning are seen as critical factors in determining their utility. In particular, the accuracy of an infinite element relies on the choice of the shape functions towards infinity and the order of approximation. On the other hand, infinite elements can be used to discretise the whole exterior domain without truncating it and, from the point of view of open boundaries, they can be seen as a local absorbing boundary condition. In this context, usually, there is also an inner conventional finite element mesh close to the object, the choice of the location of the interface between the two meshes being a critical factor in the accuracy of a coupled finite/infinite elements discretisation.

In this work we use scaled Laguerre functions for the implementation of absorbing layer boundary conditions, by coupling discretisations on domains of finite size to scaled Laguerre spectral methods. As a first step, we consider as model equations the one-dimensional shallow water equations, that provide a standard model for wave propagation problems. Regarding the numerical discretisation, a semi-implicit, semi-Lagrangian approach [30–32] was chosen because of its well-known efficiency and accuracy features and because a large number of environmental models use these time discretisation techniques (e.g., [33], which describes operational forecast model that makes use
of semi-Lagrangian and spectral techniques). If a relatively small number of nodes is employed, an efficient approach to absorbing boundary conditions can be achieved, which allows one to account for large absorbing regions at a relatively low computational cost. The coupling approach is based on imposing continuity of the mass fluxes at the interface between the bounded and the unbounded domain. The same idea can be, in principle, extended to couple the Laguerre spectral method to an arbitrary discretisation on the bounded domain. In this first attempt, an explicit finite volume discretisation has been considered for the finite size domain. The proposed coupling approach generates reflections with negligible amplitude and makes possible the use of the method, at least for environmental applications, with no substantial loss of accuracy with respect to single domain discretisations.

What we propose is actually an effective way to implement numerical absorbing layers, with the aim of building a reliable and efficient tool for the practical treatment of open boundary conditions in atmospheric applications, exploiting the properties of scaled Laguerre functions, recently introduced in [34]. Thus, on the one hand, we are not interested in writing analytical expressions of correct transmissive boundary conditions as done in, for example, [12]. On the other hand, because in environmental settings waves of different speeds and amplitudes are found, our method does not attempt to identify a selective damping procedure as in [23]. Also, differently from infinite element formulations, whose shape functions are obtained by the ones of finite elements either by mapping or by multiplication by decay functions, in our approach, the equations are directly approximated in the semi-infinite domain out of the artificial boundary. Finally, it should be observed that unlike most of the existing methods, our method does not require linearisation of the equations and directly approximates the nonlinear model equations.

The rest of the paper is organised as follows. Section 2 reviews the main results concerning polynomial approximation and interpolation on semi-infinite domains by scaled Laguerre functions. In Section 3, a spectral collocation method based on scaled Laguerre functions is introduced to show that the shallow water equations can be approximated accurately and efficiently on semi-infinite domains by extensions of semi-implicit, semi-Lagrangian methods, customary in many environmental applications. The semi-infinite discretisation is then coupled to a finite volume discretisation of the shallow water equations on a finite size domain, as detailed in Section 4. Section 5 contains the results of a number of numerical tests. A first goal of these tests is to validate the pseudo-spectral, semi-implicit and semi-Lagrangian discretisation approach on semi-infinite domains as an independent tool. The coupling approach is then validated, by comparison of the coupled finite volume/pseudo-spectral model results with those of reference runs of the finite volume model on a single domain. Finally, efficiency tests are performed to assess the performance of our coupled scheme in simulating absorbing boundary condition for the absorption of a single Gaussian wave and a continuous monochromatic wavetrain. Numerical simulations show that a reasonable number of spectral base functions is sufficient to damp perturbations effectively in comparison with standard approaches found in the literature, without appreciable modification of the accuracy properties in the inner domain. Some conclusions on accuracy and efficiency are drawn in Section 6, where the perspectives for further development of the approach are also discussed.

2. SCALED LAGUERRE POLYNOMIALS AND FUNCTIONS ON THE HALF LINE

Laguerre polynomials and functions are a classical tool for polynomial approximations on semi-infinite domains, for example, [35]. In-depth analysis has been carried out to assess the efficiency and accuracy properties of these systems in spectral discretisations of initial and boundary-value problems ([36, 37] and the review in [38]). More recently, scaled Laguerre polynomials and functions were introduced in [34] for the approximation of solutions with different asymptotic behaviours. We recall here the main results for scaled Laguerre functions. A more extensive review of the results on approximation, quadrature rules, interpolation and spectral differentiation can be found in [39].
Let \( \omega_\beta(x) = e^{-\beta x} \), \( \beta > 0 \), be a weight function on \( \mathbb{R}^+ \). Scaled Laguerre polynomials (SLPs) \( \{ \mathcal{L}_n^{(\beta)} \}_{n \in \mathbb{N}} \) are a system of orthogonal polynomials in \( L^2_{\omega_\beta}(\mathbb{R}^+) \), that is
\[
\int_0^{+\infty} \mathcal{L}_n^{(\beta)}(x) \mathcal{L}_m^{(\beta)}(x) \omega_\beta(x) dx = \frac{1}{\beta} \delta_{nm}, \quad n, m \in \mathbb{N}.
\]

Scaled Laguerre functions (SLFs) \( \{ \mathcal{D}_n^{(\beta)} \}_{n \in \mathbb{N}} \) are defined by
\[
\mathcal{D}_n^{(\beta)}(x) = e^{-\beta x/2} \mathcal{L}_n^{(\beta)}(x).
\]

SLFs are orthogonal in \( L^2(\mathbb{R}^+) \), that is
\[
\int_0^{+\infty} \mathcal{D}_n^{(\beta)}(x) \mathcal{D}_m^{(\beta)}(x) dx = \frac{1}{\beta} \delta_{nm}, \quad n, m \in \mathbb{N}.
\]

Because \( \{ \mathcal{D}_n^{(\beta)} \}_{n \in \mathbb{N}} \) is a complete orthogonal basis of \( L^2(\mathbb{R}^+) \), any function \( u \) can be expanded in series of SLFs, namely
\[
\begin{align*}
u &= \sum_{k=0}^{\infty} \hat{u}_k \mathcal{D}_k^{(\beta)}, \\
S_N u &= \sum_{k=0}^{N} \hat{u}_k \mathcal{D}_k^{(\beta)},
\end{align*}
\]

where \( S_N u \) and \( \hat{u}_k \) are the \( N \)th order truncated of \( u \) and the \( k \)th Fourier–Laguerre coefficient, and \((\cdot, \cdot)\) is the dot product in \( L^2(\mathbb{R}^+) \). We remark that in our case, we will set \( \beta = 1/L \), where \( L \) stands for a typical length scale in the application of interest. If the goal is the approximation of functions on semi-infinite domains, one will have to set \( 1 \ll L \), which implies \( \beta \ll 1 \). Figure 1 shows the scaled Laguerre function \( \mathcal{D}_5^{(\beta)} \) for five values of \( \beta \).

