On a min-max theorem on bipartite graphs

Zoltán Szigeti*

September 28, 2000; revised May 3, 2002

Abstract

Frank, Sebő and Tardos [4] proved that for any connected bipartite graph \( (G, T) \), the minimum size of a T-join is equal to the maximum value of a partition of \( A \), where \( A \) is one of the two colour classes of \( G \). Their proof consists of constructing a partition of \( A \) of value \(|F|\), by using a minimum T-join \( F \). That proof depends heavily on the properties of distances in graphs with conservative weightings. We follow the dual approach, that is starting from a partition of \( A \) of maximum value \( k \), we construct a T-join of size \( k \). Our proof relies only on Tutte’s theorem on perfect matchings.

It is known [5] that the results of Lovász on 2-packing of T-cuts, of Seymour on packing of T-cuts in bipartite graphs and in grafts that cannot be T-contracted onto \((K_4, V(K_4))\), and of Sebő on packing of T-borders are implied by this theorem of Frank et al.

The main contribution of the present paper is that all of these results can be derived from Tutte’s theorem.

1 Introduction

This paper concerns matchings and T-joins. Since T-joins are generalizations of matching, the minimum weight T-join problem contains the minimum weight perfect matching problem. On the other hand, Edmonds and Johnson [2] showed that the former problem can be reduced to the latter one. Thus, these problems are - in fact - equivalent.

In matching theory lots of min-max results are known. Concerning matchings, in fact, we shall consider Tutte’s theorem [11] on the existence of perfect matchings in general graphs, and not the min-max version, the Tutte-Berge formula. Concerning T-joins, we mention the following min-max theorems: The results of Edmonds-Johnson [2], Lovász [7] on 2-packing of T-cuts, of Seymour [9], [10] on packing of T-cuts in bipartite graphs and in grafts that cannot be T-contracted onto \((K_4, V(K_4))\), of Sebő [8] on packing of T-borders and a generalization of Seymour’s theorem due to Frank, Sebő and Tardos [4]. (For the definitions and the theorems see [3] or [5].) There are some easy known implications between these results, some others can be found in [5], where we showed that the result of Frank et al. [4] implies all of these results, including the Tutte theorem.

Our aim in this paper is to demonstrate a new (surprising) implication, namely, Tutte’s theorem implies the result of Frank et al. [4], and consequently, all of these min-max results can be derived from Tutte’s theorem.

2 Definitions, notation

In this paper \( H = (V, E) \) denotes a graph where \( V \) is the set of vertices and \( E \) is the set of edges. \( G = (A, B; E) \) denotes always a bipartite connected graph and \( T \subseteq A \cup B \) a subset of

*Equipe Combinatoire, Université Paris 6, 75252 Paris, Cedex 05, France.
vertices of even cardinality. The pair \((G, T)\) is called a bipartite graft. An edge set \(F \subseteq E\) is a T-join if \(T = \{v \in A \cup B : d_F(v) \text{ is odd}\}\). The minimum size of a T-join is denoted by \(\tau(G, T)\). We mention that a bipartite graft \((G, T)\) contains always a T-join.

For a bipartite graft \((G = (A; B; E), T)\) let us introduce an auxiliary graph \(G_A := (T, E_A)\) on the vertex set \(T\), where for \(u, v \in T\), \(uv \in E_A\) if at least one of \(u\) and \(v\) belongs to \(A\) and there exists a path in \(G\) connecting \(u\) and \(v\) of length one or two.

Let \(K\) be a vertex set in \(G\). Then \(\delta(K)\) denotes the set of edges connecting \(K\) and \((A \cup B) \setminus K\). \(G[K]\) denotes the subgraph induced by \(K\). \(b^T_K\) is defined to be 0 or 1 depending on the parity of \(|T \cap K|\). \(K\) is called T-odd if \(b^T_K = 1\) and T-even if \(b^T_K = 0\). For a subgraph \(K\) of \(G\), \(\overline{K} = G[V(G) \setminus V(K)]\).

