HIGH RESOLUTION DIRECTION FINDING FROM RECTANGULAR HIGHER ORDER CUMULANT MATRICES: THE RECTANGULAR 2Q-MUSIC ALGORITHMS

Hanna Becker(1), Pascal Chevalier(2,3) and Martin Haardt(4)

(1) Université de Nice Sophia-Antipolis, CNRS, I3S, UMR7271, 06900 Sophia-Antipolis, France.
(2) CNAM, Laboratoire CEDRIC, 292 rue Saint-Martin, 75141 Paris Cédex 3, France.
(3) Thales-Communications-Security, AMS/TCP, 4 Av. des Louvresses, 92622 Gennevilliers, Cédex, France.
(4) Technische-Universität Ilmenau, Helmholtzplatz 2, PF 100565, D-98684 Ilmenau Germany

ABSTRACT

Recently, the 2q-MUSIC (q ≥ 2) direction finding algorithm has been developed for non-Gaussian sources and square arrangements of the 2qth-order data statistics, to overcome the main limitations of MUSIC and to improve the performance of 4-MUSIC for multiple sources. To further improve the performance of the 2q-MUSIC algorithm, the purpose of this paper is to extend the latter to rectangular arrangements of the data statistics, giving rise to rectangular 2q-MUSIC algorithms. It is shown in particular that rectangular arrangements of the higher order (HO) data statistics allow to optimize the compromise between performance and maximal number of sources to be processed. Besides, it also allows a complexity reduction for a given level of performance. These results, completely new, should open new perspectives for HO array processing.

Index Terms— Higher order, Virtual Array, Rectangular, Arrangements, 2q-MUSIC.

1. INTRODUCTION

Fourth-Order (FO) direction finding methods, such as 4-MUSIC [1], have been developed for more than two decades for non-Gaussian sources, to overcome the limitations of second-order (SO) methods, such as MUSIC [2]. Recently, in order to still increase the performance of 4-MUSIC, the MUSIC method has been extended to an arbitrary even order 2q (q ≥ 1) for square arrangements of the 2qth-order data statistics, giving rise to the so-called 2q-MUSIC algorithm [3]. It has been shown in [3] that 2q-MUSIC offers increasing performance with q, in terms of resolution, robustness to modelling errors and the number of sources to be processed. This performance increase is directly linked to a virtual increase of both the effective aperture and the number of sensors, N, of the array, introducing the HO virtual array (VA) concept presented in [4]. It has been proved in [4] that 2q-MUSIC can process up to O(Nq) sources. Furthermore, it has been shown recently in [5] that by arranging the 2qth order data statistics in a (N2q×1) vector, c2q,x, it is possible to build a non uniform linear array of N identical sensors, called 2q-level nested array, giving rise to a 2qth order VA corresponding to a uniform linear array of O(N2q) virtual sensors. Using a spatial smoothing algorithm [6], it is then possible to estimate the directions of arrival (DOA) of O(N2q) sources, instead of O(Nq), from the “covariance like” matrix c2q,x c2q,x H, where H means transpose and conjugate. This result generates a more general question consisting in wondering whether it may be useful in practice to consider arbitrary rectangular arrangements of the HO data statistics instead of square ones for HO direction finding from an arbitrary array of sensors and not specific ones only. The purpose of this paper is precisely to answer this important question by extending, for arbitrary arrays of sensors, both the HO VA concept and the 2q-MUSIC algorithm to arbitrary rectangular arrangements of the 2qth order data statistics, giving rise to rectangular 2q-MUSIC algorithms. It is shown in particular that rectangular arrangements of the HO data statistics allow to optimize the compromise between performance and maximal number of sources to be processed. Besides, it also allows a complexity reduction for a given level of performance. These results, completely new, should allow the development of new methods for HO array processing.

2. HYPOTHESES AND DATA STATISTICS

2.1. Hypotheses and notations

We consider an array of N narrow-band (NB) sensors and we call x(t) the vector of complex amplitudes of the signals at the output of these sensors. Each sensor is assumed to receive the contribution of P zero-mean stationary NB sources corrupted by an additive noise. We assume that the P sources can be divided into G groups, with PG sources in the group g, such that the sources in each group are assumed to be statistically dependent, but not perfectly coherent, while sources belonging to different
groups are assumed to be statistically independent. Of course, \( P \) is the sum of the \( P_g \) over all the groups. Under these assumptions, the observation vector can be written as follows

\[
x(t) = \sum_{p=1}^{P} s_p(t) a_p + n(t) = \sum_{g=1}^{G} A_g s_g(t) + n(t)
\]

where \( n(t) \) is the noise vector, assumed zero-mean and Gaussian and \( s_g(t) \), independent of \( n(t) \), is the complex amplitude of source \( p \). Furthermore, \( A_g \) is the \((N \times P_g)\) matrix of steering vectors of sources belonging to the \( g \)th group and \( s_g(t) \) is the corresponding \((P \times 1)\) source vector. Without any coupling between sensors and for a plane wave propagation, component \( n \) of vector \( a_p \) can be written as

\[
d_{g\nu} = \exp\{j2\pi k_\nu \theta \nu \nu \} / \lambda
\]

Here, \( \lambda \) is the wavelength of the coordinates of sensor \( n, k_\nu \theta \nu \nu \) is the vector of \( \{x_n, y_n, z_n\}^T \) is the vector of the coordinates of sensor \( n, k_\nu \theta \nu \nu \) is the wave vector of source \( p \), and \( \{\theta_p, \phi_p\} \) are the azimuth and elevation angles of source \( p \).

