Reflection and refraction at the surface of an isotropic chiral medium: eigenvalue–eigenvector solution using a 4 × 4 matrix method

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Fresnel reflection amplitude coefficients (fractional amplitudes) at the surface of an isotropic, intrinsically nonmagnetic chiral medium are derived on the basis of the Drude–Condon model of optical activity. The eigenvalue–eigenvector solution is obtained with use of Berreman's 4 × 4 matrix method. Self-consistent results are obtained when the calculations are based on a new 4 × 4 matrix for reflection from an isotropic chiral medium.

1. INTRODUCTION

Optical activity[1,2] (rotation of the polarization plane) and circular dichroism[3] (differential absorption of left and right circularly polarized light) have been used for a long time to study the gyrotropy of matter by transmission. In this way information on internal molecular structure and gross molecular arrangements has been obtained. The field eigenstates in chiral media are generally either left circularly polarized or right circularly polarized, with each polarization having a distinct wave number. The medium, either natural or induced, distinguishes left and right circularly polarized light. Gyrotropy is a property of a chiral medium, and it can be induced in an initially achiral medium by an external magnetic field, as in the Faraday effect[4]; by an applied electric field, as in the electrogyratory effect[5,6]; or by mechanical stress.[7] A three-dimensional chiral object is an object that cannot be brought into congruence with its mirror image by translation or rotation, so that a collection of chiral objects will form a medium that is characterized by right or left handedness[8]; i.e., gyrotropy is a property of a special noncentrosymmetric medium.

Gyrotropy also manifests itself in the reflection of electromagnetic waves from a chiral medium, and in recent years attention concerning this reflection has been revived. Reflection of electromagnetic waves at achiral–chiral planar interfaces has received particular attention.[9–31]

Silverman,[10–12] Lalov,[14] Lalov and Miteva,[15] Lakhtakia *et al.*,[16] and Bassiry *et al.*[17] derived the Fresnel reflection amplitude coefficients for specular reflection at the surface of a chiral medium. Expressions for the reflectance and transmittance of a wave normally incident on an isotropic, optically active slab situated in a dielectric medium were obtained by Bokut and colleagues.[18,19] Specular reflection and transmission from an optically active medium have also been studied theoretically.[20] Lakhtakia *et al.*[21–26] did a parametric study of the microwave reflection characteristics of the interface and the reflection and transmission of normally incident waves on a chiral slab with linear property variations. Lalov and Miteva[27,28] and Cory and Rosenhonse[29] studied the reflection and transmission of waves at chiral–achiral planar interfaces. Also, experimental configurations to measure chiral effects in specularly reflected light, with use of a photoelastic modulator have been proposed.[30,31] Such experiments will test some fundamentals of chiral electrodynamics that cannot be obtained with transmission studies. Various applications also have been proposed.[32–36]

The objectives of this paper are to derive the Fresnel reflection amplitude coefficients (the fractional amplitudes) by use of a different approach that is based on a corrected Berreman matrix for reflection from an isotropic chiral medium. The new 4 × 4 matrix is based on the Drude–Condon model of optical activity and leads to self-consistent results. Jaggard and Sun[37] derived a 4 × 4 matrix technique for chiral multilayers that is also applicable to a single chiral interface. The technique presented in this paper is different from that of Jaggard and Sun in that the reflection amplitudes are calculated for circular polarization states rather than for linear polarization states.

2. DRUDE–CONDON MODEL OF OPTICAL ACTIVITY AND THE CONSTITUTIVE EQUATIONS OF A CHIRAL MEDIUM

We use the time-dependence factor $\exp(+i\omega t)$ and the two Maxwell equations in the SI system of units:

$$\mathrm{rot}\ \mathbf{E} = -\partial \mathbf{B}/\partial t\,, \tag{1a}$$

$$\mathrm{rot}\ \mathbf{H} = \partial \mathbf{D}/\partial t\,, \tag{1b}$$

where **E**, **D**, and **H**, as usual, are the vectors of the electromagnetic field. To include the effect of the field on matter, the constitutive (material) equations are supplemented:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \qquad \mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})\,, \tag{2}$$

where **P** and **M** are the electric and the magnetic polar-

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izations, respectively, and ϵ_0 and μ_0 are the permittivity and the permeability of vacuum, respectively.

Furthermore, Maxwell's equations are generally held to be inviolable, as also are the boundary conditions on the continuity of the tangential **E** and **H** fields at the bimaterial interface. Hence the properties of matter must enter solely through the constitutive equations.

Condon[38] noted that the essential feature of the material relations is that part of the polarization **P** be dependent on $\partial\mathbf{H}/\partial t$ and part of the magnetization **M** be dependent on $\partial\mathbf{E}/\partial t$:

$$\mathbf{D} = \epsilon\mathbf{E} - \overline{g}\,\frac{\partial\mathbf{H}}{\partial t}\,, \tag{3a}$$

$$\mathbf{B} = \mu\mathbf{H} + \overline{g}\,\frac{\partial\mathbf{E}}{\partial t}\,, \tag{3b}$$

where $\overline{g}$ is the gyrotropic parameter. Hence the constitutive equations of optical activity must contain both polarization and magnetization terms.

According to Paul Drude,[39]

$$\text{rot }\mathbf{H} = \epsilon\,\frac{\partial\mathbf{E}}{\partial t} + \frac{\overline{g}}{\mu}\,\frac{\partial}{\partial t}\,\text{rot }\mathbf{E} = \epsilon\,\frac{\partial\mathbf{E}}{\partial t} + \frac{\overline{g}}{\mu}\,\frac{\partial}{\partial t}\left(-\frac{\partial\mathbf{B}}{\partial t}\right),$$

or

$$-\text{rot }\mathbf{H} = -i\omega\epsilon\mathbf{E} + (+i\omega\overline{g})(i\omega\mathbf{H})\,. \tag{4a}$$

To take into account the coupling between the electric and magnetic field quantities, which results in the polarization and magnetization of the medium as a result of the applied fields, and also to fulfill the breakdown, we write

$$\text{rot }\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} + \frac{\overline{g}}{\epsilon}\,\frac{\partial}{\partial t}\,\text{rot }\mathbf{H} = -\mu\,\frac{\partial\mathbf{H}}{\partial t} + \frac{\overline{g}}{\epsilon}\,\frac{\partial}{\partial t}\left(\frac{\partial\mathbf{D}}{\partial t}\right),$$

or

$$-\text{rot }\mathbf{E} = i\omega\mu\mathbf{H} + (-i\omega\overline{g})(i\omega\mathbf{E})\,. \tag{4b}$$

The Drude–Condon model of optical activity is widely accepted by physicists. Microscopic models, both classical and quantum mechanical,[40] have been presented to justify it, and in this work our calculations are based on this model.