From this point on, we drop the \( \beta \) subscripts and superscripts for the sake of simplicity, the dependence on \( \beta \) of the quantities being understood. Because we focus on problems on semi-infinite domains, we use Gauss–Radau quadrature formulas [40, 41], whose nodes include the
left endpoint of the domain. Referring to integrations on the half line $I = [0, +\infty)$, the scaled Gauss–Laguerre–Radau quadrature rule based on SLFs reads as follows:

$$\int_0^{+\infty} p(x)dx = \sum_{j=0}^{N} p(x_j)\hat{\omega}_j \quad \forall p \in \hat{P}_{2N},$$

where

$$x_0 = 0, \{x_j\}_{j=1}^{N} \text{ are the zeros of } \partial_x \mathcal{L}_{N+1}(x);$$

$$\hat{\omega}_j = \frac{e^{\beta x_j}}{\beta (N+1)[\mathcal{L}_N(x_j)]^2}, \quad 0 \leq j \leq N,$$

and:

$$\hat{P}_N(I) = \{ u \mid u = ve^{-\beta x/2}, v \in \mathbb{P}_N \},$$

where $\mathbb{P}_N(I)$ is the space of polynomials of degree at most $N$ in $I$. As for the efficient computation of nodes and weights, see the discussion in [39]. The interpolation operator $\hat{I}_N : C([0, +\infty)) \to \hat{\mathbb{P}}_N$ is defined in this setting by

$$\hat{I}_N u(x) = \sum_{n=0}^{N} \tilde{u}_n \hat{\mathcal{L}}_n(x) \in \hat{\mathbb{P}}_N,$$

the coefficients $\{\tilde{u}_n\}_{n=0}^{N}$ being determined by the forward discrete transform:

$$\tilde{u}_n = \sum_{j=0}^{N} u(x_j) \hat{\mathcal{L}}_n(x_j)\hat{\omega}_j \quad 0 \leq j \leq N.$$  

Results on convergence of the interpolation error can be found in [34]. Next, one can define a discrete derivative of $u$ using the analytical derivative of the interpolating polynomial, that is [34, 41]:

$$u^{(m)} = \hat{D}^m u, \quad m \geq 1,$$

where:

$$\hat{D} = \left[ \hat{d}_{ij} \right]_{i,j=0,\ldots,N} = \left[ \hat{h}'_j(x_i) \right]_{i,j=0,\ldots,N},$$

$$u^{(m)} = \left( u^{(m)}(x_0), u^{(m)}(x_1), \ldots, u^{(m)}(x_N) \right)^T, \quad u = u^{(0)},$$

$\{\hat{h}_j(x)\}_{j=0}^{N}$ being the Lagrange interpolation functions relative to the nodes $\{x_j\}_{j=0}^{N}$, defined by the interpolation conditions:

$$\hat{h}_j \in \hat{\mathbb{P}}_N, \hat{h}_j(x_i) = \delta_{ij} \quad 0 \leq i, j \leq N.$$  

Lagrange interpolation functions can be easily computed from Lagrange interpolation polynomials relative to $\{x_j\}_{j=0}^{N}$ by

$$\hat{h}_j(x) = \frac{e^{-\beta x/2}}{e^{-\beta x_j/2}} \hat{h}_j(x).$$

In particular, the derivative reads as follows:

$$\hat{h}'_j(x_i) = \frac{e^{-\beta x_{i}/2}}{e^{-\beta x_j/2}} \left[ \hat{h}'_j(x_i) - \frac{\beta}{2} \delta_{ij} \right].$$
Therefore, the entries of the differentiation matrix (12) have the following form:

\[
\hat{d}_{ij} = \begin{cases} 
\frac{\mathcal{X}_{N+1}(x_j)}{(x_j-x_i)\mathcal{X}_{N+1}(x_j)} & \text{if } i \neq j \\
0 & \text{if } i = j \neq 0 \\
-\beta \frac{N+1}{2} & \text{if } i = j = 0
\end{cases} 
\]

(16)

3. A SEMI-IMPLICIT, SEMI-LAGRANGIAN SPECTRAL COLLOCATION METHOD FOR THE SHALLOW WATER EQUATIONS ON SEMI-INFINITE DOMAINS

The shallow water equations describe waves arising from perturbations of the free surface of a shallow layer of fluid within a constant gravitational field [30, 42]. In the spatial and temporal domain \( \mathbb{R}^+ \times \mathbb{R}^+ \), they can be written as

\[
\begin{align*}
\frac{\partial \eta}{\partial t} + \frac{\partial (Hu)}{\partial x} + \gamma \eta &= 0 & x \in \mathbb{R}^+, t > 0 \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial \eta}{\partial x} + \gamma u &= 0 & x \in \mathbb{R}^+, t > 0 \\
\frac{\partial (Hu)}{\partial x}(0, t) &= f(t) \\
\eta(x, 0) &= \eta_0(x) \\
u(x, 0) &= u_0(x)
\end{align*}
\]

(17)

Together with initial conditions and, for example, Neumann boundary conditions on the fluid discharge at the left endpoint. Other boundary conditions could also be considered. Earlier, \( \eta \) represents the water surface elevation measured from the undisturbed water surface \( z = 0 \), \( H = \eta_z + \eta \) the total water depth, and \( z_b \) the water depth measured from the undisturbed water surface (Figure 2). Moreover, we have already taken into account in both equations a damping reaction term that will be used to represent the absorbing layer.

As for the continuity Equation (17a), we discretise it by the \( \theta \)-method in time [40] and by spectral collocation in space. The advective nonlinearity in the momentum Equation (17b) is discretised by adopting a semi-Lagrangian approach (for a review, see [32]). The value of the solution at time \( n+1 \) at the mesh points is expressed in terms of the solution values at time \( n \) at those points that within a single step are transported by the flow onto the computational mesh, that is

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = Du \frac{u^{n+1} - u^n}{\Delta t},
\]

(18)

Figure 2. Vertical section of free-surface flow.
where \( u^n_i = u^n(x_i) \) and \( x_i \) denotes the foot of the characteristic. \( x_i \) can be determined by numerical solution of the following equation:

\[
\frac{dx}{dt} = u^n(x) \tag{19}
\]

backward in time with initial data equal to the mesh nodes’ coordinates. This equation is solved by an explicit forward Euler method combined with a substepping approach [30, 31], whereas the velocity values at the point \( x_i \) can be retrieved for instance by cubic interpolation, that is

\[
(u^n_i) \approx \begin{cases} 
\sum_{k=i-2}^{i+1} u(x_k) \prod_{j=i-1}^{i+1} \frac{x_j - x_k}{x_j - x_k} & \text{if } u^n_i \geq 0 \\
\sum_{k=i-1}^{i+2} u(x_k) \prod_{j=i-1}^{i+2} \frac{x_j - x_k}{x_j - x_k} & \text{if } u^n_i < 0 
\end{cases} \tag{20}
\]

The total depth variable \( H \) in the mass flux expression (17a) is treated explicitly, which amounts to a linearisation in time.

Finally, the proposed discretisation of system (17a−c) can be written as follows:

\[
\begin{align*}
\left( \frac{\eta_{n+1}^j - \eta_n^j}{\Delta t} + \theta \frac{\partial}{\partial x} \left( H_N^n u_{n+1}^j \right) + \left( 1 - \theta \right) \frac{\partial}{\partial x} \left( H_N^n u_n^j \right) + \gamma \eta_{n+1}^j \right) (x_j) &= 0 \\
\left( u_{n+1}^j - u_n^j \right) + \theta \frac{\partial \eta_{n+1}^j}{\partial x} + \left( 1 - \theta \right) \frac{\partial \eta_n^j}{\partial x} + \gamma u_n^j \right) (x_j) &= 0 \\
\frac{\partial H_N^n u_{n+1}^j}{\partial x}(x_0) &= f_{n+1} \\
\eta_0^0 &= \eta_{0,N} \in \mathbb{P}_N \\
u_0^0 &= u_{0,N} \in \mathbb{P}_N 
\end{align*}
\]

the first two equations holding collocationwise for \( j = 1, \ldots, N \). As seen earlier, if a function \( u(x) \) is identified as the vector containing its values in the scaled Gauss–Laguerre–Radau collocation nodes, that is \( \mathbf{u} = [u(x_j)]_{j=0}^N \), the derivative of \( u(x) \) with respect to \( x \) is approximated by the vector obtained multiplying \( \mathbf{u} \) by the pseudo-spectral differentiation matrix, namely \( \mathbf{u}' = \hat{D} \mathbf{u} \), where, dropping from now on the hat on \( D \)

\[
D_{ij} = \hat{h}_j'(x_i) \quad i, j = 0, \ldots, N. \tag{21}
\]

