

# Variational multi-fluid dynamics and causal heat conductivity

BY N. ANDERSSON<sup>1</sup> AND G. L. COMER<sup>2</sup>

<sup>1</sup> *School of Mathematics, University of Southampton, UK*

<sup>2</sup> *Department of Physics and Center for Fluids at All Scales, Saint Louis University, USA*

We discuss heat conductivity from the point of view of a variational multi-fluid model, treating entropy as a dynamical entity. We demonstrate that a two-fluid model with a massive fluid component and a massless entropy can reproduce a number of key results from extended irreversible thermodynamics. In particular, we show that the entropy entrainment is intimately linked to the thermal relaxation time that is required to make heat propagation in solids causal. We also discuss non-local terms that arise naturally in a dissipative multi-fluid model, and relate these terms to those of phonon hydrodynamics. Finally, we formulate a complete heat conducting two-component model and discuss briefly the new dissipative terms that arise.

**Keywords:** Fluid dynamics; Heat conductivity

## 1. Introduction

Heat conductivity is a central problem in thermodynamics. It is well known that the classical description in irreversible thermodynamics, essentially Fourier's law, has unattractive features. In particular, it predicts an instantaneous propagation of thermal signals. This is in contradiction to the expected hyperbolic nature of physical laws. In fact, the associated non-causality would be completely unacceptable within a relativistic theory. Resolving this issue has been a main motivating factor behind the development of extended irreversible thermodynamics (Jou et al, 1993; Müller & Ruggeri, 1993), a model which introduces additional dynamical fields in order to retain hyperbolicity and causality. This model has proved useful in various application areas, ranging from superfluid systems to heat conduction in solids.

In this paper we consider the problem of heat conductivity within the flux-conservative multi-fluid framework developed by Andersson & Comer (2006) (with the corrections discussed by Haskell et al (2009)). This model builds on the non-dissipative variational model developed by Prix (2004), and represents the natural non-relativistic counterpart to Carter's convective variational hydrodynamics in general relativity (Carter, 1989), see Andersson & Comer (2007) for a recent review. These models have so far primarily been used to investigate the dynamics of compact stars (see Andersson & Comer (2001); Glampedakis et al (2007); Glampedakis & Andersson (2009); Passamonti et al (2009) for details). Yet, one would expect the general framework to be universally relevant. It is therefore of some interest to consider applications in other problem areas. This serves several purposes. First of all, it is often the case that the elegant geometric view of a relativistic analysis

(where time and space are treated “equally”) simplifies the description of a complex system. The close link between the variational multi-fluid model and the relativistic counterpart may lead to insights that would be hard to reach otherwise. Secondly, it is interesting to learn from the requirements of models for a range of different systems. Ultimately, this will improve our understanding of the general multi-fluid framework, and the role of the various parameters in the model.

In this paper we focus on the simplest “conducting” system, with a single species of particle together with a massless entropy component. If the entropy drifts relative to the particles, the system is heat conducting. Throughout the discussion we will assume that the entropy can be treated as a (massless) fluid. In essence, this means that the “phonon mean free path” is not too large. We aim to demonstrate that our flux-conservative formulation for this system captures key aspects of extended thermodynamics. We illustrate this by writing down a model for heat conductivity in a rigid solid and comparing to various results in the literature. This exercise makes it clear that our model incorporates a finite propagation speed for heat. We also learn that the associated relaxation timescale is directly linked to the entrainment between particles and entropy. This entropy entrainment also played a central role in our recent discussion of finite temperature superfluids (Andersson & Comer, 2009). In fact, the model we consider here is formally equivalent to our recent model for superfluid Helium, the key element being that the massless entropy component is allowed to flow relative to the particles in the system. However, the discussion here differs in that we do not impose the irrotationality constraint associated with a superfluid system. Instead, we focus on the heat conductivity. We wish to understand to what extent this, conceptually rather elegant, model captures the complex physics associated with heat flow. As an interesting by-product, the present discussion suggests how the Helium model could be extended (using a three-fluid model) to account for the thermal conductivity associated with the interaction between rotons and phonons that dominates at higher temperatures (Khalatnikov, 1965). This is an interesting problem since the development of a causal model for this phenomenon is still outstanding. Our analysis also hints at the key ingredients of a causal relativistic model for heat conductivity. The main lesson is that the entropy entrainment must be retained in order to avoid pathological behaviour.