Other sets of constitutive relations are also used. When Eqs. (3a) and (1a) are combined, one obtains

$$\mathbf{D} = \epsilon(\mathbf{E} + \beta\nabla\times\mathbf{E})\,, \tag{5a}$$

and similarly,

$$\mathbf{B} = \mu(\mathbf{H} + \beta\nabla\times\mathbf{H})\,, \tag{5b}$$

where $\beta = \overline{g}/\epsilon\mu$ is a pseudoscalar. Equations (5a) and (5b) are also used by some authors, and it is evident that the value of **D** (respectively, **B**) at any given point depends not only on the value of **E** (respectively, **H**) at that particular point but also on the behavior of **E** (respectively, **H**) in the vicinity of this point; i.e., **D** (respectively, **B**) depends on the derivatives of **E** (respectively, **H**). This relation between **D** and **E** (respectively, between **B** and **H**) is called spatial dispersion.[41]

Since **D** and **E** are polar vectors and **B** and **H** are axial vectors, it follows that ϵ and μ in the above equations are true scalars and that $\overline{g}$ is a pseudoscalar; i.e., the handedness of the medium is manifested by the quantity $\overline{g}$.

3. SOLVING THE PROBLEM OF SPECULAR LIGHT REFLECTION AND TRANSMISSION AT AN ACHIRAL–CHIRAL INTERFACE WITH USE OF THE 4 × 4 MATRIX METHOD

We apply the general and powerful 4×4 matrix method,[42–48] using the Drude–Condon model, to solve the problem of reflection and transmission of polarized light at the surface of an isotropic, spatially dispersive, nonferromagnetic ($\mu = 1$), lossless, optically active medium.

Usually the polarized light incident on the reflective surface is resolved into components parallel (p light) and perpendicular (s light) to the plane of incidence, and these two problems are solved independently. In the case of reflection from gyrotropic media, the incident light with its electric vector entirely in the plane of incidence (p light) gives rise to reflected p light and also to s light with a component of its electric vector perpendicular to this plane. In much the same way, the incident s light on reflection gives rise to s and p light. Thus the reflection problem should be solved as a single problem and not as two independent problems, and the 4×4 matrix method is suitable for the solution of such problems because it was developed for this purpose. On the other hand, the reflection properties of a substance can be easily represented with this method by a 2×2 complex matrix, used in the Jones calculus.[49,50]

In Cartesian coordinates Eqs. (1) can be combined into a single matrix equation (or six differential equations, assuming no macroscopic charge density and no macroscopic current density):

$$\begin{bmatrix} 0 & 0 & 0 & 0 & -\partial/\partial z & \partial/\partial y \\ 0 & 0 & 0 & \partial/\partial z & 0 & -\partial/\partial x \\ 0 & 0 & 0 & -\partial/\partial y & \partial/\partial x & 0 \\ 0 & \partial/\partial z & -\partial/\partial y & 0 & 0 & 0 \\ -\partial/\partial z & 0 & \partial/\partial x & 0 & 0 & 0 \\ \partial/\partial y & -\partial/\partial x & 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} E_x \\ E_y \\ E_z \\ H_x \\ H_y \\ H_z \end{bmatrix} = i\omega \begin{bmatrix} D_x \\ D_y \\ D_z \\ B_x \\ B_y \\ B_z \end{bmatrix}, \tag{6}$$

or, in brief,

$$RG = i\omega C\,. \tag{6'}$$

Here **R**, the rotor matrix, is a 6×6 symmetric matrix operator. It can be divided into four 3×3 matrices:

$$R = \begin{bmatrix} 0 & \text{rot} \\ -\text{rot} & 0 \end{bmatrix}. \tag{7}$$

If we ignore nonlinear effects, a linear relation exists between G and C as

$$C = MG\,, \tag{8}$$

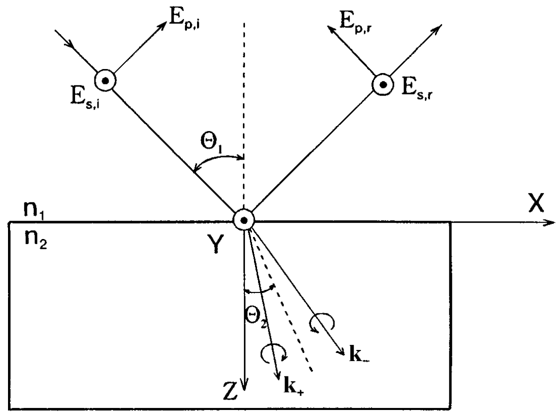

Fig. 1. Schematic diagram of polarized light reflected from an isotropic, optically active medium, showing the coordinate system for the surface and the directions of the electric fields for incident and reflected light. Generally, two elliptical polarizations are transmitted in an optically active medium with different angles of refraction and wave vectors $\mathbf{k}_+$ and $\mathbf{k}_-$.

where the 6×6 matrix M possesses all the optical properties of the medium. The matrix M, called the optical matrix, may be written in the form

$$M = \begin{bmatrix} \epsilon & \rho \\ \rho' & \mu \end{bmatrix}, \tag{9}$$

where $\epsilon = (M_{ij})$ and $\mu = (M_{i+3,j+3})$ are the dielectric and the permeability tensors, respectively, and $\rho = (M_{i,j+3})$ and $\rho' = (M_{i+3,j})$ are the optical rotation tensors (i, j = 1, 2, 3).

For our case of isotropic media the constitutive parameters ϵ, μ, ρ, and ρ' are scalar quantities.

According to Eqs. (4a) and (4b),

$$\rho = +i\omega\overline{g} = +ig, \qquad \rho' = -i\omega\overline{g} = -ig, \tag{10}$$

which place the constitutive equations in the form put forward by Jaggard *et al.*,[37] where g is the chirality admittance.

When C from Eq. (8) is replaced in Eq. (6′),

$$RG = i\omega MG. \tag{11}$$

On the other hand, G can be divided into the time-dependent part $\exp(i\omega t)$ and the spatial part Γ. Then Eq. (11) becomes

$$R\Gamma = i\omega M\Gamma, \tag{12}$$

which is the spatial wave equation for frequency ω.

We assume that the plane of incidence is the x–z plane, so the propagation vector in the y direction $k_y = 0$, and any changes of the field components in the y direction are absent ($\partial/\partial y = 0$). Our problem is the reflection and transmission of a monochromatic plane electromagnetic wave, incident obliquely at an angle θ_1 from an optically inactive medium ($Z < 0$) with coefficient of refraction n_1 on an isotropic optically active medium ($Z > 0$) with coefficient of refraction n_2, for which M is a function only of Z (Fig. 1). We are dealing with monochromatic solutions with a time dependence $\exp(+i\omega t)$ and a dependence on x of $\exp(-i\omega\xi x/c)$; i.e., $\exp(-ik_x x) = \exp(-i\omega\xi x/c)$ and $\xi = ck_x/\omega = ck \sin\theta_1/\omega = n_1 \sin\theta_1$. So

$$\partial/\partial x = -i\xi\omega/c. \tag{13}$$

Therefore Eq. (12) becomes

$$\begin{bmatrix} 0 & 0 & 0 & 0 & -\partial/\partial z & 0 \\ 0 & 0 & 0 & \partial/\partial z & 0 & i\omega\xi/c \\ 0 & 0 & 0 & 0 & -i\omega\xi/c & 0 \\ 0 & \partial/\partial z & 0 & 0 & 0 & 0 \\ -\partial/\partial z & 0 & -i\omega\xi/c & 0 & 0 & 0 \\ 0 & -i\omega\xi/c & 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \\ \Gamma_4 \\ \Gamma_5 \\ \Gamma_6 \end{bmatrix} = i\omega \begin{bmatrix} \varepsilon & 0 & 0 & ig & 0 & 0 \\ 0 & \varepsilon & 0 & 0 & ig & 0 \\ 0 & 0 & \varepsilon & 0 & 0 & ig \\ -ig & 0 & 0 & 1 & 0 & 0 \\ 0 & -ig & 0 & 0 & 1 & 0 \\ 0 & 0 & -ig & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \\ \Gamma_4 \\ \Gamma_5 \\ \Gamma_6 \end{bmatrix}. \tag{14}$$