Therefore,

\[
u'(x_i) = \sum_{j=1}^{N} \hat{h}_j'(x_i)u(x_j) \quad i = 1, \ldots, N, \tag{22}
\]

which can be expressed in matrix notation as \( \mathbf{u}' = \hat{D} \mathbf{u} \), where \( \mathbf{u} \) and the matrix \( D \) are split in the following way:

\[
\mathbf{u} = \begin{bmatrix} u_0 \\ \mathbf{u} \end{bmatrix}, \quad D = \begin{bmatrix} D_{00} & \hat{d} \\ \hat{d} & D_N \end{bmatrix}. \tag{23}
\]

Moreover, the boundary condition reads collocationwise as

\[
d^0 H_N^n u_{n+1}^0 = D_{00} \left( H_N^n u_{n+1}^0 \right)_0 + \hat{d} \left( \hat{H}_N^n \mathbf{u}_{n+1} \right) = f_{n+1}, \tag{24}
\]
where \( \hat{H}_N^n = \text{diag} \left\{ \left( H_N^n \right)_i \right\}_{i=1}^N \),

\[
\hat{H}_N^n = \begin{bmatrix}
(\hat{H}_N^n)_0 & 0 & \cdots & 0 \\
0 & \hat{H}_N^n & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \hat{H}_N^n \\
\end{bmatrix},
\]

(25)

and \( d^0 \) denotes the first row of the matrix \( D \). These discrete boundary conditions imply that

\[
(1 + \gamma_0 \Delta t) (\eta_N^{n+1})_0 = (\eta_N^n)_0 - \Delta t \left[ \theta f^{n+1} + (1 - \theta) f^n \right],
\]

\[
(1 + \gamma_0 \Delta t) (\eta_N^{n+1})_0 = (\eta_N^n)_0 - g \Delta t \theta \left[ D_{00} (\eta_N^{n+1})_0 + \hat{d}_n^{n+1} \right] - g \Delta t (1 - \theta) \left[ D_{00} (\eta_N^n)_0 + \hat{d}_n^n \right],
\]

(26)

where \( \gamma_0 = \gamma(x_0) \). Furthermore, setting

\[
\Gamma = \text{diag} \left\{ \left[ \gamma(x_i) \right]_{i=1}^N \right\}, \quad I_{N+1} = I_N + \Delta t \Gamma,
\]

the discrete equations for the internal nodes read:

\[
\begin{align*}
I_{N+1} \hat{u}_N^{n+1} & = \left( \hat{u}_N^n \right) + g \Delta t \theta \left[ D_N \hat{\eta}_N^{n+1} + \hat{d} (\eta_N^{n+1})_0 \right] \\
- g \Delta t (1 - \theta) \left[ D_N \hat{\eta}_N^n + \hat{d} (\eta_N^n)_0 \right] \\
I_{N+1} \hat{\eta}_N^{n+1} & = \hat{\eta}_N^n - \theta \Delta t \left[ D_N \hat{H}_N^n \hat{u}_N^{n+1} + \hat{d} \left( H_N^n u_N^{n+1} \right)_0 \right] \\
- (1 - \theta) \Delta t \left[ D_N \hat{H}_N^n \hat{u}_N^n + \hat{d} \left( H_N^n u_N^n \right)_0 \right] \\
\end{align*}
\]

(28)

where \( \left( \hat{u}_N^n \right) = \left\{ \left\{ \left( u_N^n \right)_i \right\} \right\}_{i=1}^{N} \). Replacing \( \hat{u}_N^{n+1} \), \( (\eta_N^{n+1})_0 \) and \( (\eta_N^n)_0 \) in the second equation, one obtains an equation for the elevation at the internal nodes, which after further reductions reads:

\[
\begin{align*}
\left\{ I_{N+1} - \theta^2 g \Delta t^2 G \right\} \hat{\eta}_N^{n+1} & = \left\{ I_N + \theta (1 - \theta) g \Delta t^2 G \right\} \hat{\eta}_N^n \\
& - \theta \Delta t D_N \hat{H}_N^n I_{N+1}^{-1} \left( \hat{u}_N^n \right) + \theta^2 g \Delta t^2 (\eta_N^{n+1})_0 G_0 \\
& - (1 - \theta) \Delta t D_N \hat{H}_N^n \hat{u}_N^n \\
& - \Delta t \left( H_N^n \right)_0 (u_N^n)_0 \hat{d} \left( \frac{1 + (1 - \theta) \gamma_0 \Delta t}{1 + \gamma_0 \Delta t} \right) \\
& + \theta (1 - \theta) g \Delta t^2 (\eta_N^n)_0 G_0,
\end{align*}
\]

(29)

where

\[
G = D_N \hat{H}_N^n I_{N+1}^{-1} D_N + \frac{(H_N^n)_0}{1 + \gamma_0 \Delta t} \hat{d},
\]

(30)

\[
G_0 = D_N \hat{H}_N^n I_{N+1}^{-1} \hat{d} + \frac{(H_N^n)_0}{1 + \gamma_0 \Delta t} D_{00} \hat{d}.
\]

(31)

Replacing expression (29) in the equation for the velocity in (28) gives the solution of the problem at time \( n + 1 \).

4. COUPLING FINITE AND SEMI-INFINITE DOMAINS

We will now describe a classical finite volume method for the numerical approximation of the shallow water equations in a finite domain and show how to couple it with the previously introduced spectral discretisation on a semi-infinite domain. This will allow us to obtain an appropriate
setting for the implementation of absorbing boundary condition. We remark that the finite volume discretisation is used as an example, being a standard approximation technique for the shallow water equations [43], and different methods could be employed in the finite domain. The conservative formulation of shallow water equations reads:

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \quad (32)$$

where:

$$U = \begin{bmatrix} H \\ Hu \end{bmatrix}, \quad F(U) = \begin{bmatrix} Hu \\ Hu^2 + \frac{1}{2}gH^2 \end{bmatrix}. \quad (33)$$

$U$ is the vector of the conserved quantities (depth and discharge). $F(U)$ is the flux function. We want to solve (32) in the spatial and temporal domain $[x_1, x_2] \times [0, T]$. We divide the interval $[x_1, x_2]$ in $N_x$ cells of uniform length $\Delta x$, denoted by $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, $i = 1, \ldots, N_x$, the centres of the cells being the points $x_i$, $i = 1, \ldots, N_x$. The numerical solution is approximated by cell averages of the solution over grid cells. These averages are then updated at each time step through the fluxes crossing the cell boundary points $x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}$, the interfaces. As for the time discretisation, we use a uniform mesh $t^n = n\Delta t$, $n = 0, \ldots, T/\Delta t$ and a fully explicit method. We define $U^n_i$, the numerical approximation to the cell average at time $t^n$ over the $i$th cell by

$$U^n_i \approx \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} U(x, t^n) \, dx, \quad (34)$$

and the approximation to the flux at the interface $x_{i-\frac{1}{2}}$ by

$$F^n_{i-\frac{1}{2}} \approx \frac{1}{\Delta t} \int_{t^n_{i-1}}^{t^n_i} F\left(U\left(x_{i-\frac{1}{2}}, t\right)\right) \, dt. \quad (35)$$

$F^n_{i-\frac{1}{2}}$ is the numerical flux. A finite volume method in conservation form reads [44]:

$$U^{n+1}_i = U^n_i - \frac{\Delta t}{\Delta x} \left( F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}} \right). \quad (36)$$

For the simulations presented in this paper, we have employed Rusanov’s first-order numerical flux [44, p. 233]:

$$F^n_{i-\frac{1}{2}} = \frac{1}{2} \left( F(U^n_{i-1}) + F(U^n_i) \right) - a_{i-1/2} \left( U^n_i - U^n_{i-1} \right). \quad (37)$$

where:

$$a_{i-1/2} = \max \left( |F'(u)| \right) \quad (38)$$

the maximum being taken over all states between $U^n_{i-1}$ and $U^n_i$. However, we have also experimented with a number of first and second order fluxes, including flux limited versions of the Lax–Wendroff flux. Apart from the obvious differences in accuracy over the finite domain, both the order of magnitude of the resulting errors at the interface and the effectiveness of the absorbing layer described and validated in Section 5.3 appear to be independent of the choice of the numerical flux.