## 2. Multi-Fluid formulation

We consider a simple system with two dynamical degrees of freedom. We distinguish between the mass carrying “atoms” and the massless “entropy”. The former will be identified by constituent index  $n$ , it has mass  $m$ , number density  $n$  and flows with a velocity  $v_n^i$ , while the latter is represented by  $s$ , with number density  $s$  (the entropy per unit volume) and a flow given by  $v_s^i$ . The canonical momentum (density) for each fluid is determined by the variational analysis of Prix (2004). This leads to

$$\pi_i^n = mnv_i^n - 2\alpha w_i^{ns} , \quad (2.1)$$

for the atoms and

$$\pi_i^s = 2\alpha w_i^{ns} , \quad (2.2)$$

for the entropy. This latter relation encodes the inertia of heat. We see that the heat’s inertia vanishes if the entrainment between the two components, quantified

by  $\alpha$ , vanishes. In these expressions we have used the relative velocity  $w_i^{\text{ns}} = v_i^{\text{n}} - v_i^{\text{s}}$ . We recall that the variational analysis takes as its starting point an energy functional  $E$ , representing the equation of state. The coefficient  $\alpha$  then follows from

$$\alpha = \left. \frac{\partial E}{\partial w_{\text{ns}}^2} \right|_{n,s} . \quad (2.3)$$

As discussed by Andersson & Comer (2006, 2009) the momentum equations for the dissipative two-component system can be written

$$f_i^{\text{n}} = \partial_t \pi_i^{\text{n}} + \nabla_j (v_{\text{n}}^j \pi_i^{\text{n}} + D_{ij}^{\text{n}}) + n \nabla_i \left( \mu_{\text{n}} - \frac{1}{2} m v_{\text{n}}^2 \right) + \pi_j^{\text{n}} \nabla_i v_{\text{n}}^j , \quad (2.4)$$

and

$$f_i^{\text{s}} = \partial_t \pi_i^{\text{s}} + \nabla_j (v_{\text{s}}^j \pi_i^{\text{s}} + D_{ij}^{\text{s}}) + s \nabla_i T + \pi_j^{\text{s}} \nabla_i v_{\text{s}}^j . \quad (2.5)$$

Here the terms  $D_{ij}^{\text{n}}$  and  $D_{ij}^{\text{s}}$  represent the dissipation and  $f_{\text{n}}^i$  and  $f_{\text{s}}^i$  are the respective forces acting on the matter and entropy. The chemical potentials are determined by

$$\mu_{\text{n}} = \left. \frac{\partial E}{\partial n} \right|_{s, w_{\text{ns}}^2} , \quad (2.6)$$

and

$$\mu_{\text{s}} = \left. \frac{\partial E}{\partial s} \right|_{n, w_{\text{ns}}^2} \equiv T , \quad (2.7)$$

where we have identified the temperature,  $T$ , in the usual way (note that we use units such that Boltzmann's constant is equal to unity,  $k_B = 1$ )<sup>†</sup>.

Since we have no particle creation or destruction, mass conservation leads to

$$\partial_t n + \nabla_j (n v_{\text{n}}^j) = 0 . \quad (2.8)$$

At the same time the entropy can increase, so we have

$$\partial_t s + \nabla_j (s v_{\text{s}}^j) = \Gamma_{\text{s}} , \quad (2.9)$$

where the second law of thermodynamics requires  $\Gamma_{\text{s}} \geq 0$ . Finally, the dissipative terms are constrained by the fact that we consider a closed system. As discussed by Andersson & Comer (2006), this means that we must have

$$f_i^{\text{s}} = -f_i^{\text{n}} , \quad (2.10)$$

and

$$D_{ij}^{\text{s}} = D_{ij} - D_{ij}^{\text{n}} . \quad (2.11)$$

Here,  $D_{ij}$  is the total ‘‘dissipation’’. It is distinguished by the fact that it is symmetric in its indices whereas  $D_{ij}^{\text{n}}$  and  $D_{ij}^{\text{s}}$  do not have to be.

<sup>†</sup> It should be pointed out that the notion of temperature in a non-equilibrium system is non-trivial, see Casas-Vázquez & Jou (2003) for a thorough discussion. In the present analysis we assume that the temperature is obtained from (2.7), i.e. in the same way as in thermal equilibrium. This ‘‘operational’’ definition seems the most natural in the present context.

Following the steps taken in Andersson & Comer (2009), i.e. constructing the dissipative terms from the relevant thermodynamic fluxes and taking account of the Onsager symmetry, we arrive at<sup>‡</sup>

$$-f_i^n = 2\mathcal{R}^{\text{nn}}w_i^{\text{ns}} + 2\mathcal{S}^{\text{nn}}W_i^{\text{ns}} , \quad (2.12)$$

$$-D_{ij}^n = \mathcal{S}^{\text{nn}}\epsilon_{ijk}w_{\text{ns}}^k + g_{ij}(\zeta^{\text{nn}}\Theta_{\text{ns}} + \zeta^n\Theta_s) + 2\eta^{\text{nn}}\Theta_{ij}^{\text{ns}} + 2\eta^n\Theta_{ij}^s + \sigma^{\text{nn}}\epsilon_{ijk}W_{\text{ns}}^k , \quad (2.13)$$

and

$$-D_{ij} = g_{ij}(\zeta^n\Theta_{\text{ns}} + \zeta\Theta_s) + 2\eta^n\Theta_{ij}^{\text{ns}} + 2\eta\Theta_{ij}^s . \quad (2.14)$$

Here we have defined the expansion

$$\Theta_s = \nabla_j v_s^j , \quad (2.15)$$

the trace-free shear

$$\Theta_{ij}^s = \frac{1}{2} \left( \nabla_i v_j^s + \nabla_j v_i^s - \frac{2}{3}g_{ij}\Theta_s \right) , \quad (2.16)$$

and the ‘‘vorticity’’

$$W_s^i = \frac{1}{4}\epsilon^{ijk}(\nabla_j v_k^s - \nabla_k v_j^s) , \quad (2.17)$$

of the entropy flow. The quantities  $\Theta_{\text{ns}}$ ,  $\Theta_{ij}^{\text{ns}}$  and  $W_{\text{ns}}^k$  are constructed from  $w_{\text{ns}}^i$  analogously.