We shall need the following operation applied for grafts. For a connected subgraph \(K\) of \(G\), by T-contracting \(K\) we mean the graft \((G', T')\) obtained from \((G, T)\) where \(G' = G/K\) (that is \(K\) is contracted into one vertex \(v_K\)) and \(T' = T - V(K)\) if \(b^T_K = 0\) and \(T' = T - V(K) + \{v_K\}\) if \(b^T_K = 1\).

In what follows a component of a graph means a connected component. For \(X \subseteq V(G)\), \(K(G - X)\) denotes the set of components of \(G - X\) and \(K_T(G - X)\) denotes the set of T-odd components of \(G - X\). Let \(q_T(G - X) = |K_T(G - X)|\).

We denote by \(P_A := \{u : u \in A\}\) the partition of \(A\) where the elements of \(P_A\) are the vertices in \(A\) as singletons. The value of a (sub)partition \(P = \{A_1, \ldots, A_k\}\) of \(A\) is defined to be

\[
val(P) = \sum_{A_i \in P} \{q_T(G - A_i) : A_i \in P\},
\]

in other words,

\[
val(P) = \sum_{A_i \in P} \{b^T_K : K \in \bigcup_{A_i \in P} K(G - A_i)\}.
\]

Theorem 1 (Frank, Sebő, Tardos) If \((G, T)\) is a bipartite graft with \(G = (A; B; E)\), then

\[
\tau(G, T) = \max\{val(P) : P \text{ is a partition of } A\}.
\]

In order to be able to prove Theorem 1 by induction we will have to prove a slightly stronger result than Theorem 1. To present it we need some definitions. An edge set \(C\) of a connected graph \(G\) is called bicut if \(G - C\) has exactly two connected components. Note that each edge of a tree is a bicut. Let \(P = \{A_1, \ldots, A_k\}\) be a partition of \(A\) and let \(Q = \{B_1, \ldots, B_l\}\) be a partition of \(B\). Then \(P \cup Q\) is called a bi-partition of \(A \cup B\) in \(G\). Let us denote by \(G/(P \cup Q)\) the bipartite graph obtained from \(G\) by identifying the vertices in \(R\) for every member \(R \in P \cup Q\) and by taking the underlying simple graph. A bi-partition \(P \cup Q\) of \(A \cup B\) is called admissible if

(i) \(F := G/(P \cup Q)\) is a tree, and

(ii) for each edge \(e\) of \(F\), the edge set of \(G\) that corresponds to \(e\) forms a bicut of \(G\).

By Claim 4, for any bipartite graft there exists an admissible bi-partition.

Theorem 2 If \((G, T)\) is a bipartite graft with \(G = (A; B; E)\), then

\[
\tau(G, T) = \max\{val(P) : P \cup Q \text{ is an admissible bi-partition of } A \cup B\}.
\]
bi-partition of $A \cup B$ of maximum value $k$, we construct a $T$-join of size $k$. Our proof applies induction. Taking a special optimal admissible bi-partition either we can use induction for some contracted graphs (and here we need admissibility of the bi-partition) or we can apply Tutte’s theorem on perfect matchings, namely a graph $H$ has a perfect matching if and only if $q_V(H - X) \leq |X|$ for every vertex set $X$ of $V(H)$.

We must mention two papers on this topic. Kostochka [6] and Ageev and Kostochka [1] proved results similar to Theorem 2. Their proof technique is different from the present one.

3 Preliminary results

Claim 3 Let $(G = (A, B; E), T)$ be a bipartite graft.

$(a)$ Then the bi-partition $P \cup Q$ of $A \cup B$ satisfies (i) where $P := \{a : a \in A\}$ and $Q := \{B\}$.

$(b)$ If $X \subseteq A$, then the bi-partition $P \cup Q$ of $A \cup B$ satisfies (i) where $P := \{a : a \in A - X\} \cup \{X\}$ and $Q := \{K \cap B : K \in \mathcal{K}(G - X)\}$.

The following claim (whose proof is left for the reader) shows that for any bipartite graft there exists an admissible bi-partition.

Claim 4 Let $(G = (A, B; E), T)$ be a bipartite graft.