A straightforward second-order in time extension of (36) is obtained by the two-stage Runge–Kutta–Heun method [45], which reads

$$\tilde{U}^{n+1}_i = U^n_i - \frac{\Delta t}{\Delta x} \left( F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}} \right), \quad (39)$$

$$U^{n+1}_i = U^n_i - \frac{\Delta t}{2\Delta x} \left( F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}} + \tilde{F}^{n+1}_{i+\frac{1}{2}} - \tilde{F}^{n+1}_{i-\frac{1}{2}} \right). \quad (40)$$

where $\tilde{F}^{n+1}_{i-\frac{1}{2}}$ is the numerical flux (35) computed using the first stage solutions $\tilde{U}^{n+1}_i, \tilde{U}_{i-1}^{n+1}$. 

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Next, in order to use the spectral semi-infinite discretisation described in Section 3 as an absorbing layer to extend the finite domain discretisation, we need a strategy to couple the two domains, that is, to connect the right endpoint of the finite domain to the left endpoint of the semi-infinite one, the contact point being the interface. A representation of the coupling setting is displayed in Figure 3.

Multidomain discretisations of hyperbolic problems require suitable conditions on the fluxes or the characteristic variables at the interface between the two domains. For a detailed discussion on the topic, we refer to [46, 47]. In our case, we found that a simple condition on continuity of the mass flux provides sufficient accuracy for the purpose of the implementation of an absorbing layer in a multidomain framework.

Recalling the discrete formulation (24) of the Neumann boundary condition for the semi-infinite part:

\[
\frac{d^0 H_N^nu_N^{n+1}}{D_{00}(H_N^nu_N^{n+1})_0 + \delta (\vec{H}_N^nu_N^{n+1})} = f^{n+1},
\]  

(41)

the coupling is obtained assuming \( f^{n+1} \) to be computed with the values of the bounded domain. For simplicity, to obtain this approximation we use a first-order one-sided approximation; that is, we impose

\[
f^{n+1} = \frac{(hu)^{n+1}(x_{N_x}) - (hu)^{n+1}(x_{N_x-1})}{\Delta x},
\]

(42)

where \( \Delta x \) is the finite domain uniform grid spacing. The quantities on the right hand side of (42) result from time advancing the finite volume scheme according to (36), whereas the numerical flux at the last interface of the left domain \( x_{N_x+1/2} \) is computed taking as right states the depth and discharge values at the left endpoint \( x_0^{(\beta)} \) of the semi-infinite domain. Therefore, the solution of the coupled scheme is obtained by the following algorithm:

For each \( t^n, n = 0, \ldots, T/\Delta t - 1 \):

1. Advance the finite volume scheme by (36) or (40), having computed \( F_{N_x+\frac{1}{2}}^n \) assuming a right state:

\[
U_{N_x+1}^n = (h(x_0), h(x_0)u(x_0)).
\]

(43)

2. Compute

\[
f^{n+1} = \frac{(hu)^{n+1}(x_{N_x}) - (hu)^{n+1}(x_{N_x-1})}{\Delta x},
\]

(44)
and replace it in
\[ d^n_0 \tilde{H}_N^n \tilde{u}_N^{n+1} = \left[ D_{00} (H_N^n \tilde{u}_N^{n+1})_0 + \tilde{d} (\tilde{H}_N^n \tilde{u}_N^{n+1}) \right] = f^{n+1}. \] (45)

3. Advance the spectral collocation scheme by solving system (28).

5. NUMERICAL TESTS

In this section, we present some results obtained with the numerical schemes described earlier in a series of relevant test cases. First, we test the accuracy of the stand-alone spectral collocation approximation of shallow water equations, as described in Section 3. Next, we validate the coupling approach described in Section 4 by matching the proposed spectral discretisation on the semi-infinite domain to the standard finite volume approximation of the shallow water equations also described in Section 4. We compare the results with those of a full finite volume approximation and show that the interface is sufficiently transparent to the propagation of waves. These results will provide a motivation for the use of the proposed spectral semi-infinite discretisation to implement an absorbing layer for the approximation of open boundary conditions for waves departing from the finite domain. The semi-infinite absorbing layer is tested using an approach and error indicator similar to those proposed in [18]. In particular, we show that use of a small number of spectral base functions to the right of the artificial interface is sufficient to damp outgoing waves without reflections either at the interface or at infinity. Furthermore, we compare quantitatively the efficiency of our approach with a standard implementation of the absorbing layer in the context of the finite volume method, demonstrating the superior performance of the semi-infinite absorbing layer.

5.1. Validation of the spectral collocation method

The objective of this section is to validate the proposed semi-implicit and semi-Lagrangian spectral collocation discretisation of the shallow water equations, meant as an example of a generic hyperbolic initial and boundary value problem.

As for the domain data, setting \( \beta = 10^{-3} \) and \( N = 180 \) internal nodes, the rightmost node is placed at \( x_N \approx 6.91 \cdot 10^5 \) m. We consider homogeneous Neumann boundary condition on the discharge, that is expression (17c) with \( f = 0 \). As initial condition on the free surface elevation, we consider the following profile:

\[ \eta_0(x) = \tilde{\eta} \exp \left[ -\left( \frac{x - x_0}{\sigma} \right)^2 \right], \] (46)

that is, a Gaussian hump of amplitude \( \tilde{\eta} \) centred in \( x = x_0 \) with standard deviation \( \sigma \). The initial velocity is set to zero. Specifically, the hump is centred at \( x_0 = 2.4 \cdot 10^5 \) m, with \( \sigma = 2 \cdot 10^4 \) m. We solve system (28) semi-implicitly with \( \theta = 0.55 \) and the following data:

\[ T = 1.5 \cdot 10^4 \text{ s}, \quad \Delta t = 7.5 \text{ s}, \quad \tilde{H} = 10 \text{ m}, \quad \tilde{u} = 0 \text{ m/s}, \quad \tilde{\eta} = 0.1 \text{ m}. \] (47)

The corresponding maximum Courant number is 3.66. Furthermore, we use cubic interpolation for reconstruction of the trajectories, see expression (20). Figure 4 shows a detail on the initial depth, and Figure 5, the elevations and velocities at times \( T/2 \) and \( T \) computed by the numerical scheme in comparison with the exact values obtained in the linear regime by classical arguments [44]. Although our scheme approximates the nonlinear equations, the comparison shown is meaningful because the initial perturbation in the elevation is small. The hump splits into two crests propagating at the correct velocity \( c = \sqrt{gH} \), and an agreement with the expected values is achieved. For this simulation, the relative \( L^2 \) and \( L^\infty \) errors are 0.032 and 0.033, respectively, for both elevation and velocity. Furthermore, we report in Table I the condition numbers of the matrix relative to system (29) for different choices of the number of nodes. No significant conditioning problems arise, ruling out in our case one of the usual flaws of spectral collocation methods [35]. This test shows that our discretisation based on scaled Laguerre functions provides a reliable tool for the approximation of hyperbolic differential problems on unbounded domains.

Finally, we remark that performing a convergence test in this setting is not a trivial task. Design issues are given by the very inhomogeneous distribution of the spectral Laguerre nodes and the fact that the domain size changes significantly with $N$ as the value of $\beta$ is held constant. In addition, it is not easy to single out the spectral spatial rate of convergence in the framework of a scheme which is only second-order accurate in time, as documented in [39] with convergence tests on parabolic
Table I. Condition numbers of the matrix of the system (29) for spectral-collocation approximation of shallow water equations, homogeneous Neumann boundary conditions.

<table>
<thead>
<tr>
<th>N</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
<th>128</th>
<th>180</th>
</tr>
</thead>
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<tr>
<td>$K_2$</td>
<td>1.008</td>
<td>1.03</td>
<td>1.1</td>
<td>1.39</td>
<td>2.53</td>
<td>7.03</td>
<td>12.88</td>
</tr>
</tbody>
</table>

Table II. Relative $L^2$ and $L^\infty$ errors for the stand-alone spectral collocation scheme, $\beta = 0.001$, $T = 10$ s, $\Delta t = 0.001$ s.