So far, the development has been formal. The model was developed by combining the non-dissipative equations of motion from the variational analysis (Prix, 2004) with the general form for the dissipative terms assuming a quadratic deviation from thermodynamic equilibrium. We have accounted for the Onsager symmetry between the dissipative terms, and imposed the conservation of total angular momentum. This reduces the problem to one with nine, as yet unspecified, dissipation coefficients;  $\mathcal{R}^{\text{nn}}$ ,  $\mathcal{S}^{\text{nn}}$ ,  $\zeta^n$ ,  $\eta^n$ ,  $\zeta^{\text{nn}}$ ,  $\eta^{\text{nn}}$ ,  $\sigma^{\text{nn}}$ ,  $\zeta$  and  $\eta$ .

It should be noted that a number of, essentially, analogous formulations for multi-component systems exist in the literature, see for example Drew & Passman (2007); Rajagopal & Tao (1995). However, most of these are not arrived at by variational arguments, and hence they do not distinguish the fluxes from the canonically conjugate momenta. This means that the entrainment effect tends to be ignored. In fact, the novelty of our approach relates to the entrainment between particles and entropy. From an intuitive point of view, this encodes the inertia of heat and allows us to assign an effective mass to the entropy component. It is well known that these concepts are central to causal models of heat conductivity, and we will soon see why this is so.

<sup>‡</sup> We use a coordinate basis to represent tensorial relations. In other words, we distinguish between co- and contra-variant objects,  $v_i$  and  $v^i$ , respectively. Indices, which range from 1 to 3, can be raised and lowered with the (flat space) metric  $g_{ij}$ , i.e.,  $v_i = g_{ij}v^j$ . Derivatives are expressed in terms of the covariant derivative  $\nabla_i$  which is consistent with the metric in the sense that  $\nabla_i g_{kl} = 0$ . This formulation has great advantage when one wants to discuss the geometric nature of the different dissipation coefficients. We also use the volume form  $\epsilon_{ijk}$  which is completely antisymmetric, and which has only one independent component (equal to  $\sqrt{g}$  in the present case).

### 3. The standard approach to heat conductivity

It is natural to begin the discussion of heat conductivity by sketching the classic approach to the problem. This provides a useful contrast to the more general model that we will develop.

The following argument is more or less identical to that provided by Prix (2004). First we rewrite (2.9) as

$$\partial_t s + \nabla_i [s v_n^i + s w_{sn}^i] = \Gamma_s . \quad (3.1)$$

In general,  $\Gamma_s > 0$  if the system is out of equilibrium. Next we introduce the heat flux vector

$$q^i = s T w_{sn}^i , \quad (3.2)$$

such that

$$\partial_t s + \nabla_i \left[ s v_n^i + \frac{q^i}{T} \right] = \Gamma_s . \quad (3.3)$$

Keeping only the linear friction term associated with  $\mathcal{R}^{nn}$ , and using eq. (2.9) from Andersson & Comer (2009), i.e.

$$T \Gamma_s = -f_i^n w_{ns}^i - D_i^j \nabla_j v_s^i - D_i^{nj} \nabla_j w_{ns}^i , \quad (3.4)$$

we arrive at

$$\Gamma_s = -\frac{1}{T} f_i^n w_{ns}^i = \frac{2}{T} \mathcal{R}^{nn} w_{ns}^2 = 2 \mathcal{R}^{nn} \left( \frac{q}{sT} \right)^2 \geq 0 . \quad (3.5)$$

The second law of thermodynamics thus requires that  $\mathcal{R}^{nn} \geq 0$ . In the following we will often omit the subscripts on the resistivity coefficient  $\mathcal{R}^{nn}$ . This makes the equations somewhat clearer and we do not believe that it should cause any confusion.

Finally, if we consider the case of vanishing entrainment (cf. Prix (2004)) then it follows from (2.5) that

$$f_i^s = s \nabla_i T = -f_i^n = 2 \mathcal{R} w_i^{ns} = -2 \frac{\mathcal{R}}{sT} q_i , \quad (3.6)$$

or

$$q_i = -\frac{1}{2} \frac{s^2 T}{\mathcal{R}} \nabla_i T . \quad (3.7)$$

Comparing to Fourier's law we identify the thermal conductivity as

$$\kappa = \frac{1}{2} \frac{s^2 T}{\mathcal{R}} . \quad (3.8)$$

This completes the traditional description of the heat conductivity problem. However, for reasons that we have already discussed, this analysis is not entirely satisfactory. Fortunately, the multi-fluid approach advocated here allows for a more general view. This alternative description has a richer phenomenology and recovers a number of features of extended irreversible thermodynamics (Jou et al, 1993; Müller & Ruggeri, 1993). Illustrating this is one of the main purposes of our discussion.