Equation (14) may be written as four first linear differential equations and two linear algebraic equations in which

$$\Gamma_1 = E_x,\ \Gamma_2 = E_y,\ \Gamma_3 = E_z,\ \Gamma_4 = H_x,\ \Gamma_5 = H_y,\ \Gamma_6 = H_z.$$

Then we follow Berreman's work[46] to construct the 4×4 matrix Δ that contains the optical parameters of the reflecting material. For Δ we find

$$\Delta = \begin{bmatrix} 0 & 1 - \dfrac{{n_2}^2 \sin^2\theta_2}{{n_2}^2 - g^2} & -ig\left(1 + \dfrac{{n_2}^2 \sin^2\theta_2}{{n_2}^2 - g^2}\right) & 0 \\ {n_2}^2 & 0 & 0 & -ig \\ ig & 0 & 0 & 1 \\ 0 & ig\left(1 + \dfrac{{n_2}^2 \sin^2\theta_2}{{n_2}^2 - g^2}\right) & {n_2}^2\left(1 - \dfrac{{n_2}^2 \sin^2\theta_2}{{n_2}^2 - g^2}\right) & 0 \end{bmatrix}. \tag{15}$$

Matrix (15) is different from Berreman's matrix,[46,49] which is based only on Drude's equation (4a) and does not take into account Eq. (4b); i.e., it describes only one side of the problem. Berreman's matrix leads to inequality of the off-diagonal Fresnel reflection amplitude coefficients, which is not correct (see Appendix A).

If we use the two algebraic equations to eliminate E_z and H_z, the remaining four differential equations can be written as a single matrix equation:

$$\frac{\partial}{\partial z}\begin{bmatrix} E_x \\ H_y \\ E_y \\ -H_x \end{bmatrix} = -i\omega\Delta \begin{bmatrix} E_x \\ H_y \\ E_y \\ -H_x \end{bmatrix}, \tag{16}$$

which will be abbreviated

$$\partial\psi/\partial z = -i\omega\Delta\psi\,, \tag{16'}$$

with the column vector

$$\psi(z) = \begin{bmatrix} E_x \\ H_y \\ E_y \\ -H_x \end{bmatrix}.$$

Δ is the 4×4 matrix that depends on the elements of ϵ, the elements of the optical rotation tensors ρ and ρ', and ξ.

Equation (16) contains only the tangential components of **E** and **H**; and, as we know, they are usually used in matching boundary conditions when one is calculating reflection coefficients.

We denote the dependence of ψ on the z coordinate as $\exp(-ik_z z) = \exp(-i\omega z\zeta/c)$, and the solution of Eq. (16) is $\zeta\psi = \Delta\psi$. So our problem is an eigenvalue problem. The eigenvalues ζ will allow us to calculate the z component of the propagation vector (k_z), and the eigenvectors ψ will give us the tangential (x and y) components of the fields for each polarization state. Then the z components of the fields may be calculated from the two linear algebraic equations.

In the case of complex n_2 (i.e., absorption), angle θ_2 will also be complex, and it cannot be interpreted as the angle of refraction for complex n_2. No mathematical difficulties arise in that case.[51]

From matrix Δ we find for our problem the following eigenvalues:

$$\zeta = \pm({n_2}^2 \cos^2 \theta_2 + g^2 \pm 2gn_2)^{1/2}. \tag{17}$$

From Eq. (17) we need to take only the two positive values of ζ, because these represent light with a positive z component of velocity. To simplify our future calculations we do the substitutions

$$\delta = {n_2}^2 \cos^2 \theta_2 + g^2 - 2gn_2\,, \tag{18}$$

$$\sigma = {n_2}^2 \cos^2 \theta_2 + g^2 + 2gn_2\,. \tag{19}$$

Therefore the two eigenvalues that we need may be written in the abbreviated form

$$\zeta_\delta = \delta^{1/2}, \qquad \zeta_\sigma = \sigma^{1/2}. \tag{20}$$

The eigenvectors associated with these eigenvalues are found from the nonlinear system of equations based on Eq. (15). The system has a nonzero solution because its determinant is zero:

For δ polarization

$$\begin{aligned} r_1 &= \psi_{\delta 1} = E_x = 1, \\ r_2 &= \psi_{\delta 2} = H_y = n_2(n_2 - g)/\sqrt{\delta}\,, \\ r_3 &= \psi_{\delta 3} = E_y = -i(n_2 - g)/\sqrt{\delta}\,, \\ r_4 &= \psi_{\delta 4} = -H_x = -in_2\,; \end{aligned}$$

For δ polarization

$$\begin{aligned} r_1 &= \psi_{\sigma 1} = E_x = 1\,, \\ r_2 &= \psi_{\sigma 2} = H_y = n_2(n_2 + g)/\sqrt{\sigma}\,, \\ r_3 &= \psi_{\sigma 3} = E_y = i(n_2 + g)/\sqrt{\sigma}\,, \\ r_4 &= \psi_{\sigma 4} = -H_x = in_2\,. \end{aligned} \tag{21}$$

The first column of these eigenvectors we call the δ polarization, and it represents left elliptically polarized light; i.e., it has an electric vector that rotates counterclockwise for an observer looking into the beam. The second column of eigenvectors (the σ polarization) represents right elliptically polarized light. However, the difference in the amplitudes of the r_1 and r_3 components arises because of oblique incidence. The modes are circularly polarized at normal incidence. In this way the problem of reflection and refraction is solved as an eigenvalue–eigenvector problem.

The linear combination of these two eigenvectors gives the total transmitted light into the sample:

$$\psi_t = k_1\psi_\delta + k_2\psi_\sigma\,, \tag{22}$$

where ψ_t denotes the vector containing the components of the total transmitted fields and constants k_1 and k_2 are proportional to the amplitudes of the waves for the two component polarizations.