<table>
<thead>
<tr>
<th>N</th>
<th>CFL</th>
<th>$\varepsilon^{rel}_2(\eta)$</th>
<th>$\varepsilon^{rel}_\infty(\eta)$</th>
<th>$\varepsilon^{rel}_2(\mathbf{u})$</th>
<th>$\varepsilon^{rel}_\infty(\mathbf{u})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>1.11E−04</td>
<td>7.83E−06</td>
<td>3.90E−05</td>
<td>8.33E−03</td>
<td>8.81E−03</td>
</tr>
<tr>
<td>64</td>
<td>1.75E−04</td>
<td>2.86E−06</td>
<td>3.91E−06</td>
<td>2.36E−05</td>
<td>3.78E−05</td>
</tr>
<tr>
<td>96</td>
<td>2.62E−04</td>
<td>2.86E−06</td>
<td>3.80E−06</td>
<td>4.29E−06</td>
<td>5.88E−06</td>
</tr>
</tbody>
</table>

equations. Table II reports the results of a run with $x_0 = 2.5 \cdot 10^4$ m and $\sigma = 5 \cdot 10^3$ m for increasing values of $N$, with data as in the previous test. It is evident that the time integration error prevents the total error from approaching spectral accuracy. Moreover, it is worth reminding that we are currently comparing a solution of the linear problem with a discretisation of the full nonlinear equations. Further convergence tests of the Laguerre spectral collocation methods performed in the context of elliptic problems can be found in [36].

5.2. Validation of the coupling approach

To test the accuracy of the coupling strategy described in Section 4, we compare the coupled finite/semi-infinite discretisation with a stand-alone finite volume approximation. To this end, we consider again the initial condition (46) for free surface elevation, the initial velocity being again set to zero. We compare the results of a full finite volume approximation with those of the coupled finite volume/spectral collocation approximation, considering the full finite volume scheme as the reference solution. We recall that the finite volume approximation is based on the conservative formulation of the shallow water equations (Equations (32) and (36)). The results presented here have been obtained with Rusanov’s numerical flux and Heun’s second-order method for time integration. The simulations to obtain the reference solution are performed on an extended domain to prevent spurious reflections at the boundary. As for the spectral part, the spectral collocation scheme described in Section 3 is used for space discretisation, using cubic interpolation for the reconstruction of the velocity at the foot of the characteristics in the semi-Lagrangian method. In this setting, the aim is to show that no spurious phenomena occur as a wave crosses the interface. Indeed, we check that in the finite domain, the relative error of the coupled discretisation with respect to the full finite volume one is small enough to validate the coupling approach. The lack of spurious phenomena arising from the interface will give a motivation for the use of the spectral discretisation on the semi-infinite domain to implement an absorbing layer boundary condition. We consider the coupled approximation described in Section 4, with the following data:

\[ \tilde{\eta} = 1 \text{ m}, \quad H_N^0 = 10 \text{ m}, \quad x_{\frac{1}{2}} = 0, \quad x_{N_{\chi} + \frac{1}{2}} = x_0 = 10000 \text{ m}, \]

\[ \beta = 1/400, \quad N_{\chi} = 1250, \quad N = 180, \quad x_N = 2.86 \cdot 10^5 \text{ m}, \]

where we recall that $x_{\frac{1}{2}}$ is the left endpoint of the finite domain, $x_{N_{\chi} + \frac{1}{2}} = x_0$ is the interface between the two domains, $N_{\chi}$ is the number of cells of the finite volume discretisation of the left domain, $N$ is the number of nodes of the spectral semi-infinite collocation approximation and $x_N$ is the rightmost spectral node.

**Test 1 - ingoing wave** We centre the initial perturbation at $x = 12000$ m, to the right of the interface, placed at $x_0 = 10000$ m. Figure 6 on the left shows the initial depth for two values of $\sigma$. 

We apply the coupling algorithm described in Section 4 until the final time \( T = 400 \) s. The initial hump splits into two waves travelling in opposite directions. The left crest crosses the interface and, propagating at the correct velocity \( \sqrt{gH} \approx 10 \text{ m/s} \), reaches at \( t = T \) the point \( x \approx 8000 \text{ m} \), whereas the right crest symmetrically travels the same length in the positive direction. The result obtained with the coupled model is compared with that of a full finite volume approximation in a domain 20 km wide with the same initial data. Grid spacing is chosen so that the first 10 km are covered by the same number of nodes as for the finite part of the interval used by coupled scheme, that is \( N_x = 1250 \) in the first 10 km. With a view to the use of the coupled discretisation to simulate open boundary conditions, we want reflections into the finite domain generated at the interface to be minimised. Table III shows, for \( \sigma_1 = 1000 \text{ m} \), \( \sigma_2 = 500 \text{ m} \) and \( C \approx 0.6 \), the relative \( L^1 \), \( L^2 \) and \( L^\infty \) errors of the coupled model for free surface elevation and velocity over the first 10 km, using, as a reference solution to the same quantities the one computed by the full finite volume discretisation. Figure 7 shows a comparison of computed final depth and velocity at final time \( T \) by the coupled and full finite volume schemes. For this example, we considered \( C \approx 0.6 \), \( \theta = 1 \), \( \sigma = 1000 \text{ m} \) (error values boxed in Table III). For all practical purposes, the solution computed by the coupled formulation approximates well the one computed by the reference method. Relative errors are higher for the shorter wavelength case and for the velocity variable. Nonetheless, on the whole, they are all acceptable in an environmental modelling context, [31].

**Test 2 - outgoing wave** We then centre the initial perturbation \( \eta_0 \), again with \( \eta = 1 \text{ m} \), in the finite domain at \( x_0 = 5000 \text{ m} \), see Figure 6 on the right. We set a final time \( T = 1000 \text{ s} \), sufficient for all the waves to propagate out of the finite domain. Therefore, elevation and velocity variables are expected to be zero in the finite domain. The other parameters are set as in the previous test.

![Figure 6. Initial depth for ingoing (left) and outgoing (right) wave for coupling validation tests.](image)

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>( \eta )</th>
<th>( u )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_1 = 1000 \text{ m} )</td>
<td>( 7.22 \times 10^{-05} )</td>
<td>0.0082</td>
</tr>
<tr>
<td>( \sigma_2 = 500 \text{ m} )</td>
<td>( 6.93 \times 10^{-05} )</td>
<td>0.156</td>
</tr>
<tr>
<td>( \sigma_1 = 1000 \text{ m} )</td>
<td>( 1.03 \times 10^{-04} )</td>
<td>0.0117</td>
</tr>
<tr>
<td>( \sigma_2 = 500 \text{ m} )</td>
<td>( 1.67 \times 10^{-04} )</td>
<td>0.038</td>
</tr>
</tbody>
</table>

*Table III. Coupling validation results, ingoing wave, \( C \approx 0.6 \).*
This test corresponds to the typical situation in which an absorbing layer is employed, namely to damp perturbations coming from the inner domain. Absolute $L^1$, $L^2$ and $L^\infty$ errors are reported in Table IV for $\sigma_1$ and $\sigma_2$ with $C \approx 0.6$. $L^1$ and $L^2$ errors have been rescaled by the domain length and its square root, respectively, to obtain average values. Further tests with other values of the Courant number, leading to comparable results, are reported in [39]. These values confirm the feasibility of the present approach to couple the finite and semi-infinite domains, and justify the implementation of an absorbing layer boundary condition in the context of the coupled model.