$(a)$ If there is no cut vertex in $A$ then $P \cup Q$ is an admissible bi-partition of $A \cup B$, where $P := \{a : a \in A\}$ and $Q := \{B\}$.

$(b)$ If there is a cut vertex $v \in A$, that is $G$ can be decomposed into two connected bipartite subgraphs $G_1 = (A_1, B_1; E_1)$ and $G_2 = (A_2, B_2; E_2)$ with exactly one vertex in common, namely $v$, then let us denote by $(G_1, T_1)$ and $(G_2, T_2)$ the two grafts obtained from $(G, T)$ by $T$-contracting $V(G_2)$ and $V(G_1)$. If for $i = 1, 2$, $P_i \cup Q_i$ is an admissible bi-partition of $A_i \cup B_i$ and $v \in A'_i$, then $P \cup Q$ is an admissible bi-partition of $A \cup B$, where $P := (P_1 - A'_1) \cup (P_2 - A'_2) \cup \{A'_1 \cup A'_2\}$ and $Q := Q_1 \cup Q_2$.

The definition of an admissible bi-partition implies at once the following claim.

Claim 5 Let $P \cup Q$ be an admissible bi-partition of $A \cup B$.

$(a)$ $K \in \mathcal{K}_T(G - A_i)$ for some $A_i \in \mathcal{P}$ if and only if $R \in \mathcal{K}_T(G - B_j)$ for some $B_j \in \mathcal{Q}$.

$(b)$ $\text{val}(P) = \text{val}(Q)$.

Claim 6 Let $P$ be a partition of $A$ and $F$ a $T$-join in a bipartite graft $(G = (A, B; E), T)$.

$(a)$ Then $\text{val}(P) \leq |F|$.

$(b)$ Moreover, if $\text{val}(P) = |F|$, then for every component $K$ of $G - A_i$ for any $A_i \in \mathcal{P}$, $|\delta(K) \cap F| = b^P_K$.

Proof. Let $\mathcal{R} := \bigcup_{A_i \in \mathcal{P}} \mathcal{K}(G - A_i)$. By parity, for each $K \in \mathcal{R}$,

$$b^P_K \leq |\delta(K) \cap F|.$$ 

Since for $K_1, K_2 \in \mathcal{R}$, $\delta(K_1) \cap \delta(K_2) = \emptyset$, we have

$$\text{val}(P) = \sum_{K \in \mathcal{R}} b^P_K \leq \sum_{K \in \mathcal{R}} |\delta(K) \cap F| \leq |F|.$$

$\square$
Claim 7 For every partition $\mathcal{P}$ of $A$ in a bipartite graft $(G = (A, B; E), T)$, 
\[ \text{val}(\mathcal{P}) \equiv |T \cap A| \pmod{2}. \]

Proof. Since $|T|$ is even, for each $A_i \in \mathcal{P}$, $q_T(G - A_i) \equiv |T \cap A_i| \pmod{2}$. Thus
\[ \text{val}(\mathcal{P}) = \sum_{A_i \in \mathcal{P}} q_T(G - A_i) \equiv \sum_{A_i \in \mathcal{P}} |T \cap A_i| = |T \cap A|. \]

We shall deal with some bi-partitions along the proofs. The admissibility of these bi-partitions can always be easily verified. The following easy fact may be useful.

Claim 8 Let $X$ be a subset of vertices of a connected graph $H$. Let $K$ be a component of $H - X$. If $X$ is contained in one of the components of $H - K$, then $H - K$ is connected.

Claim 9 Let $H$ be a connected graph with $|V(H)|$ even. If $X$ is a minimal vertex set with $q_V(H - X) > |X| + 2$, then for every component $K$ of $H - X$, $H - K$ is connected.

Claim 10 Let $(G = (A, B; E), T)$ be a bipartite graft. If the auxiliary graph $G_A$ has a perfect matching $M$ then $G$ contains a $T$-join of cardinality $|T \cap A|$.

Proof. For every edge $uv \in M$ there exists an $(u, v)$-path in $G$ of length at most two. Since $M$ is a matching these paths are edge disjoint. The union $F$ of these paths is a $T$-join of $G$ because $M$ covers all the vertices of $T$. By construction, $|F| = |T \cap A|$.