The next step of our problem is to include the boundary conditions in it; i.e., at the boundary between the achiral and the chiral materials the tangential components of the **E** and the **H** fields must be continuous:

$$\psi_i + \psi_r = \psi_t\,. \tag{22'}$$

Usually the reflection problems at oblique incidence are solved by resolving the incident and reflected fields into components parallel to the plane of incidence (p light) and components perpendicular to this plane (s light). So if ψ_i stands for the incident light, $E_{p,i}$ and $E_{s,i}$ are the amplitudes of the electric fields of incident light for these two polarizations (p and s, see Fig. 1). Therefore we write

$$\psi_i = E_{p,i}\begin{bmatrix} \cos\theta_1 \\ n_1 \\ 0 \\ 0 \end{bmatrix} + E_{s,i}\begin{bmatrix} 0 \\ 0 \\ 1 \\ n_1\cos\theta_1 \end{bmatrix}. \tag{23}$$

Similarly for the reflected light the two polarizations (p and s) may be represented as

$$\psi_r = E_{p,r}\begin{bmatrix} -\cos\theta_1 \\ n_1 \\ 0 \\ 0 \end{bmatrix} + E_{s,r}\begin{bmatrix} 0 \\ 0 \\ 1 \\ -n_1\cos\theta_1 \end{bmatrix}. \tag{24}$$

If we apply the four boundary conditions in order to relate the six quantities $E_{p,i}$, $E_{s,i}$, $E_{p,r}$, $E_{s,r}$, and the coefficients k_1 and k_2, we obtain four linear homogeneous equations. These may be represented in matrix form:

$$\begin{bmatrix} \cos\theta_1 & 0 & -\cos\theta_1 & 0 & \psi_{\delta 1} & \psi_{\sigma 1} \\ 0 & 1 & 0 & 1 & \psi_{\delta 3} & \psi_{\sigma 3} \\ n_1 & 0 & n_1 & 0 & \psi_{\delta 2} & \psi_{\sigma 2} \\ 0 & n_1\cos\theta_1 & 0 & -n_1\cos\theta_1 & \psi_{\delta 4} & \psi_{\sigma 4} \end{bmatrix} \times \begin{bmatrix} E_{p,i} \\ E_{s,i} \\ E_{p,r} \\ E_{s,r} \\ -k_1 \\ -k_2 \end{bmatrix} = 0 . \quad (25)$$

In the large matrix the second and third rows are deliberately interchanged. Also, for the next calculations the large matrix is broken into six 2×2 matrices. So Eq. (25) may be written in abbreviated form:

$$\begin{bmatrix} M_1 & M_2 & M_3 \\ M_4 & M_5 & M_6 \end{bmatrix} \begin{bmatrix} E_i \\ E_r \\ k \end{bmatrix} = 0 , \quad (25')$$

where

$$E_i = \begin{bmatrix} E_{p,i} \\ E_{s,i} \end{bmatrix}, \qquad E_r = \begin{bmatrix} E_{p,r} \\ E_{s,r} \end{bmatrix}, \qquad k = \begin{bmatrix} -k_1 \\ -k_2 \end{bmatrix}.$$

The matrix that we need is the reflection matrix; it is found by elimination of k:

$$M_1 E_i + M_2 E_r + M_3 k = 0 ,$$

$$M_4 E_i + M_5 E_r + M_6 k = 0 ,$$

or, finally,

$$[M_3 M_6^{-1} M_5 - M_2]^{-1} [M_1 - M_3 M_6^{-1} M_4] E_i = E_r .$$

So the reflection matrix r in $E_r = rE_i$ is

$$r = [M_3 M_6^{-1} M_5 - M_2]^{-1} [M_1 - M_3 M_6^{-1} M_4] . \quad (26)$$

Carrying out the detailed calculations, we obtain the following results for the Fresnel reflection amplitude coefficients:

$$r_{pp} = \frac{2\sqrt{\delta\sigma}\, n_1 n_2 + [(n_2\sqrt{\delta} + n_2\sqrt{\sigma} + g\sqrt{\delta} - g\sqrt{\sigma})(n_1^2 - n_2^2) - 2 n_1 n_2 \cos\theta_1 (n_2^2 - g^2)]\cos\theta_1}{2\sqrt{\delta\sigma}\, n_1 n_2 + [(n_2\sqrt{\delta} + n_2\sqrt{\sigma} + g\sqrt{\delta} - g\sqrt{\sigma})(n_1^2 + n_2^2) + 2 n_1 n_2 \cos\theta_1 (n_2^2 - g^2)]\cos\theta_1} , \quad (27)$$

$$r_{ps} = i\, \frac{2 n_1 n_2 [(\delta - \sigma)(n_2^2 + g^2) + 2 g n_2 (\delta + \sigma)]\cos\theta_1}{2\sqrt{\delta\sigma}\, n_1 n_2 + [(n_2\sqrt{\delta} + n_2\sqrt{\sigma} + g\sqrt{\delta} - g\sqrt{\sigma})(n_1^2 + n_2^2) + 2 n_1 n_2 \cos\theta_1 (n_2^2 - g^2)]\cos\theta_1} , \quad (28)$$

$$r_{sp} = -i\, \frac{2 n_1 n_2 [(\delta - \sigma)(n_2^2 + g^2) + 2 g n_2 (\delta + \sigma)]\cos\theta_1}{2\sqrt{\delta\sigma}\, n_1 n_2 [(n_2\sqrt{\delta} + n_2\sqrt{\sigma} + g\sqrt{\delta} - g\sqrt{\sigma})(n_1^2 + n_2^2) + 2 n_1 n_2 \cos\theta_1 (n_2^2 - g^2)]\cos\theta_1} , \quad (29)$$

$$r_{ss} = \frac{-2\sqrt{\delta\sigma}\, n_1 n_2 + [(n_2\sqrt{\delta} + n_2\sqrt{\sigma} + g\sqrt{\delta} - g\sqrt{\sigma})(n_1^2 - n_2^2) + 2 n_1 n_2 \cos\theta_1 (n_2^2 - g^2)]\cos\theta_1}{2\sqrt{\delta\sigma}\, n_1 n_2 + [(n_2\sqrt{\delta} + n_2\sqrt{\sigma} + g\sqrt{\delta} - g\sqrt{\sigma})(n_1^2 + n_2^2) + 2 n_1 n_2 \cos\theta_1 (n_2^2 - g^2)]\cos\theta_1} . \quad (30)$$

We note that when $g = 0$, then $\delta = \sigma$, r_{ss} and r_{pp} reduce to the usual p and s Fresnel reflection amplitude coefficients, and the off-diagonal elements of the reflection matrix (r_{sp} and r_{ps}) become zero. Figures 2–7 are plots of r_{ss}, r_{ss}^2, r_{pp}, r_{pp}^2, $r_{sp} = r_{ps}$, and $r_{sp}^2 = r_{ps}^2$ versus angle of incidence θ_1, with Eqs. (27)–(30) used for $Bi_{12}SiO_{20}$ ($g = 10^{-5}$, $n = 2.583$, $\lambda = 550$ nm).