#### 4. The extended thermodynamics view

The analysis in the previous section focussed on the entropy conservation law. The spirit of the discussion was very much that of classic irreversible thermodynamics. We will now depart from this view, and ask what we can learn by taking the two-fluid model, e.g. the entropy inertia, at face value. The massless entropy flow and the associated entrainment then play central roles.

Let us consider the entropy momentum equation (2.5) a bit closer. We can rewrite this equation as

$$\begin{aligned} (\partial_t + v_n^j \nabla_j) \pi_i^s - \nabla_j \left( \frac{1}{2\alpha} \pi_s^j \pi_i^s \right) - \pi_j^s \nabla_i \left( \frac{1}{2\alpha} \pi_s^j \right) \\ + s \nabla_i T + \pi_i^s (\nabla_j v_n^j) + \pi_j^s \nabla_i v_n^j + \nabla_j D^{sj}_i = f_i^s . \end{aligned} \quad (4.1)$$

For simplicity, we first include only the resistive contribution to the dissipation, i.e., we set

$$D_{ij}^s = 0 , \quad \text{and} \quad f_i^s = 2\mathcal{R}w_i^{ns} , \quad (4.2)$$

as before. Then we have

$$\begin{aligned} (\partial_t + v_n^j \nabla_j) \pi_i^s - \nabla_j \left( \frac{1}{2\alpha} \pi_s^j \pi_i^s \right) - \pi_j^s \nabla_i \left( \frac{1}{2\alpha} \pi_s^j \right) \\ + s \nabla_i T + \pi_i^s (\nabla_j v_n^j) + \pi_j^s \nabla_i v_n^j = \frac{\mathcal{R}}{\alpha} \pi_i^s , \end{aligned} \quad (4.3)$$

where we recall the definition of the entropy momentum, (2.2).

It turns out to be instructive to view this equation in the context of extended irreversible thermodynamics (Jou et al, 1993; Müller & Ruggeri, 1993). To facilitate the comparison, let us focus on the simplified case of heat conduction in a rigid solid. In that case we have  $v_n^i = 0$  and our momentum equation simplifies to

$$\partial_t \pi_i^s - \nabla_j \left( \frac{1}{2\alpha} \pi_s^j \pi_i^s \right) - \pi_j^s \nabla_i \left( \frac{1}{2\alpha} \pi_s^j \right) + s \nabla_i T = \frac{\mathcal{R}}{\alpha} \pi_i^s , \quad (4.4)$$

Moreover, as before it makes sense to introduce the heat flux  $q^i$  as

$$w_{sn}^i = \frac{1}{sT} q^i = -\frac{1}{2\alpha} \pi_i^s . \quad (4.5)$$

In the following we will often choose to keep the relative velocity as the main variable (we also omit the constituent indices, using  $w^i = w_{sn}^i$  for clarity). This simplifies many of the equations as it suppresses factors of  $sT$ . Expressing all results in terms of the heat flux vector is trivial given (4.5).

It follows that

$$-\partial_t (2\alpha w_i) - \nabla_j (2\alpha w^j w_i) - \alpha \nabla_i w^2 + s \nabla_i T + 2\mathcal{R}w_i = 0 . \quad (4.6)$$

This is one of our main results.

Consider the limit of vanishing entrainment. Letting  $\alpha \rightarrow 0$  we are left with

$$w_i = -\frac{s}{2\mathcal{R}} \nabla_i T , \quad (4.7)$$

or

$$q_i = -\frac{s^2 T}{2\mathcal{R}} \nabla_i T = -\kappa \nabla_i T, \quad (4.8)$$

where we have identified the thermal conductivity,  $\kappa$ , as in the previous section. In the limit of vanishing entrainment we recover Fourier's law, as expected.

If we instead linearise the equation (with respect to thermal equilibrium,  $q^i = 0$ ) we find that

$$-\partial_t \left( \frac{2\alpha}{sT} q_i \right) + s \nabla_i T + \frac{2\mathcal{R}}{sT} q_i = 0. \quad (4.9)$$

For constant parameters<sup>†</sup>, this can be written

$$\tau \partial_t q_i + q_i = -\kappa \nabla_i T, \quad (4.10)$$

where we have introduced the thermal relaxation time,  $\tau$ , according to

$$\tau = -\frac{\alpha}{\mathcal{R}}. \quad (4.11)$$

Eq. (4.10) is known as the Cattaneo equation (Cattaneo, 1948). It resolves the “paradox” associated with instantaneous propagation of heat predicted in classic irreversible thermodynamics, see Jou & Casas-Vázquez (1988) for a brief discussion. It also leads to the presence of a second sound in solids, an effect that has been observed in laboratory experiments on dielectric crystals, see for example the discussion by Ruggeri et al (1996). As described by Jou et al (1993), the Cattaneo equation provided key stimulus for the development of extended irreversible thermodynamics.