5.3. Validation of the absorbing boundary conditions implemented in the coupled model

In this section, we present the numerical results obtained by implementing an absorbing layer boundary condition in the context of the coupled model. More specifically, the absorbing layer is simulated by a damping reaction coefficient placed in the semi-infinite part. More specifically, we will employ the following sigmoid-like damping function:

$$\gamma(x) = \frac{\Delta \gamma}{1 + \exp\left(\frac{\alpha L_0 - x}{\sigma}\right)}.$$  (50)

Figure 8 displays $\gamma(x)$ in $[0, L_0]$ for $L_0 = 3.77 \cdot 10^5$ m, $\alpha = 0.7$, $\Delta \gamma = 0.005$, $\sigma = L_0/18$. The parameters’ meanings are as follows: $L_0 = x_N$ denotes the position of the last node employed by the spectral collocation method, $\alpha$ determines the position of the sigmoid, centred in $x = \alpha L_0$, and $\sigma$ determines the slope of the damping function.

$\Delta \gamma$ can be naturally interpreted as the inverse of the time scale over which outgoing perturbations are effectively damped. This parameter could be chosen in order to tune the effectiveness of the damper to the characteristics of the typical outgoing waves, as done, for example, in [18]. However,
it will be seen in the following that the approach we propose is quite robust with respect to the choice of this parameter. In particular, in the following tests, we set the value of $\sigma$ so as to obtain a damping region $L_0/2$ wide: by setting $\sigma = L_0/18$ throughout, we found a good compromise, on the one hand, to damp waves effectively; on the other hand, to prevent reflections caused by an excessive steepness of $\gamma$ (see also [18]).

In the following tests, we want to demonstrate that, on the one hand, waves can be effectively damped in the semi-infinite domain within a finite time interval towards the external solution, assumed to be zero for both elevation and velocity. On the other hand, it will be shown that, reducing the number of spectral collocation nodes, the coupling strategy does not give rise to numerical instability at the interface, while the damper retains its effectiveness at a reduced computational cost. We perform two tests, the first with a single outgoing wave, the second on a continuous wavetrain.

**Single Gaussian perturbation** We consider again a finite domain of length $D = 10$ km and let evolve through the interface an initially motionless Gaussian elevation perturbation given by (46), with $\eta = 1$ m and $H^0_N = 10$ m. We expect both elevation and velocity to be damped by the relaxation term, which however should not produce spurious waves neither in the finite nor in the semi-infinite part. The performance of this absorbing layer open boundary condition is assessed in terms of maximum values of residual elevation, velocity and energy at the final time. The results of this test may be compared with those obtained with currently employed absorbing layers; see, for example, the same test in [18], in which the absorbing layer is placed in the external part of the computational grid.

To enhance the effectiveness of the absorbing layer, we use linear interpolation for the computation of the characteristics in semi-Lagrangian method and $\theta = 1$ in (29). For a decreasing number of nodes, we report in Table V the absolute $L^\infty$ norm of elevation and velocity at final time $T = 5000$ s with $x = 7000$ m as the centre of the initial perturbation and four choices for the number of nodes used by the spectral collocation discretisation. With $N = 40, 30, 20, 10$ we set $L = 280$ ($\beta \approx 0.00357$), $\sigma = 500$ m, $\Delta \gamma = 2$ and $\alpha = 0.4$. $x_N$ is the position of the last node, expressed in metres, $N_x$ is the number of points in the finite grid and $T/\Delta t$ is the number of time steps. Moreover, as in [18], we consider an energy-like error at the final time $T$, whose discretized form is given by:

$$
\mathcal{E}_{EN} = \frac{1}{N_x} \sum_i \frac{1}{2} \left[ g \left( \eta_i - [\eta_{ref}] \right)^2 + H \left( u_i - [u_{ref}] \right)^2 \right].
$$

(51)
Table V. Maximum residual elevation, velocity, and mean square root energy error for the single Gaussian perturbation test, $T = 5000$ s, coupled finite volume/spectral collocation discretisation.

| $N$ | $x_N$       | $T/\Delta t$ | $N_x$ | CFL | $|\eta(T)|_\infty$ | $|u(T)|_\infty$ | $\mathcal{E}_{EN}$ |
|-----|-------------|--------------|-------|-----|-------------------|-----------------|-------------------|
| 40  | 4.04E04     | 3000         | 400   | 0.66| 2.81E--04         | 2.77 E--04      | 7.71E--07         |
| 30  | 2.96E04     | 2500         | 300   | 0.60| 3.56E--04         | 3.51E--04       | 1.23E--06         |
| 20  | 1.91E04     | 2000         | 200   | 0.51| 5.22E--04         | 5.14E--04       | 2.65E--06         |
| 10  | 8.87E03     | 1100         | 100   | 0.48| 6.31E--04         | 6.16E--04       | 3.82E--06         |

In these computations, the zero solution is taken as a reference both for elevation and velocity, because the aim is to show that outgoing waves are damped effectively with negligible reflection produced at the boundary between the finite and semi-infinite domain running our computational model for long times. Residual values in Table V confirm the reliability of our approach, proving that the absorbing layer is effective as no significant spurious disturbances arise at the interface between the finite and the semi-infinite domain.

Outgoing wavetrain To provide further evidence of the effectiveness of the proposed absorbing layer approach, we test our scheme in the more demanding case of a continuous wavetrain generated by a periodic forcing at the leftmost point of the finite domain. Specifically, we run our coupled finite volume/spectral collocation scheme with zero elevation and velocity initial data, imposing a Dirichlet boundary condition on momentum at the left endpoint of the finite domain of the form $(H \mathbf{u})(0,t) = A \sin(\omega t)$, where $A$ is the amplitude and $\omega$ the frequency of the discharge prescribed at inflow. The implementation of the Dirichlet boundary condition in a finite volume setting is standard, details can be found, for example, in [44]. The generated wavetrain propagates through the interface between the finite and semi-infinite domain, in which perturbations get damped by the sponge layer. During this process, if the coupling and the damping are not performed properly, spurious reflection can occur at the interface that can modify the amplitude and phase of the waves in the finite domain. With a view to atmospheric applications, this simulation can be seen as a simplified test for the vertical propagation of gravity waves ([15] and the more recent tests in [48], where the absorbing layer approximately requires one quarter of the computational cost of the whole simulation). We want to show that, even after a relatively long simulation, reflections in the finite domain due to the presence of the interface and the damper are sufficiently small and that the approach has a low computational cost. To this end, we compare over the finite domain the solution produced by our coupled scheme with the one of a reference run of a finite volume scheme with the same resolution on a larger domain.

Here, we consider a finite domain 5000 m wide and two values for the frequency of the periodic source, setting $\omega = 2\pi k/T$ where, as before, $T$ denotes the final time. For $N = 30$ and $N = 15$ spectral nodes, respectively, and for different choices of $A$ and $k$, Tables VI and VII show the $L^2$ ($\mathcal{E}_{2}^{rel}$) and $L^\infty$ ($\mathcal{E}_{\infty}^{rel}$) errors on surface elevation $\eta$ and velocity $u$ computed by the coupled scheme relative to those computed by the finite volume scheme. The energy error (51) is included as well. Also indicated in the tables are the number of grid points in the finite domain $N_x$, the value of the

Table VI. Outgoing wavetrain test results, $N = 30$.

<table>
<thead>
<tr>
<th>$A$</th>
<th>$k$</th>
<th>$N_x$</th>
<th>$\beta$</th>
<th>$x_N^{(\beta)}$</th>
<th>$\mathcal{E}_{2}^{rel}(\eta)$</th>
<th>$\mathcal{E}_{\infty}^{rel}(\eta)$</th>
<th>$\mathcal{E}_{2}^{rel}(u)$</th>
<th>$\mathcal{E}_{\infty}^{rel}(u)$</th>
<th>$\mathcal{E}_{EN}$</th>
<th>CFL</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30</td>
<td>600</td>
<td>0.0143</td>
<td>7.42E03</td>
<td>0.0131</td>
<td>0.0246</td>
<td>0.0081</td>
<td>0.0111</td>
<td>5.29E--06</td>
<td>0.38</td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>1200</td>
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<td>3.71E03</td>
<td>0.0047</td>
<td>0.0093</td>
<td>0.0065</td>
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<td>0.75</td>
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<tr>
<td>2</td>
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<td>0.0286</td>
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<tr>
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<td>60</td>
<td>1200</td>
<td>0.0286</td>
<td>3.71E02</td>
<td>0.009</td>
<td>0.0114</td>
<td>0.005</td>
<td>0.0063</td>
<td>2.56E--05</td>
<td>0.75</td>
</tr>
</tbody>
</table>

Table VII. Outgoing wavetrain test results, $N = 15$.