4 The proof of Theorem 2

Let $(G, T)$ be a counterexample with minimum number of vertices in $G$. By Claim 6(a), for any admissible bi-partition $\mathcal{P} \cup \mathcal{Q}$ of $A \cup B$, $\text{val}(\mathcal{P}) \leq \tau(G, T)$, so $\text{val}(\mathcal{P}) < \tau(G, T)$.

Lemma 11 $G$ is 2-connected.

Proof. Suppose that $G$ contains a cut vertex $v$, by symmetry we may suppose that $v \in A$. We use the notation of Claim 4. For $i = 1, 2$, $(G_i, T_i)$ is a bipartite graft and $|A_i \cup B_i| < |A \cup B|$ so there exists an admissible bi-partition $\mathcal{P}_i \cup \mathcal{Q}_i$ of $A_i \cup B_i$ with
\[ \tau(G_i, T_i) = \text{val}(\mathcal{P}_i). \] (5)

Clearly,
\[ \tau(G, T) = \tau(G_1, T_1) + \tau(G_2, T_2). \] (6)

Let $\mathcal{P} \cup \mathcal{Q}$ be the admissible bi-partition of $A \cup B$ defined in Claim 4(b). Note that
\[ \text{val}(\mathcal{P}) = \text{val}(\mathcal{P}_1) + \text{val}(\mathcal{P}_2). \] (7)

Then, by (6), (5) and (7), $\tau(G, T) = \text{val}(\mathcal{P})$ showing that $(G, T)$ is not a counterexample.

Let us denote by MAX the maximum value of an admissible bi-partition of $A \cup B$. Observe that $\text{MAX} \geq |T \cap A|$ and $\text{MAX} \geq |T \cap B|$. The first comes from the admissible bi-partition $\mathcal{P} = \{v : v \in A\}, \mathcal{Q} = \{B\}$, the other one from $\mathcal{P} = \{A\}, \mathcal{Q} = \{v : v \in B\}$. These bi-partitions are admissible by Claim 4(a).

CASE 1. First suppose that $\text{MAX} = |T \cap A|$ (or $\text{MAX} = |T \cap B|$).
Lemma 12 If the auxiliary graph $G_A$ has no perfect matching then there exists an admissible bi-partition $\mathcal{P} \cup \mathcal{Q}$ of $A \cup B$ with $\text{val}(\mathcal{P}) > |T \cap A|$.

Proof. By Tutte’s Theorem, there exists a set $X \subset T$ so that $q_T(G_A - X) > |X|$. Let us take a minimal such set.

We claim that $X \cap B = \emptyset$. Suppose that $a \in X \cap B$. Suppose that $a$ is connected to two odd components $K_1$ and $K_2$ of $G_A - X$. Then, by the definition of $G_A$, there is an edge between $K_1$ and $K_2$, that is they cannot be different components of $G_A - X$. Thus $a$ is connected to at most one odd component of $G_A - X$. Hence $q_T(G_A - (X - a)) \geq q_T(G_A - X) - 1 \geq |X| > |X - a|$, contradicting the minimality of $X$.

Let us denote by $B_1$ the set of vertices in $B - T$ that has at least one neighbour in $A \cap T$ and let $B_2 := B - T - B_1$. Let $G_1 := G[T \cup B_1]$ and $G_2 := G[(A - T) \cup B_2]$. Note that by the definition of $G_A$ there is a bijection between the components of $G_A - X$ and the components of $G_1 - X$ different from isolated vertices in $B_1$. Moreover, the $T$ parity of the corresponding components are the same. Let $R = K(G_2)$. Note that if $R \in \mathcal{R}$ then there is no edge between $R \cap B_2$ and $A \cap T$. We distinguish two cases.