The above exercise shows that the variational multi-fluid formalism contains a key element of extended thermodynamics. Moreover, we have a natural interpretation of the entropy entrainment,  $\alpha$ . In terms of the relaxation time  $\tau$  we have

$$\alpha = -\tau \mathcal{R} = -\frac{s^2 T}{2} \left( \frac{\tau}{\kappa} \right). \quad (4.12)$$

That is, we should expect to have  $\alpha \leq 0$ .

Is it surprising that the Cattaneo equation can be deduced from the variational two-fluid model? Not really. In retrospect the result is, more or less, obvious. This is easy to see if we compare the nature of the variational energy functional  $E$  (Prix, 2004) to the generalised entropy used in extended irreversible thermodynamics (Jou & Casas-Vázquez, 1988). In the variational case, we have  $E = E(n, s, w^2)$  and in the case of a solid (taking  $n$  to be constant) it follows that

$$dE = \frac{\partial E}{\partial s} ds + \frac{\partial E}{\partial w^2} dw^2 = T ds + \alpha dw^2 = T ds + \alpha d \left( \frac{q}{sT} \right)^2. \quad (4.13)$$

Meanwhile, one of the tenets of extended irreversible thermodynamics is the generalised entropy

$$s(u, q) = s_0(u) - \frac{1}{2} \beta(T) q^2, \quad (4.14)$$

<sup>†</sup> There is no physical reason why  $\alpha/sT$  should be “constant”. We only make the assumption in order to facilitate a direct comparison with the Cattaneo equation. The discussion of Morro & Ruggeri (1988) makes it quite clear that the parameter needs to be temperature dependent in order for this kind of model to be able to reproduce data from second sound experiments.

where  $u$  is the internal energy. As discussed by Alvarez et al (2008), this assumption is common to a number of alternative approaches to irreversible thermodynamics. The entropy satisfies the generalised Gibbs identity

$$ds = \frac{1}{T} du - \beta dq^2 . \quad (4.15)$$

It is now apparent that, once we identify  $u = E$ , (4.13) and (4.15) contain similar information. The key point is that the energy/entropy of the system depends on the heat flux. This is natural since the heat flux corresponds to the flow of energy relative to the matter current. The variational approach provides a slightly different perspective on the problem, but the predicted dynamics should be equivalent.

Before moving on, it is worth discussing the role of the non-linear terms that were discarded in (4.9). The variational analysis naturally leads to the presence of quadratic terms in the heat flux. Due to their origin, i.e. the entropy momentum equation, these terms (essentially) take the same form as the non-linear terms in the standard Euler equation. In other words, they do not represent the most general non-linearities that one might envisage (in the dissipative problem). In the interest of clarity, we have chosen not to compare the non-linear terms in our model to various attempts at constructing non-linear heat-conducting models, see for example Morro & Ruggeri (1987); Ruggeri et al (1996); Jou et al (2004); Lebon et al (2008b) and Llebot et al (1983). Such a comparison would obviously be interesting given that non-linearities are relevant for the development of both shocks and turbulence. However, our main initial aim is to establish the viability of the multi-fluids approach to the heat problem. For this purpose, the evidence provided by the linear comparison should be adequate.

## 5. Including “non-local” terms

The model discussed in the previous section is the simplest in a hierarchy of possible heat conductivity models. By relaxing the assumptions that led to (4.10) we can easily obtain more complicated models. Such models are interesting, because (4.10) may not provide a faithful representation of all relevant phenomena. It is obviously important to compare a more general model to analogous efforts in the literature. This comparison provides further insight into the usefulness of the multi-fluid framework and the interpretation of the various coefficients.

So far we have neglected the viscous stresses in the entropy momentum equation. Let us now relax this assumption, i.e., account also for  $D_{ij}^s$ . In order to simplify the analysis somewhat, we will still assume that  $\mathcal{S}^{nn} = \sigma^{nn} = 0$ . That is, we ignore the coupling to vorticity.

We now need

$$D_{ij}^s = D_{ij} - D_{ij}^n = -g_{ij}[(\zeta - \zeta^n)\Theta_s + (\zeta^n - \zeta^{ns})\Theta_{ns}] - 2[(\eta - \eta^n)\Theta_{ij}^s + (\eta^n - \eta^{nn})\Theta_{ij}^{ns}] . \quad (5.1)$$

Rewriting this expression in terms of the particle velocity  $v_n^i$  and the relative flow  $w_{ns}^i$  we have

$$D_{ij}^s = -g_{ij}(\zeta - \zeta^n)\Theta_n - 2(\eta - \eta^n)\Theta_{ij}^n - g_{ij}\bar{\zeta}\Theta_{ns} - 2\bar{\eta}\Theta_{ij}^{ns} , \quad (5.2)$$

where we have introduced

$$\bar{\zeta} = 2\zeta^n - \zeta^{ns} - \zeta , \quad \text{and} \quad \bar{\eta} = 2\eta^n - \eta^{ns} - \eta . \quad (5.3)$$