<table>
<thead>
<tr>
<th>$A$</th>
<th>$k$</th>
<th>$N_x$</th>
<th>$\beta$</th>
<th>$x_N^{(\beta)}$</th>
<th>$\xi_2^{\text{rel}}(\eta)$</th>
<th>$\xi_2^{\text{rel}}(u)$</th>
<th>$\xi_2^{\text{rel}}(u)$</th>
<th>$\xi_\infty^{\text{rel}}(\eta)$</th>
<th>$\xi_\infty^{\text{rel}}(u)$</th>
<th>$\xi_\infty^{\text{rel}}(u)$</th>
<th>$\xi_{EN}$</th>
<th>CFL</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30</td>
<td>600</td>
<td>0.0286</td>
<td>1.74E03</td>
<td>0.0169</td>
<td>0.0234</td>
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<td>1200</td>
<td>0.0571</td>
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<td>0.0082</td>
<td>0.0048</td>
<td>0.0079</td>
<td>8.17E+06</td>
<td>0.76</td>
<td>0.76</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>30</td>
<td>600</td>
<td>0.0286</td>
<td>1.74E03</td>
<td>0.0214</td>
<td>0.0338</td>
<td>0.0101</td>
<td>0.0168</td>
<td>4.95E+05</td>
<td>0.38</td>
<td>60</td>
<td>0.38</td>
</tr>
<tr>
<td></td>
<td>1200</td>
<td>0.0571</td>
<td>8.72E02</td>
<td>0.0061</td>
<td>0.0106</td>
<td>0.0053</td>
<td>0.0088</td>
<td>4.15E+05</td>
<td>0.76</td>
<td>0.76</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>30</td>
<td>600</td>
<td>0.0286</td>
<td>1.74E03</td>
<td>0.0314</td>
<td>0.0505</td>
<td>0.0153</td>
<td>0.0269</td>
<td>4.27E+04</td>
<td>0.39</td>
<td>60</td>
<td>0.39</td>
</tr>
<tr>
<td></td>
<td>1200</td>
<td>0.0571</td>
<td>8.72E02</td>
<td>0.0087</td>
<td>0.0121</td>
<td>0.0053</td>
<td>0.0092</td>
<td>2.48E+05</td>
<td>0.77</td>
<td>0.77</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Courant number $CFL$, the value of $\beta$, and the extension of the layer, given by the position of the last spectral node $x_N$. The number of time steps $T/\Delta t$ and the value of $\alpha$ (for which, see formula (50) and Figure 8) have been held fixed at 16000 and 0.3, respectively. Finally, the value of $\Delta y$ for these simulations was set to 2. Figures 9 and 10 show a comparison between the two schemes in terms of computed depth in the finite domain at the final time $T = 5000$ s. The plots refer to the two simulations with boxed error values in the Tables VI and VII. The absorbing layer (not visible) is attached to the right of the interface placed at the point $x = 5000$ m. It should be remarked that the simulation has been carried out in a nonlinear regime, as clearly displayed by the progressive...
steepening of the free surface, in order to make this a more stringent test for all the components of the numerical method. The wavenumber-dependent reduction of the wavetrain amplitude is mostly due to nonlinear and dispersive effects and should not be attributed to numerical diffusion. Indeed, at the given spatial resolution, essentially the same pattern is observed, independently of the numerical flux employed in the finite domain.

On the whole, results show that the accuracy of the solution in the finite domain is only marginally affected by the presence of the absorbing layer, at least for the typical standards of environmental applications. As for the efficiency of the scheme, we achieve results comparable with currently used absorbing layers [12, 18], with the grid points in the layer being 5–1.25% of the ones in the finite domain. In this respect, in the tables, we adjusted the spatial resolution and the time step to the value of k, because waves of smaller period have to be better resolved. It is interesting to notice that the number of grid points in the finite domain can be increased without modifying the number of spectral collocation points in the absorbing layer, whose resolution can be adjusted simply by tuning the value of \( \beta \). This feature of the proposed scheme allows for a low computational cost and, in perspective, makes our approach attractive for atmospheric modelling applications.

5.4. Comparison with a standard damping approach

In this final section, we provide a comparison of our scheme with a more standard implementation of absorbing boundary conditions. Here, we want to show that the efficiency and accuracy gain with our method compensates the additional cost entailed by the implementation of a spectral collocation method with respect to a classical single-domain finite difference or finite volume method. The ratio between the number of points employed for the damping and the ones in the internal computational domain gives a measure of efficiency, whereas accuracy is assessed by estimating the reflectivity of the absorbing layer. For this comparison, we choose the same framework as proposed in [18] to assess the effectiveness of different profiles for the damping coefficients. We compare our method with a standard damping strategy, obtained considering a finite volume numerical scheme with \( N_x \) internal grid points and additional \( N \) grid points attached to the rightmost internal point, maintaining uniform grid spacing. These \( N \) points act in this setting as the absorbing layer, with the same damping coefficient \( \gamma \) employed in the previous tests. For the single domain model, a simple outflow boundary condition is then employed [44], adding a ghost cell at the right boundary of the absorbing layer. For instance, the case of our coupled model with \( N = 40 \) spectral nodes, \( N_x = 400 \) grid points in the finite domain will be compared with a single-domain finite volume model with \( N_x = 400 \), \( N = 40 \) grid points for the absorbing layer.

We consider the initial setting of [18] (Figure 11 and their Figure 4) with a domain \( D = 10000 \) m wide, letting a Gaussian perturbation as of Section 5.3 be placed at the centre of the second half of the finite domain, at \( x_0 = 7500 \) m. We run our scheme until the final time \( T = D/2 \sqrt{gH} \), so that any spurious reflection may be evaluated at the initial point \( x_0 \), and we use \( \Delta y = 20 \). Also, as a measure of the performance of the absorbing layer, we refer to [18] and define a reflection ratio as follows:

\[
\rho = \frac{\epsilon_{EN}(T)}{\epsilon_{WEN}(T)}
\]

(52)

where \( \epsilon_{EN}(T) \) refers to expression (51) computed with our coupled scheme, whereas \( \epsilon_{WEN}(T) \) denotes the same quantity computed with a wall condition \( u = 0 \) at the right boundary. The value of \( \rho \), \( L^\infty \) norms of elevation and velocity along with energy error at final time \( T \) computed with our coupled finite volume/spectral collocation scheme are reported in Table VIII. Consistently with [18], errors have been computed in \([D/2, D]\). Considering the accuracy standards in most environmental applications and the performances of currently employed absorbing layers [12, 18], the results shown are quite satisfactory. A good damping efficiency is achieved by employing at most one tenth of the total number of nodes for the absorbing layer, with very small spurious effects arising.

Table IX shows results of the same test with the aforementioned single-domain finite volume setting. In all the four cases considered, the absorbing layer thickness and the CFL number are
Figure 11. Initial setup as in [18] with $N = 10$. The dashed line marks the interface between the finite domain on the left and the semi-infinite domain on the right. The squares mark the Laguerre collocation points.

Figure 12. Depth at time $T = D/2\sqrt{gH}$ computed with the single domain (solid line) and the proposed coupled finite volume/spectral collocation (dashed line) absorbing layer strategies.

1000 m and 0.7, respectively, whereas $\alpha = 0.4$, $\sigma = 500$ and $\tilde{\eta} = 1$. The high values reveal the presence of a large-amplitude spuriously reflected wave in the finite domain, displayed as a solid line in Figure 12 for the 10 nodes case (boxed values in Tables VIII and IX). By contrast, the scheme we propose (dashed line) triggers only a tiny reflection. Although a full sensitivity analysis with respect to the choice of the damping coefficient is beyond the scope of this work, we remark that numerical experiments performed with different damping profiles (such as those considered in [18]) lead to similar results in terms of reflected perturbations for the single domain approach.