Case I. First suppose that $X = \emptyset$, that is $q_T(G_1) \geq 1$, in other words $q_T(G - (A - T)) \geq 1$. Let $R_1 \subseteq \mathcal{R}$ be a minimal subset of $\mathcal{R}$ so that $q_T(G - A') \geq 1$, where $A' := \{R \cap A : R \in R_1\}$. Let $\mathcal{P} = \{u : u \in A - A'\} \cup \{A'\}$ and let $\mathcal{Q} = \{R \cap B : R \in K(G - A')\}$. By Claim 3(b), $\mathcal{P} \cup \mathcal{Q}$ satisfies (i). Since $A' \subseteq A - T$, $|(V(G - A') \cap T)|$ is even so $q_T(G - A') \geq 2$ and, by the minimality of $R_1$, each such component has at least one neighbour in every $R \in R_1$. Since $G$ is 2-connected and for every $R \in R_1$, $G[R]$ is connected, it follows that for every $D \in K(G - A')$, $G - D$ is connected, that is (ii.) is also satisfied, so $\mathcal{P} \cup \mathcal{Q}$ is an admissible bi-partition and

$$\text{val}(\mathcal{P}) = \sum_{A_i \in \mathcal{P}} q_T(G - A_i) \geq \sum_{t \in A - A'} b_t^T + q_T(G - A') \geq |T \cap A| + 2.$$ 

Case II. Secondly suppose that $X \neq \emptyset$. By the minimality of $X$, $X \subset V(G')$ where $G' \in K(G_1)$. Let $R_1 \subseteq \mathcal{R}$ be a minimal subset of $\mathcal{R}$ so that all the components of $G' - X$ rest in different components of $G - A'' - X$, where $A'' := \{R \cap A : R \in R_1\}$. Let $\mathcal{P} := \{X \cup A''\} \cup \{u : u \in A - (X \cup A'')\}$ and let $\mathcal{Q} := \{R \cap B : R \in K(G - X - A'')\}$. By Claim 3(b), $\mathcal{P} \cup \mathcal{Q}$ satisfies (i). For each $R \in R_1$, $G[R]$ is connected and, by the minimality of $R_1$, $R$ has neighbours in at least two different components of $G - X - A''$. Moreover, by Claim 9, for each $K \in K(G' - X)$, $G' - K$ is connected, hence $(G - \bigcup\{R : R \in R_1\}) - K'$ is connected, where $K' \in K(G - X - A'')$ that contains $K$. It follows that $X \cup A''$ is contained in one of the components of $G - K'$. Thus, by Claim 8 and by 2-connectivity, $\mathcal{P} \cup \mathcal{Q}$ is an admissible bi-partition of $A \cup B$ and

$$\text{val}(\mathcal{P}) = \sum_{A_i \in \mathcal{P}} q_T(G - A_i) = \sum_{t \in A - X - A''} b_t^T + q_T(G - (X \cup A''))$$

$$= |A \cap T| - |X| + q_T(G_A - X) > |T \cap A|.$$ 

By Lemma 12, $G_A$ (or $G_B$, resp.) has a perfect matching and thus, by Claim 10, $G$ contains a T-join of cardinality $|T \cap A| (|T \cap B|$, resp.). By Claim 6, the proof of the theorem is complete.

CASE 2. Secondly suppose that MAX $> |T \cap A|$ and MAX $> |T \cap B|$. Then, by Lemma 11, every optimal admissible bi-partition contains a set $A_1$ with $1 < |A_1| < |A|$. Let us choose an optimal admissible bi-partition $\mathcal{P} \cup \mathcal{Q}$ of $A \cup B$ so that such a set $A_1$ of $\mathcal{P}$ is as large as possible. Let $K \in K(G - A_1)$ so that $|V(K)| \geq 2$. (Since $|A_1| < |A|$ such a set exists.) Then, by Claim 5, $K \in K(G - B_j)$ for some $B_j \in \mathcal{Q}$ and $|V(K)| \geq 2$. Let us denote by $(G_1, T_1)$ and $(G_2, T_2)$ the two bipartite grafts obtained from $(G, T)$ by T-contracting the connected subgraphs $K$ and $K$, respectively. The colour classes of $G_r$ will be denoted by $A'$ and $B'$, while the contracted
vertex of \( G_r \) is denoted by \( v_r \) for \( r = 1, 2 \). Let \( \mathcal{P}_1 := \{ A_k \in \mathcal{P} : A_k \subseteq A^1 \} \) and \( \mathcal{Q}_1 := \{ B_l \in \mathcal{Q} : B_l \subseteq B^1 \} \cup \{ v_1 \} \). Let \( \mathcal{P}_2 := \{ A_k \in \mathcal{P} : A_k \subseteq A^2 \} \cup \{ v_2 \} \) and \( \mathcal{Q}_2 := \{ B_l \in \mathcal{Q} : B_l \subseteq B^2 \} \). The admissibility of the bi-partition \( \mathcal{P} \cup \mathcal{Q} \) implies the following Claim.