4. DISCUSSION

The 4×4 matrix analysis shows that two elliptical waves are refracted into the chiral medium. These two waves have different indices of refraction and therefore propagate with different velocities. The two indices of refraction for the gyrotropic medium must satisfy

$$n_{\pm}^2 = \xi^2 + \zeta^2 = n_1^2 \sin^2\theta_1 + n_2^2 \cos^2\theta_2 + g^2 \pm 2 g n_2$$

or

$$n_{\pm} = (n_2^2 + g^2 \pm 2 g n_2)^{1/2} = \pm(n \pm g) , \quad (31)$$

where the plus in front of the parentheses has physical significance only.

The difference in refractive indices is

$$\Delta n = n_+ - n_- = 2g . \quad (32)$$

Using the two algebraic equations, we can resurrect the z components of the fields. For δ polarization,

$$E_z = -i\, \frac{n_2 g \sin\theta_2}{n_2^2 - g^2} E_y - \frac{n_2 \sin\theta_2}{n_2^2 - g^2} H_y = -\frac{n_2 \sin\theta_2}{\sqrt{\delta}} ,$$

$$H_z = \frac{n_2^3 \sin\theta_2}{n_2^2 - g^2} E_y - i\, \frac{g n_2 \sin\theta_2}{n_2^2 - g^2} H_y = -i\, \frac{n_2^2 \sin\theta_2}{\sqrt{\delta}} . \quad (33)$$

Finally, the components of the electric and magnetic fields for δ polarization are

$$E_x = 1, \qquad H_x = i n_2 ,$$

$$E_y = -i\, \frac{n_2 - g}{\sqrt{\delta}} , \qquad H_y = \frac{n_2 (n_2 - g)}{\sqrt{\delta}} ,$$

$$E_z = -\frac{n_2 \sin\theta_2}{\sqrt{\delta}} , \qquad H_z = -i\, \frac{n_2^2 \sin\theta_2}{\sqrt{\delta}} . \quad (34)$$

For σ polarization,

$$E_z = -i\, \frac{n_2 g \sin\theta_2}{n_2^2 - g^2} E_y - \frac{n_2^2 \sin\theta_2}{n_2^2 - g^2} H_y = -\frac{n_2 \sin\theta_2}{\sqrt{\sigma}} ,$$

$$H_z = \frac{n_2^3 \sin\theta_2}{n_2^2 - g^2} E_y - i\, \frac{n_2 g \sin\theta_2}{n_2^2 - g^2} H_y = +i\, \frac{n_2^2 \sin\theta_2}{\sqrt{\sigma}} . \quad (35)$$

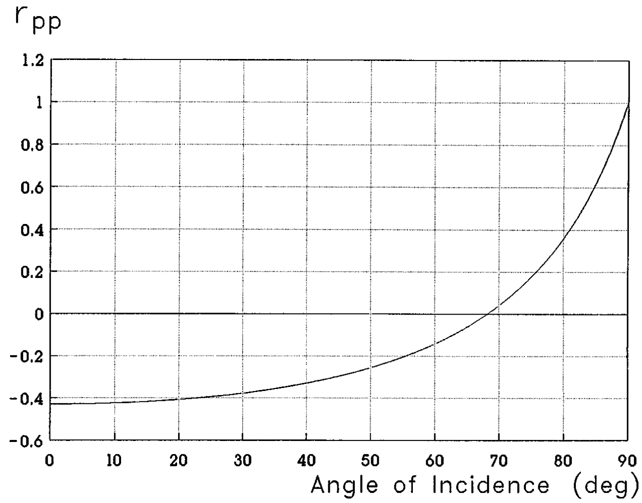

Fig. 2. Coefficient of reflection r_{pp} versus angle of incidence θ_1 for $Bi_{12}SiO_{20}$ ($g = 10^{-5}$, $n = 2.583$, $\lambda = 550$ nm).

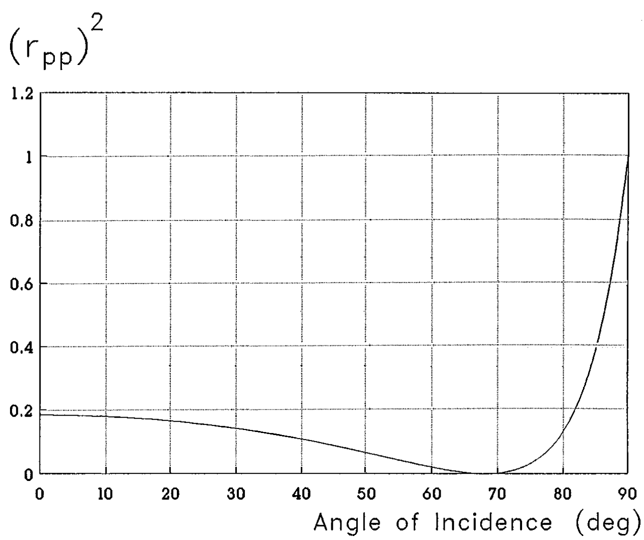

Fig. 3. Coefficient of reflection $(r_{pp})^2$ versus angle of incidence θ_1 for $Bi_{12}SiO_{20}$ ($g = 10^{-5}$, $n = 2.583$, $\lambda = 550$ nm).

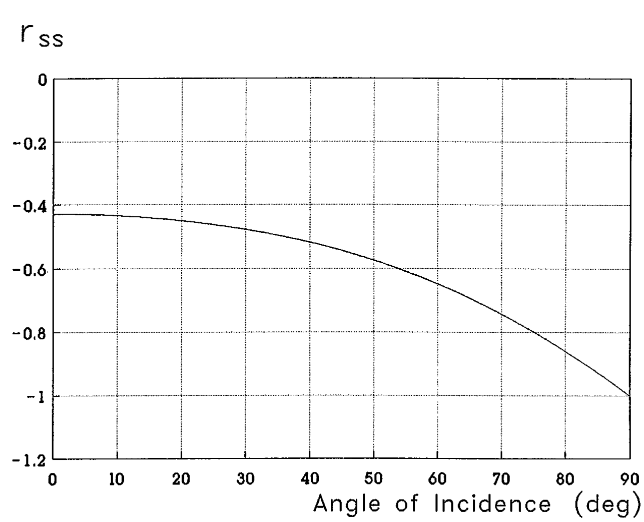

Fig. 4. Coefficient of reflection r_{ss} versus angle of incidence θ_1 for $Bi_{12}SiO_{20}$ ($g = 10^{-5}$, $n = 2.583$, $\lambda = 550$ nm).