Again focussing on the case of a rigid solid, the first two terms in (5.2) vanish and we also have

$$\Theta_{\text{ns}} = -\nabla_i w^i, \quad (5.4)$$

and

$$\Theta_{ij}^{\text{ns}} = \frac{1}{2} \left[ \nabla_i w_j + \nabla_j w_i - \frac{2}{3} \nabla_l w^l \right], \quad (5.5)$$

Adding the relevant terms to (4.6), taking  $\bar{\zeta}$  and  $\bar{\eta}$  to be constant, for simplicity, we have

$$\begin{aligned} -\partial_t(2\alpha w_i) - \nabla_j(2\alpha w^j w_i) - \alpha \nabla_i w^2 + s \nabla_i T \\ + 2\mathcal{R} w_i + \bar{\eta} \nabla^2 w_i + \left( \bar{\zeta} + \frac{1}{3} \bar{\eta} \right) \nabla_i \nabla_j w^j = 0. \end{aligned} \quad (5.6)$$

We can simplify things further by i) assuming that all parameters are constant, and ii) linearising in  $q^i$  (the caveats regarding these assumptions remain as before). This leads to the equation

$$-\frac{2\alpha}{sT} \partial_t q_i + s \nabla_i T + \frac{2\mathcal{R}}{sT} q_i + \frac{\bar{\eta}}{sT} \nabla^2 q_i + \frac{1}{sT} \left( \bar{\zeta} + \frac{1}{3} \bar{\eta} \right) \nabla_i \nabla_j q^j = 0, \quad (5.7)$$

or, making use of the thermal relaxation time and the thermal conductivity,

$$\tau \partial_t q_i + \kappa \nabla_i T + q_i + l_\eta^2 \nabla^2 q_i + \left( l_\zeta^2 + \frac{1}{3} l_\eta^2 \right) \nabla_i \nabla_j q^j = 0. \quad (5.8)$$

We have also introduced the two lengthscales

$$l_\eta^2 = \frac{\bar{\eta}}{2\mathcal{R}}, \quad (5.9)$$

and

$$l_\zeta^2 = \frac{\bar{\zeta}}{2\mathcal{R}}. \quad (5.10)$$

These can be taken to represent the mean-free path associated with the dissipative terms.

This result can be directly compared to the ‘‘phonon hydrodynamics’’ model developed by Guyer & Krumhansl (1966) (see Llebot et al (1983), and Cimmelli (2007) for alternative descriptions). Their model is the most celebrated attempt to account for non-local heat conduction effects. It accounts for interaction of phonons with each other and the lattice. Resistive terms are represented by  $\tau$  while momentum conserving interactions are associated with  $l_\eta$  and  $l_\zeta$ . Our model, (5.8), completely reproduces the Guyer & Krumhansl (1966) result, once we set

$$l_\zeta^2 = \frac{1}{3} l_\eta^2. \quad (5.11)$$

This leads to

$$\tau \partial_t q_i + \kappa \nabla_i T + q_i + l_\eta^2 (\nabla^2 q_i + 2 \nabla_i \nabla_j q^j) = 0. \quad (5.12)$$

The usefulness of this result is due to the fact that it can be used both in the collision dominated and the ballistic phonon regime. In the former, the resistivity

dominates, the nonlocal terms can be neglected and heat propagates as waves. In the opposite regime, the momentum conserving interactions are dominant and we can neglect the thermal relaxation. In this regime, heat propagates by diffusion. The transition between these two extremes has recently been discussed by Vásquez & Márkus (2009).

Interestingly, the non-local heat conduction model may be useful in the description of nano-size systems. If a system has characteristic size  $L$ , and  $l_\eta/L \gg 1$ , then one would not necessarily expect a fluid model to apply. Nevertheless, Alvarez et al (2009) have argued that the expected behaviour of the thermal conductivity as the size of the system decreases (as discussed by Alvarez & Jou (2007) one would expect the “effective” conductivity to scale as  $L/l_\eta$ ) can be reproduced from (5.8) provided that the appropriate slip condition for  $q^i$  is applied at the boundaries. A key part of this analysis is the close analogy between (5.8) and the Navier-Stokes equation. In the latter case it is well-known that an applied non-slip condition at a surface leads to the formation of a viscous boundary layer that dominates the dissipation of the bulk flow. It appears that the heat problem is quite similar in the ballistic phonon regime, although the required slip condition is different. This is an interesting problem that requires more detailed study.

## 6. The general two-component model

At this point we have demonstrated that the multi-fluid formalism, with one fluid representing the massless entropy, reproduces a number of non-trivial results for heat conductivity in rigid solids. However, this is a simplified problem since one of the degrees of freedom in the system was “frozen”. In order to complete the model, we will now relax this assumption and allow  $v_n^i \neq 0$ . This leads to a system of equations with interesting applications. In particular, one could imagine modelling systems with spatial transitions to superfluidity (as in a neutron star core). In one regime, the system would be dominated by resistivity, while the thermal relaxation timescale determines the dynamics elsewhere. A unified model for this problem could prove very useful indeed.