**Claim 13**  
(a) \( \mathcal{P}_r \cup \mathcal{Q}_r \) is an admissible bi-partition of \( A' \cup B' \) in \( G_r, r = 1, 2 \).

(b) \( \text{val}(G,T)(\mathcal{P}) = \text{val}(G,\mathcal{T}_1)(\mathcal{P}_1) - b_{v_1}^{Ti_1} + \text{val}(G,\mathcal{T}_2)(\mathcal{P}_2) \).

**Lemma 14**  
For \( r = 1, 2 \), \( \mathcal{P}_r \cup \mathcal{Q}_r \) is an optimal admissible bi-partition of \( A' \cup B' \) in \( (G_r, T_r) \).

**Proof.** By Claim 13(a), only the optimality must be verified. By symmetry, it is enough to prove it for \( r = 2 \). Suppose that \( \mathcal{P}' \cup \mathcal{Q}' \) is an admissible bi-partition of \( A^2 \cup B^2 \) in \( G_2 \) with \( \text{val}(G,\mathcal{T}_2)(\mathcal{P}') > \text{val}(G,\mathcal{T}_2)(\mathcal{P}_2) \). Let us denote by \( X \) that member of \( \mathcal{P}' \) that contains \( v_2 \). Since \( \mathcal{P}_1 \cup \mathcal{Q}_1 \) and \( \mathcal{P}' \cup \mathcal{Q}' \) are admissible bi-partitions and \( K \) is connected, \( \mathcal{P}_2 := (\mathcal{P}_1 - A_1) \cup (\mathcal{P}' - X) \cup \{(X - v_2) \cup A_1\} \), \( \mathcal{Q}_2 := (\mathcal{Q}_1 - \{v_1\}) \cup \mathcal{Q}' \) is an admissible bi-partition of \( A \cup B \) in \( G \). By Claim 13(b),

\[
\text{val}(G,T)(\mathcal{P}') = \text{val}(G,\mathcal{T}_1)(\mathcal{P}_1) - b_{v_1}^{Ti_1} + \text{val}(G,\mathcal{T}_2)(\mathcal{P}_2) > \text{val}(G,\mathcal{T}_1)(\mathcal{P}_1) - b_{v_1}^{Ti_1} + \text{val}(G,\mathcal{T}_2)(\mathcal{P}_2) = \text{val}(G,T)(\mathcal{P}),
\]

a contradiction.

**Lemma 15**  
If \( K \) is \( T \)-odd, then for every edge \( v_2u \) of \( G_2 \), \( \mathcal{P}_2 \cup \mathcal{Q}_2 \) is an optimal admissible bi-partition of \( A^2 \cup B^2 \) in \( (G_2, T_2) \) of value \( \text{val}(G,\mathcal{T}_2)(\mathcal{P}_2) - 1 \), where \( T_2 := T_2 \cup \{v_2, u\} \).