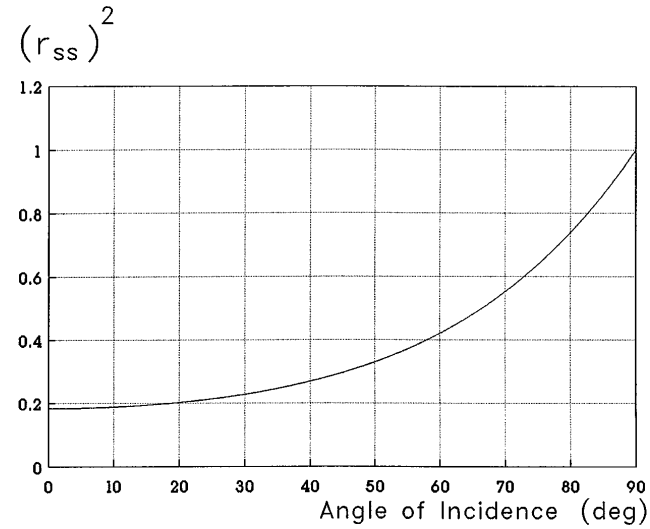

Fig. 5. Coefficient of reflection $(r_{ss})^2$ versus angle of incidence θ_1 for $Bi_{12}SiO_{20}$ ($g = 10^{-5}$, $n = 2.583$, $\lambda = 550$ nm).

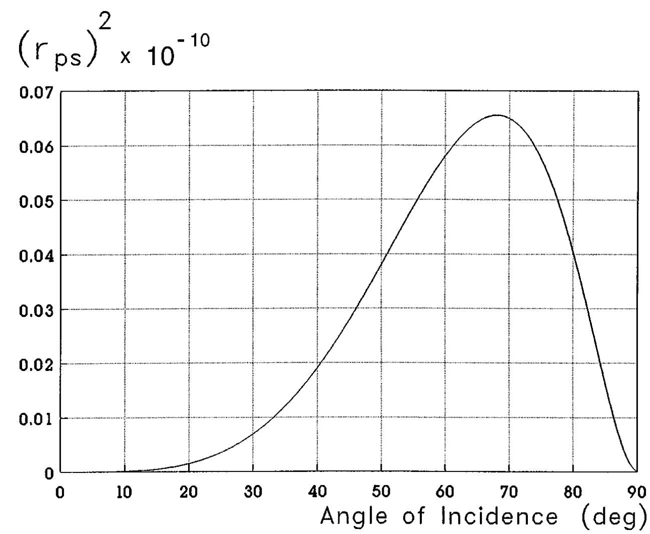

Fig. 6. Coefficient of reflection $(r_{ps})^2 = (r_{sp})^2$ versus angle of incidence θ_1 for $Bi_{12}SiO_{20}$ ($g = 10^{-5}$, $n = 2.583$, $\lambda = 550$ nm).

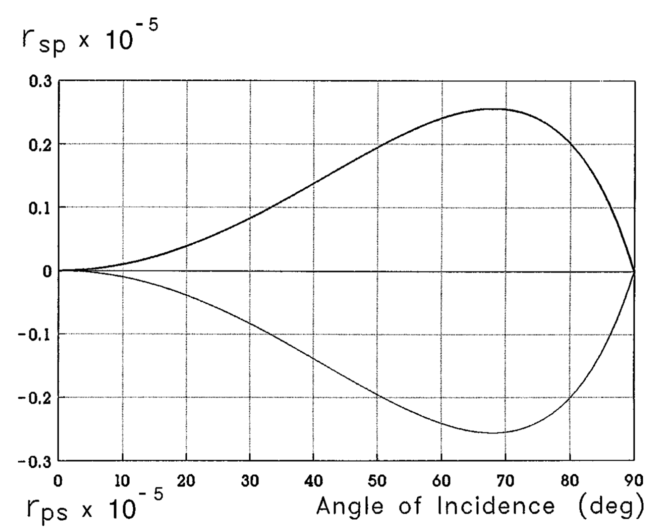

Fig. 7. Coefficients of reflection r_{ps} and $r_{sp} = -r_{ps}$ versus angle of incidence θ_1 for $Bi_{12}SiO_{20}$ ($g = 10^{-5}$, $n = 2.583$, $\lambda = 550$ nm).

So the components of σ polarization are

$$E_x = 1, \qquad H_x = -in_2\,,$$
$$E_y = i\,\frac{n_2 + g}{\sqrt{\sigma}}\,, \qquad H_y = \frac{n_2(n_2 + g)}{\sqrt{\sigma}}\,,$$
$$E_z = -\frac{n_2 \sin\theta_2}{\sqrt{\sigma}}\,, \qquad H_z = +i\,\frac{{n_2}^2 \sin\theta_2}{\sqrt{\sigma}}\,. \tag{36}$$

For both polarizations $\mathbf{E} \cdot \mathbf{H} = 0$.

Now we find the angle between the propagation vector and the surface normal for the two refracted waves:

$$\tan\theta_{\pm} = \frac{\xi}{\zeta} = \frac{n_2 \sin\theta_2}{({n_2}^2 \cos^2\theta_2 + g^2 \pm 2gn_2)^{1/2}} = \frac{\tan\theta_2}{\sqrt{1 + (g^2 \pm 2gn_2)/({n_2}^2 \cos^2\theta_2)}}\,. \tag{37}$$

Finally, we calculate the Poynting vector for each polarization, taking the cross product of the real parts of the complex fields and remembering that each field component is multiplied by $\exp(+i\omega t)$.

For δ polarization we find that

$$S_x \sim \frac{{n_2}^2(n_2 - g)\sin\theta_2}{\delta}\,,$$
$$S_y \sim 0\,,$$
$$S_z \sim \frac{n_2(n_2 - g)}{\delta^{1/2}}\,,$$

but for σ polarization they are

$$S_x \sim \frac{{n_2}^2(n_2 + g)\sin\theta_2}{\sigma}\,,$$
$$S_y \sim 0\,,$$
$$S_z \sim \frac{n_2(n_2 + g)}{\sigma^{1/2}}\,.$$

5. CONCLUSION

Using the 4 × 4 matrix method we derived the Fresnel reflection amplitude coefficients for an achiral–chiral interface at an arbitrary angle of incidence. The electric and magnetic amplitudes for the two circular polarizations propagating in the chiral medium, their angles of refraction, indices of refraction, and Poynting vectors were also found. The 4 × 4 matrix for reflection from an optically active medium is a corrected Berreman matrix, with the same coupling in the constitutive relation for the magnetic field as appears for the electric field taken into account.