Let us return to the general two-fluid model and focus on the matter degree of freedom. As in the case of superfluid Helium (Andersson & Comer, 2009), it is natural to work with the total momentum equation. By combining (2.4) and (2.5) we have

$$\begin{aligned} \partial_t(\pi_i^n + \pi_i^s) + \nabla_l(v_n^l \pi_i^n + v_s^l \pi_i^s) + n \nabla_i \mu_n + s \nabla_i T \\ - n \nabla_i \left( \frac{1}{2} m v_n^2 \right) + \pi_l^n \nabla_i v_n^l + \pi_l^s \nabla_i v_s^l = -\nabla_j D^j_i . \end{aligned} \quad (6.1)$$

We can rewrite this using

$$\pi_i^n + \pi_i^s = \rho v_i^n , \quad (6.2)$$

$$\pi_l^n \nabla_i v_n^l + \pi_l^s \nabla_i v_s^l = n \nabla_i \left( \frac{1}{2} m v_n^2 \right) - 2\alpha w_l^{ns} \nabla_i w_{ns}^l , \quad (6.3)$$

and

$$v_n^l \pi_i^n + v_s^l \pi_i^s = m n v_n^l v_i^n - 2\alpha w_{ns}^l w_i^{ns} . \quad (6.4)$$

We also use the fact that the pressure follows from (Andersson & Comer, 2009)

$$\nabla_i p = n \nabla_i \mu_n + s \nabla_i T - \alpha \nabla_i w_{ns}^2. \quad (6.5)$$

Combining these results we arrive at

$$\partial_t(\rho v_i^n) + \nabla_l(\rho v_n^l v_i^n) - \nabla_l(2\alpha w_{ns}^l w_i^{ns}) + \nabla_i p = -\nabla_j D_i^j. \quad (6.6)$$

Using our definition for the heat flux, (4.5), we see that if we linearise in  $w_i$  (or equivalently,  $q_i$ ) then the problem simplifies considerably. If we also use the continuity equation (2.8), we arrive at an equation that resembles the standard Navier-Stokes result (Landau & Lifshitz, 1959);

$$\rho(\partial_t + v_n^l \nabla_l) v_i^n + \nabla_i p = -\nabla_j D_i^j. \quad (6.7)$$

The right-hand side is, however, different. Keeping  $\mathcal{S}^{nn} = \sigma^{nn} = 0$  (as before) we find that

$$-\nabla_j D_i^j = -\left(\tilde{\zeta} + \frac{1}{3}\tilde{\eta}\right) \nabla_i(\nabla_j w_{ns}^j) - \tilde{\eta} \nabla^2 w_i^{ns} + \left(\zeta + \frac{1}{3}\eta\right) \nabla_i(\nabla_j v_n^j) + \eta \nabla^2 v_i^n, \quad (6.8)$$

where we have defined

$$\tilde{\zeta} = \zeta^n - \zeta, \quad \text{and} \quad \tilde{\eta} = \eta^n - \eta. \quad (6.9)$$

For simplicity, we have also assumed that the equilibrium configuration is uniform. This leads to the final result

$$\begin{aligned} & \rho(\partial_t + v_n^l \nabla_l) v_i^n + \nabla_i p \\ &= \left(\zeta + \frac{1}{3}\eta\right) \nabla_i(\nabla_j v_n^j) + \eta \nabla^2 v_i^n - \frac{1}{sT} \left(\tilde{\zeta} + \frac{1}{3}\tilde{\eta}\right) \nabla_i(\nabla_j q^j) - \frac{\tilde{\eta}}{sT} \nabla^2 q_i. \end{aligned} \quad (6.10)$$

The first two terms on the right-hand side are familiar from the Navier-Stokes equation (Landau & Lifshitz, 1959), but the last two terms are new. They represent the dissipative coupling between the total momentum and the heat flux. An interesting question concerns whether there are situations where these terms have decisive impact on the dynamics. Are there, for example, situations where the last term is similar in magnitude to the second term?

To complete the model, we need the momentum equation for the heat flux. Starting from (4.1), linearising in the heat flux and using the results from the previous section, we find that

$$\begin{aligned} & (\partial_t + v_n^j \nabla_j) q_i + \frac{1}{\tau} q_i + q_i \nabla_j v_n^j + q_j \nabla_i v_n^j \\ &= -\frac{\kappa}{\tau} \nabla_i T - \frac{sT}{2\alpha} \left[ \left(\tilde{\zeta} + \frac{1}{3}\tilde{\eta}\right) \nabla_i(\nabla_j v_n^j) + \tilde{\eta} \nabla^2 v_i^n \right] \\ & \quad + \frac{1}{2\alpha} \left[ \left(\tilde{\zeta} + \frac{1}{3}\tilde{\eta}\right) \nabla_i(\nabla_j q^j) + \tilde{\eta} \nabla^2 q_i \right]. \end{aligned} \quad (6.11)$$

Comparing to the corresponding equation for a rigid solid, we recognize several terms. Some dissipative terms are, however, new.