Finally, we report the result of additional runs with the same setting as in Table VIII, that is, $x_0 = 7500$ m, $\alpha = 0.4$, $\sigma = 500$ and $\tilde{\eta} = 1$. We considered different values of the scaling factor $\beta$ and the DOFs $N$ for our semi-infinite absorbing layer. Reflection ratios obtained in $[D/2, D]$ are reported in Table X. The values shown confirm the good performance of our method in comparison with existing absorbing layers. In particular, setting the values of $N$ and $\beta$ according to the resolution in the finite domain allows us to reach good efficiency in terms of the ratio between...
Table VIII. Maximum residual elevation, velocity, mean square root energy error and reflection ratios for the single Gaussian perturbation test, \( T = D/2\sqrt{gH}, x_0 = 7500 \text{m} \), coupled finite volume/spectral collocation scheme.

<table>
<thead>
<tr>
<th>N</th>
<th>( x_N )</th>
<th>( \Delta t )</th>
<th>( N_x )</th>
<th>( CFL )</th>
<th>( \eta(T) )</th>
<th>( u(T) )</th>
<th>( E_{EN} )</th>
<th>( \rho )</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>4.04E04</td>
<td>300</td>
<td>400</td>
<td>0.66</td>
<td>1.46E–03</td>
<td>1.20E–03</td>
<td>2.3E–06</td>
<td>3.15E–03</td>
</tr>
<tr>
<td>30</td>
<td>2.96E04</td>
<td>225</td>
<td>300</td>
<td>0.67</td>
<td>1.64E–03</td>
<td>1.33E–03</td>
<td>3.04E–06</td>
<td>3.74E–03</td>
</tr>
<tr>
<td>20</td>
<td>1.91E04</td>
<td>150</td>
<td>200</td>
<td>0.68</td>
<td>1.47E–03</td>
<td>1.10E–03</td>
<td>3.95E–06</td>
<td>4.49E–03</td>
</tr>
<tr>
<td>10</td>
<td>8.87E03</td>
<td>75</td>
<td>100</td>
<td>0.71</td>
<td>5.97E–03</td>
<td>5.42E–03</td>
<td>3.85E–05</td>
<td>0.0188</td>
</tr>
</tbody>
</table>

Table IX. Maximum residual elevation, velocity, and mean square root energy error for the single Gaussian perturbation test, \( T = D/2\sqrt{gH}, x_0 = 7500 \text{m} \), single-domain finite volume scheme.

<table>
<thead>
<tr>
<th>N</th>
<th>( \Delta t )</th>
<th>( N_x )</th>
<th>( \eta(T) )</th>
<th>( u(T) )</th>
<th>( E_{EN} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>300</td>
<td>400</td>
<td>0.2583</td>
<td>0.2545</td>
<td>0.1316</td>
</tr>
<tr>
<td>30</td>
<td>225</td>
<td>300</td>
<td>0.240</td>
<td>0.237</td>
<td>0.120</td>
</tr>
<tr>
<td>20</td>
<td>150</td>
<td>200</td>
<td>0.212</td>
<td>0.209</td>
<td>0.104</td>
</tr>
<tr>
<td>10</td>
<td>75</td>
<td>100</td>
<td>0.1612</td>
<td>0.1596</td>
<td>0.075</td>
</tr>
</tbody>
</table>

Table X. Reflection ratios for the single Gaussian perturbation test, \( T = D/2\sqrt{gH}, x_0 = 7500 \text{m} \), coupled finite volume/spectral collocation test.

<table>
<thead>
<tr>
<th>( N_x )</th>
<th>( \Delta t )</th>
<th>( N )</th>
<th>( \beta )</th>
<th>( \rho )</th>
</tr>
</thead>
<tbody>
<tr>
<td>400</td>
<td>300</td>
<td>20</td>
<td>1/145</td>
<td>2.96E–03</td>
</tr>
<tr>
<td>300</td>
<td>225</td>
<td>20</td>
<td>1/190</td>
<td>4.01E–03</td>
</tr>
<tr>
<td>200</td>
<td>150</td>
<td>10</td>
<td>1/120</td>
<td>1.28E–02</td>
</tr>
<tr>
<td>110</td>
<td>75</td>
<td>13</td>
<td>1/270</td>
<td>7.61E–03</td>
</tr>
</tbody>
</table>

Comparing the results of the multidomain with those of the single domain approach, it is clear that even though the spectral collocation discretisation requires more computations per node due to its intrinsic global nature, it yields much smaller errors for the same number of nodes in the absorbing layer. Although it is not possible to estimate a precise trade-off threshold between the two approaches in the context of the present proof-of-concept implementation, the multidomain approach shows a clear potential for improvement of the present implementations of absorbing layer boundary conditions.

6. CONCLUSIONS AND FUTURE WORK

In this work, we have introduced a numerical method based on scaled Laguerre functions expansions for the approximation of wave propagation problems on semi-infinite domains. In particular,
a semi-implicit, semi-Lagrangian discretisation has been developed for the one-dimensional shallow water equations on semi-infinite domains, along with a coupling approach based on imposing continuity of mass fluxes at the interface between the bounded and unbounded domain. The semi-implicit, semi-Lagrangian approach has been chosen because of its well-known efficiency and accuracy features and because a large number of environmental models use these time discretisation techniques. On the other hand, an explicit finite volume discretisation has been considered for the finite size domain. Furthermore, by coupling discretisations on domains of finite size to scaled Laguerre spectral methods employing a relatively small number of nodes, a novel approach to absorbing boundary conditions has been achieved, which makes it possible to account for large absorbing regions at a relatively low computational cost.

A number of numerical tests has been carried out. First, the pseudo-spectral, semi-implicit, semi-Lagrangian discretisation on semi-infinite domains has been validated. Moreover, we have shown that spurious reflections due to the proposed coupling approach have relatively small amplitude and admit of the use of the method, at least for environmental applications, with no substantial loss of accuracy with respect to single domain discretisations. Next, absorbing boundary conditions have been validated, as implemented in the context of the coupled finite volume/pseudo-spectral model. Numerical simulations on single and periodic perturbations have shown that a small number of scaled Laguerre base functions is sufficient to damp efficiently the outgoing waves, with tangible improvement over single domain approaches and no accuracy loss for the solution in the finite domain.

The obtained results appear to confirm that this approach to absorbing layer boundary conditions is attractive for a number of reasons. It is therefore of interest to consider various natural extensions of the present work, in order to exploit the proposed technique in practical applications.

First, the semi-implicit, semi-Lagrangian pseudo-spectral method on semi-infinite domains can be coupled to different discretisations on the finite size domain; in particular, efficient and commonly used semi-implicit, semi-Lagrangian techniques could be considered in the finite domain, based on either a finite difference or finite element spatial discretisation.

Next, two-dimensional discretisations can be developed, in which a semi-infinite strip is discretised using Laguerre basis functions in the \(x\) direction and some other discretisation approach in the \(y\) direction. For example, tensor product multi-dimensional polynomial spaces as in [37] could be employed, or, more simply, finite difference or finite volume discretisations could be used in the transversal direction. The resulting two-dimensional vertical strips can be coupled to vertical slice atmospheric models, see, for example, [48, 49], for the purpose of assessing the effectiveness of this approach in the case of vertically propagating gravity waves.

The main target of this further development would be the comparison of the standard absorbing layer approaches (e.g., [15, 17]) with those built following the ideas proposed in this work. For these numerical models, the size of the necessary upper absorbing layer is known to increase with the increase in horizontal resolution, thus leading to an extra computational cost that can be especially high for models employing height-based coordinates, such as many high-resolution numerical weather prediction models. Hence, it would be interesting to assess whether a substantial decrease of the computational cost for these models can be achieved by extending the techniques proposed in this work.

ACKNOWLEDGEMENTS

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