APPENDIX A

In this appendix are given the corresponding formulas derived from Berreman's matrix for reflection from an isotropic optically active medium:

$$\Delta = \begin{bmatrix} 0 & \cos^2\theta_2 & -ig\sin^2\theta_2 & 0 \\ {n_2}^2 & 0 & 0 & -ig \\ 0 & 0 & 0 & 1 \\ 0 & ig & {n_2}^2 & 0 \end{bmatrix}. \tag{A1}$$

The corresponding eigenvalues are

$$\zeta = \pm\left(\frac{2{n_0}^2\cos^2\theta_2 + g^2 \pm \sqrt{4{n_0}^2 g^2 + g^4}}{2}\right)^{1/2}, \tag{A2}$$

$$\delta = \frac{1}{2}\,(2{n_0}^2\cos^2\theta_2 + g^2 - \sqrt{4g^2{n_0}^2 + g^4}\,), \tag{A3}$$

$$\sigma = \frac{1}{2}\,(2{n_0}^2\cos^2\theta_2 + g^2 + \sqrt{4g^2{n_0}^2 + g^4}\,), \tag{A4}$$

$$\zeta_\delta = \delta^{1/2}, \qquad \zeta_\sigma = \sigma^{1/2}, \tag{A5}$$

$$r_1 = \psi_{\delta 1} = -\delta^{1/2}(\delta - g^2 - {n_0}^2\cos^2\theta_2)\,,$$
$$r_2 = \psi_{\delta 2} = {n_0}^2(\delta - {n_0}^2\cos^2\theta_2)\,,$$
$$r_3 = \psi_{\delta 3} = -i{n_0}^2 g\,,$$
$$r_4 = \psi_{\delta 4} = -i\delta^{1/2}{n_0}^2 g\,,$$
$$r_1 = \psi_{\sigma 1} = E_x = -\sigma^{1/2}(\sigma - g^2 - {n_0}^2\cos^2\theta_2)\,,$$
$$r_2 = \psi_{\sigma 2} = H_y = -{n_0}^2(\sigma - {n_0}^2\cos^2\theta_2)\,,$$
$$r_3 = \psi_{\sigma 3} = E_y = i{n_0}^2 g\,,$$
$$r_4 = -H_x = i\sigma^{1/2}{n_0}^2 g\,.$$

So with the use of Eq. (A1) we find for the Fresnel reflection amplitude coefficients

$$r_{ss} = \frac{n_1 a\{[{n_2}^2(\cos^2\theta_1 - \cos^2\theta_2) - an_1\cos\theta_1 - b]b + {n_2}^4\cos^2\theta_1\cos^2\theta_2\} + {n_2}^2\cos\theta_1 bc + f - {n_1}^2\cos\theta_1(\mathrm{d}{n_2}^2\cos^2\theta_2 - e)}{n_1 a\{[{n_2}^2(\cos^2\theta_1 + \cos^2\theta_2) + an_1\cos\theta_1 + b]b + {n_2}^4\cos^2\theta_1\cos^2\theta_2\} + {n_2}^2\cos\theta_1 bc + f + {n_1}^2\cos\theta_1(\mathrm{d}{n_2}^2\cos^2\theta_2 - e)}\,,$$

$$r_{pp} = \frac{n_1 a\{[{n_2}^2(\cos^2\theta_1 - \cos^2\theta_2) + an_1\cos\theta_1 - b]b + {n_2}^4\cos^2\theta_2\} - {n_2}^2\cos\theta_1 bc - f + {n_1}^2\cos\theta_1(\mathrm{d}{n_2}^2\cos^2\theta_2 - e)}{n_1 a\{[{n_2}^2(\cos^2\theta_1 + \cos^2\theta_2) + an_1\cos\theta_1 + b]b + {n_2}^4\cos^2\theta_1\cos^2\theta_2\} + {n_2}^2\cos\theta_1 bc + f + {n_1}^2\cos\theta_1(\mathrm{d}{n_2}^2\cos^2\theta_2 - e)}\,,$$

$$r_{sp} = i\,\frac{2gn_1{n_2}^2\cos\theta_1(b + {n_2}^2\cos^2\theta_2)}{n_1 a\{[{n_2}^2(\cos^2\theta_1 + \cos^2\theta_2) + an_1\cos\theta_1 + b]b + {n_2}^4\cos^2\theta_1\cos\theta_2\} + {n_2}^2\cos\theta_1 bc + f + {n_1}^2\cos\theta_1(\mathrm{d}{n_2}^2\cos^2\theta_2 - e)}\,,$$

$$r_{ps} = i\,\frac{2n_1\cos\theta_1(b + {n_2}^2\cos^2\theta_2)(\mathrm{d}{n_2}^2\cos^2\theta_2 - e)}{g\{[n_1 ab + {n_2}^2\cos\theta_1(b + {n_2}^2\cos^2\theta_2)][n_1\cos\theta_1 a + b + {n_2}^2\cos^2\theta_2] + {n_1}^2\cos\theta_1(\mathrm{d}{n_2}^2\cos^2\theta_2 - b^2 - bg^2)\}}\,,$$

where

$$a = \sigma^{1/2} + \delta^{1/2},$$
$$b = \sigma^{1/2}\delta^{1/2},$$
$$c = \sigma^{1/2}\delta^{1/2} + 2{n_2}^2 \cos^2 \theta_2,$$
$$d = \sigma + \delta - g^2 - {n_2}^2 \cos^2 \theta_2,$$
$$e = \sigma\delta + g^2\sigma^{1/2}\delta^{1/2},$$
$$f = {n_2}^6 \cos \theta_1 \cos^4 \theta_2.$$

At $g = 0$, r_{sp} and r_{ps} become zero, while r_{ss} and r_{pp} reduce to the usual Fresnel amplitude reflection coefficients.