Before we conclude our discussion, it is worth making the following observation. In our formulation of the problem we have assumed that the entropy conservation law is used explicitly. An alternative, more common, strategy is to use the energy equation. The two approaches are, in principle, equivalent. Nevertheless, it is useful to complement our analysis with a brief consideration of the energy equation. From Andersson & Comer (2006) we have (for an isolated system) the energy equation

$$\partial_t \mathcal{U} + \nabla_i Q^i = 0 . \quad (6.12)$$

After some work, using the various definitions, we find that

$$Q^i = \left( \frac{1}{2} \rho v_n^2 - 2\alpha w_{\text{ns}}^2 + n\mu + sT \right) v_n^i - (sT + 2\alpha v_s^j w_j^{\text{ns}}) w_{\text{ns}}^i . \quad (6.13)$$

Here, we can use the fundamental relation

$$p + E = n\mu + sT , \quad (6.14)$$

to get

$$Q^i = \left( \frac{1}{2} \rho v_n^2 - 2\alpha w_{\text{ns}}^2 + p + E \right) v_n^i - (sT + 2\alpha v_s^j w_j^{\text{ns}}) w_{\text{ns}}^i . \quad (6.15)$$

We also have

$$\mathcal{U} = \frac{1}{2} \rho v_n^2 - 2\alpha w_{\text{ns}}^2 + E . \quad (6.16)$$

Combining these results, and linearising in  $w_{\text{ns}}^i$ , we immediately arrive at the standard energy equation (c.f. Landau & Lifshitz (1959))

$$\partial_t \left( \frac{1}{2} \rho v_n^2 + E \right) + \nabla_i \left[ \left( \frac{1}{2} \rho v_n^2 p + E \right) v_n^i \right] = \nabla_i (sT w_{\text{ns}}^i) = -\nabla_i q^i . \quad (6.17)$$

This confirms, at least at the linear level, the definition (3.2) of the heat flux.

We now have a “complete” model for a heat conducting fluid. It combines the equations of motion (6.10) and (6.11) with the two continuity equations (2.8) and (2.9). This model should be relevant for dynamics on timescales such that the thermal relaxation cannot be ignored. In a typical system, this would correspond to the extreme high-frequency regime. However, as we already know from the discussion of Andersson & Comer (2009) the model also applies to superfluid condensates at finite temperatures. In essence, we have a unified framework for modelling the transition to superfluidity. Finally, there may be situations where the model applies even if the thermal relaxation can be safely ignored. This would be the case when the “phonon” mean free path exceeds the size of the system. As discussed in the previous section, the dissipative terms in (6.11) then play the leading role and the crucial boundary effects may be incorporated by imposing suitable surface conditions on the heat flux. This possibility has so far been discussed only for nano-systems, but it is worth noting that it may be relevant also for large scale systems. In fact, the ballistic phonon regime should apply to cold superfluid condensates in neutron stars. To what extent the present analysis can be applied to that problem is an interesting question for the future.

## 7. Concluding remarks

We have discussed heat conductivity from the point of view of the variational multi-fluid model developed by Prix (2004) and Andersson & Comer (2006). We have shown that a two-fluid model that distinguishes between a massive fluid component and a massless entropy can reproduce a number of key results from extended irreversible thermodynamics. In particular, we have demonstrated that the entropy entrainment, that played a central role in our recent discussion of superfluid Helium (Andersson & Comer, 2009), is intimately linked to the thermal relaxation time that is required to make heat propagation in solids causal. We have also considered non-local terms that arise naturally in the dissipative multi-fluid model, and related them to models of phonon hydrodynamics. This discussion may provide useful insight into the modelling of both nano-systems and superfluids at low temperatures where the phonon mean-free path is large compared to the size of the system. Finally, we formulated a “complete” heat conducting two-fluid model and identified a number of “new” dissipative terms. Future work needs to establish whether there are physical situations where these terms play a decisive role.

What is the importance of this work? First of all, we believe that the discussion provides strong support for the main assumptions of our model: That one can treat the entropy as an additional fluid, endowed with the inertial and dynamical properties generally associated with fluids. The entrainment between particles and the massless entropy plays a key role in this approach. The connection between this entropy entrainment and the thermal relaxation time provides an immediate interpretation, and illustrates the importance, of the main parameter of the model. The simple fact that the same mathematical framework can be used to model both heat conduction and finite temperature superfluids (Andersson & Comer, 2009) is, in our view, clear evidence of the elegance and promise of the variational multi-fluid approach. Moreover, since our model has its origin in a fully relativistic variational analysis, see Andersson & Comer (2007) for a review, the present discussion suggests a promising strategy for developing a causal relativistic model for heat conductivity. This is known to be a challenging problem where a number of issues remain to be resolved.

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