**Proof.** By Claim 13(a), only the optimality must be verified. \( \text{val}(G,\mathcal{T}_2)(\mathcal{P}_2) = \text{val}(G,\mathcal{T}_2)(\mathcal{P}_2) - 1 \) because for a component \( L \) of \( G_2 - R \) with \( R \in \mathcal{P}_2 - \{v_2\} \), \( L \cap T_2 = \{L \cap T_2\} \mod 2 \) and the unique component \( K \) of \( G_2 - v_2 \) becomes \( T_2 \)-even. Suppose that \( \mathcal{P}' \cup \mathcal{Q}' \) is an admissible bi-partition of \( A^2 \cup B^2 \) in \( G_2, T_2 \) with \( \text{val}(G,\mathcal{T}_2)(\mathcal{P}') > \text{val}(G,\mathcal{T}_2)(\mathcal{P}_2) - 1 \). By Claim 7, \( \text{val}(G,\mathcal{T}_2)(\mathcal{P}') \geq \text{val}(G,\mathcal{T}_2)(\mathcal{P}_2) + 1 \). Note that since \( K \) is \( T \)-odd, \( b_{v_1}^{Ti_1} = 1 \). Let us denote by \( X \) that member of \( \mathcal{P}' \) that contains \( v_2 \). Since \( K \) and \( K \) are connected, \( \mathcal{P}_2 := (\mathcal{P}_1 - A_1) \cup (\mathcal{P}' - X) \cup \{(X - v_2) \cup A_1\} \), \( \mathcal{Q}_2 := (\mathcal{Q}_1 - \{v_1\}) \cup \mathcal{Q}' \) is an admissible bi-partition of \( A \cup B \) in \( G \).

If \( X = v_2 \) then, by Claim 13(b),

\[
\text{val}(G,T)(\mathcal{P}') = \text{val}(G,\mathcal{T}_1)(\mathcal{P}_1) + \text{val}(G,\mathcal{T}_2)(\mathcal{P}') \geq \text{val}(G,\mathcal{T}_1)(\mathcal{P}_1) + \text{val}(G,\mathcal{T}_2)(\mathcal{P}_2) + 1 > \text{val}(G,T)(\mathcal{P}),
\]

a contradiction.

If \( X \neq v_2 \), then, by Claim 13(b),

\[
\text{val}(G,T)(\mathcal{P}') \geq (\text{val}(G,\mathcal{T}_1)(\mathcal{P}_1) - 1) + (\text{val}(G,\mathcal{T}_2)(\mathcal{P}') - 1) \geq \text{val}(G,\mathcal{T}_1)(\mathcal{P}_1) - 1 + \text{val}(G,\mathcal{T}_2)(\mathcal{P}_2) = \text{val}(G,T)(\mathcal{P}),
\]

that is \( \mathcal{P}' \cup \mathcal{Q}' \) is an optimal admissible bi-partition of \( A \cup B \) in \( G \), but \(|(X - v_2) \cup A_1| > |A_1|\), contradicting the maximality of \( A_1 \).

By induction \(|V(G)| < |V(K)| \) because \(|V(K)| > 2\) and by Lemma 14, there exists a \( T_1 \)-join \( F_1 \) in \( G_1 \) with \(|F_1| = \text{val}(\mathcal{P}_1)\).

First suppose that \( K \) is a \( T \)-even component of \( G - A_1 \). Then, by Claim 14, there exists a \( T_2 \)-join \( F_2 \) in \( G_2 \) with \(|F_2| = \text{val}(\mathcal{P}_2)\). Hence, \( F := F_1 \cup F_2 \) is a \( T \)-join and, by Claim 13(b), \(|F| = |F_1| + |F_2| = \text{val}(\mathcal{P}_1) + \text{val}(\mathcal{P}_2) = \text{val}(\mathcal{P})\). By Claim 6, we are done.

Now suppose that \( K \) is a \( T \)-odd component of \( G - A_1 \). Then, by Claim 14, \(|F_1 \cap \delta(K)| = 1\). This edge corresponds to an edge \( v_2u \) in \( G_2 \). By induction \(|V(G)| < |V(K)| \) because \(|V(K)| \geq 2\) and by Lemma 15 with edge \( v_2u \), there exists a \( T_2 \)-join \( F_2 \) in \( G_2 \) with \(|F_2| = \text{val}(\mathcal{P}_2) - 1\). Then \( F := F_1 \cup F_2 \) is a \( T \)-join and, by Claim 13(b), \(|F| = |F_1| + |F_2| = \text{val}(\mathcal{P}_1) + \text{val}(\mathcal{P}_2) - 1 = \text{val}(\mathcal{P})\). By Claim 6, we are done.
References