REFERENCES

1. V. A. Kizel and V. I. Burkov, *Gyrotropy of Crystals* (Nauka, Moscow, 1980).
2. L. D. Baron, *Molecular Light Scattering and Optical Activity* (Cambridge U. Press, New York, 1982).
3. L. Velluz, M. Legrand, and M. Grosjean, *Optical Circular Dichroism* (Academic, London, 1965).
4. R. W. Ditchburn, *Light* (Blackie, London, 1963), p. 637.
5. I. S. Zheludev, "Axial tensors of third rank and the physical phenomena they describe," Sov. Phys. Crystallogr. **9,** 501–505 (1964).
6. K. Aizu, "Ferroelectric transformation of tensorial properties in regular ferroelectrics," Phys. Rev. A **133,** 1350–1359 (1964).
7. S. Balakrishna, M. Krishnamurthi, and B. Rao, *Physics of the Solid State* (Academic, London, 1969), p. 447.
8. D. Jaggard, A. R. Mickelson, and C. H. Papas, "On electromagnetic waves in a chiral medium," Appl. Phys. **18,** 211–216 (1979).
9. T. Takizawa, "Effects of optical activity on the reflectance of paratelurite TeO_2," J. Phys. Soc. Jpn. **50,** 3054–3062 (1981).
10. M. P. Silverman, "Specular light scattering from a chiral medium: unambiguous test of gyrotropic constitutive relations," Lett. Nuovo Cimento Soc. Ital. Fis. **43,** 378–382 (1985).
11. M. P. Silverman, "Test of gyrotropic constitutive relations by specular light reflection," J. Opt. Soc. Am. A **2,** (13), p. 99 (1985).
12. M. P. Silverman, "Reflection and refraction at the surface of a chiral medium: comparison of gyrotropic constitutive relations invariant or noninvariant under a duality transformation," J. Opt. Soc. Am. A **3,** 830–837 (1986).
13. M. P. Silverman and R. B. Sohn, "Effects of circular birefringence on light propagation and reflection," Am. J. Phys. **54,** 69–76 (1986).
14. I. J. Lalov, "Effects of vibrational optical activity in the reflection spectra of crystals for the frequency regions of nondegenerate vibrations," Phys. Rev. B **32,** 6884–6891 (1985).
15. I. J. Lalov and A. I. Miteva, "Reflection optical activity of uniaxial media," J. Chem. Phys. **85,** 5505–5511 (1986).
16. A. Lakhtakia, V. V. Varadan, and V. K. Varadan, "What happens to plane waves at the planar interfaces of mirror-conjugate chiral media," J. Opt. Soc. Am. **6,** 23–26 (1989).
17. S. Bassiri, C. H. Papas, and N. Engeta, "Electromagnetic wave propagation through a dielectric chiral interface and through a chiral slab," J. Opt. Soc. Am. A **5,** 1450–1459 (1988).
18. B. V. Bokut and F. I. Fedorov, "Reflection and refraction of light in optically isotropic active media," Opt. Spectrosk. **9,** 334–336 (1960).
19. B. V. Bokut and B. A. Sotski, "The passage of light through an optically active absorbing plate," Opt. Spektrosk. **14,** 117–120 (1963).
20. F. I. Fedorov, B. V. Bokut, and A. F. Konstantinova, "Optical activity of the planar classes of the middle groups," Sov. Phys. Crystallogr. **7,** 738–744 (1963).
21. A. Lakhtakia, V. V. Varadan, and V. K. Varadan, "A parametric study of microwave reflection characteristics of a planar achiral–chiral interface," IEEE Trans. Electromagn. Compat. **28,** 90–95 (1986).
22. A. Lakhtakia, V. V. Varadan, and V. K. Varadan, "The extended boundary condition method for scattering by a chiral scatterer in a chiral medium: formulation and analysis," Optik **86,** 155–161 (1991).
23. A. Lakhtakia, V. V. Varadan, and V. K. Varadan, "Scattering by periodic achiral–chiral interfaces," J. Opt. Soc. Am. A **6,** 1675–1681 (1989); errata: **7,** 951 (1990).
24. A. Lakhtakia, V. V. Varadan, and V. K. Varadan, "Eigenmodes of a chiral sphere with a perfectly conducting coating," J. Phys. D Appl. Phys. **22,** 825–829 (1989).
25. A. Lakhtakia, V. V. Varadan, and V. K. Varadan, "Reflection of plane waves at planar achiral–chiral interfaces: independence of the reflected polarization state from the incident polarization state," J. Opt. Soc. Am. A **7,** 1654–1660 (1990).
26. A. Lakhtakia, V. V. Varadan, and V. K. Varadan, "Scattering and absorption characteristics of lossy dielectric chiral, nonspherical objects," Appl. Opt. **24,** 4146–4154 (1985).
27. A. I. Miteva and I. J. Lalov, "Reflection of electromagnetic waves from isotropic optically active media," J. Condens. Matter **2,** 529–538 (1991).
28. I. J. Lalov and A. I. Miteva, "Optically active Fabry–Perot étalon," J. Mod. Opt. **38,** 395–411 (1991).
29. H. Cory and I. Rosenhonse, "Electromagnetic wave reflection and transmission at a chiral–dielectric interface," J. Mod. Opt. **38,** 1229–1241 (1991).
30. M. P. Silverman, N. Ritchie, G. M. Cushman, and B. Fisher, "Experimental configurations using optical phase modulation to measure chiral asymmetries in light specularly reflected from a naturally gyrotropic medium," J. Opt. Soc. Am. A **5,** 1852–1862 (1988).
31. E. M. Georgieva, "Jones and Mueller matrices for specular reflection from a chiral medium: determination of the basic chiral parameters using the elements of the Mueller matrix and experimental configurations to measure the basic chiral parameters," Appl. Opt. **30,** 5081–5085 (1991).
32. D. L. Jaggard, J. C. Liu, A. Grot, and P. Pelet, "Thin wire scatterers in chiral media," Opt. Lett. **16,** 781–783 (1991).
33. D. L. Jaggard, J. C. Liu, A. Grot, and P. Pelet, "Thin wire antennas in chiral media," Electron. Lett. **27,** 243–244 (1991).
34. D. L. Jaggard and N. Engeta, "'Chirosorb' as an invisible medium," Electron. Lett. **25,** 173–174 (1989).
35. D. L. Jaggard, J. C. Liu, and X. Sun, "Spherical chiroshield," Electron. Lett. **27,** 77–79 (1991).
36. J. C. Liu and D. L. Jaggard, "Chiral layers on planar surfaces," J. Electromagn. Waves Applic. **6,** 705–721 (1992).
37. D. L. Jaggard and X. Sun, "Theory of chiral multilayers," J. Opt. Soc. Am. A **9,** 804–813 (1992).
38. E. U. Condon, "Theory of optical rotatory power," Rev. Mod. Phys. **9,** 432–457 (1937).
39. P. Drude, *The Theory of Optics* (Longmans, Green, New York, 1992), Chap. 6, Eqs. (7) and (8).
40. W. Kauzmann, *Quantum Chemistry* (Academic, New York, 1957), p. 703.
41. V. M. Agranovich and V. Ginsburg, *Crystal Optics with Spatial Dispersion and Exitons* (Springer-Verlag, Berlin, 1984), Chap. 4.
42. S. Teitler and B. W. Henvis, "Reflection in stratified anisotropic media," J. Opt. Soc. Am. **60,** 830–834 (1970).
43. P. Allia, C. Oldano, and L. Trossi, "Light propagation in anisotropic stratified media in the quasi adiabatic limit," Mol. Cryst. Liq. Cryst. **143,** 17–20 (1987).
44. K. Eidner, G. Mayer, M. Schmidt, and H. Schmidel, "Optics in stratified media—the use of optical eigenmodes of uniaxial crystals in the 4×4 matrix formalism," Mol. Cryst. Liq. Cryst. **172,** 191–200 (1989).
45. D. W. Berreman and T. J. Scheffer, "Bragg reflection of light from single-domain cholesteric liquid-crystal films," Phys. Rev. Lett. **25,** 577–581 (1970).
46. D. W. Berreman, "Optics in stratified and anisotropic media: 4×4 matrix formulation," J. Opt. Soc. Am. **62,** 502–510 (1972).

47. D. W. Berreman, "Optics in smoothly varying anisotropic planar structures: application to liquid-crystal twist cells," J. Opt. Soc. Am. **63,** 1374–1380 (1973).
48. Lin-Ching and S. Teitler, "4×4 matrix formalism for optics in stratified anisotropic media," J. Opt. Soc. Am. A **1,** 703–705 (1984).
49. R. M. Azzam and N. M. Bashara, *Ellipsometry and Polarized Light,* (North-Holland, Amsterdam, 1977) (Russian translation, 1981), p. 394.
50. A. Gerrard and J. M. Burch, *Introduction to Matrix Methods in Optics* (Wiley, London, 1975).
51. M. Born and E. Wolf, *Principles of Optics* (Pergamon, Oxford, 1964), Chap. 7.