### 2.2. Statistics of the data

The HO methods discussed in this paper exploit the information contained in the 2\( q \)th order circular cumulants of the data, \( \text{Cum}[x_1(t), \ldots, x_q(t), x_{i_1+1}(t), \ldots, x_{i_2}(t)] (1 \leq i_j \leq N) \) \((1 \leq j \leq 2q)\). The latter entries have been arranged in square matrices in [4] and [3]. These entries will be arranged in this paper in rectangular matrices giving rise, in the next sections, to the extended HO VA concept and the rectangular 2\( q \)-MUSIC algorithm, respectively. In situations of practical interest, the HO statistics of the data have to be estimated from \( K \) data samples, \( \hat{x}(k) = x(kT_c) \), \( 1 \leq k \leq K \), where \( T_c \) is the sample period, using empirical estimators presented in [4] and [3].

### 3. RECTANGULAR ARRANGEMENTS OF THE DATA STATISTICS

In order to arrange the data 2\( q \)th order circular cumulants into rectangular matrices, we introduce two arbitrary integers, \( v \) and \( l \), such that \( 0 \leq v \leq 2q \) and \( \text{sup}(0, v - q) \leq l \leq \text{inf}(v, q) \). The integer \( v \) controls the size \((N^v \times N^{2q-v})\) of the rectangular 2\( q \)th order cumulant matrix (as will be seen later, in practice only arrangements for which \( v \geq q \) are of interest). The integer \( l \) controls the way the data statistics are arranged in the rectangular cumulant matrix. More precisely, for given values of \( v \) and \( l \), let us arrange the \( 2q \)-uplet, \((i_1, \ldots, i_p, i_{q+v+1}, \ldots, i_{2q})\), of indices \( i_j (1 \leq j \leq 2q) \) \((1 \leq i_j \leq N)\), into one \( v \)-uplet and one \((2q-v)\)-uplet, indexed by \( v \) and \( l \), and defined by \((i_1, i_2, \ldots, i_p, i_{q+v+1}, \ldots, i_{q+v})\) and \((i_{q+v+1}, \ldots, i_{2q}, i_1, \ldots, i_q)\), respectively. As the indices \( i_j (1 \leq j \leq 2q) \) vary from 1 to \( N \), the two \( v \)-uplet and \((2q-v)\)-uplet take \( N^v \) and \( N^{2q-v} \) values respectively. We number, in a natural way, the \( N^v \) values of the \( v \)-uplet and the \( N^{2q-v} \) values of the \((2q-v)\)-uplet by the integers \( I_{q+v} \) and \( I_{2q} \), respectively, such that \( 1 \leq I_{q+v} \leq N^v \) and \( 1 \leq I_{2q} \leq N^{2q-v} \). Using the permutation invariance of the cumulants, we deduce that \( \text{Cum}[x_1(t), \ldots, x_q(t), x_{i_1+1}(t), \ldots, x_{i_2}(t)] = \text{Cum}[x_{i_1+1}(t), \ldots, x_{i_2}(t), x_1(t), \ldots, x_{i_1+1}(t)] \), \( x_{i_1+1}(t), \ldots, x_{i_2}(t) \), \( \text{where} \) corresponds to the rectangular cumulant matrix denoted as \( C_{2q,v} \). It is easy to verify, from section 2.1 and for a Gaussian noise, that the \((N^v \times N^{2q-v})\) \( C_{2q,v} \) matrix can be written as

\[
C_{2q,v} = \sum_{g=1}^{G} [A_g \otimes A_g^{* \otimes (v - q)}] C_{2q,v} \text{e}^{(v)} \text{e}^{(q - v)} + \eta_2 \text{V}(v, l) \delta(q - 1).
\]

Here, \( C_{2q,v} \) is the \((N^v \times N^{2q-v})\) matrix of the \( v \)-th order circular cumulants of \( s(t) \), \( \eta_2 \) is the mean power of the noise per sensor, \( \text{V}(v, l) \), defined for \( q = 1 \) only, is the \((N^v \times N^{2q-v})\) \((0 \leq v \leq 2)\) normalized rectangular spatial coherence matrix of the noise such that the total input power of the noise is \( N\eta_2 \). \( \delta(.) \) is the Kronecker symbol, \( \otimes \) is the Kronecker product and \( A_g^{* \otimes (v - q)} \) is the \((N^v \times N^{2q-v})\) matrix defined by \( A_g^{* \otimes (v - q)} = A_g^{*} \otimes A_g \otimes \ldots \otimes A_g \), with a number of Kronecker product equal to \( v-1 \). In particular, for \( v = q \), \( C_{2q,q} \), reduces to the \((N^q \times N^q)\) square matrix \( C_{2q,q} \), used in [3] where it is denoted by \( C_{2q,q}(l) \).

### 4. RECTANGULAR 2\( q \)-MUSIC ALGORITHMS

#### 4.1. Hypotheses

To develop the rectangular 2\( q \)-MUSIC algorithms for the arrangement \( C_{2q,v} \), we introduce some hypotheses:

**H1:** \( P_g \leq N^q \), \( 1 \leq g \leq G \)

**H2:** \( A_g \otimes A_g^{* \otimes (v - q)} \) and \( A_g \otimes A_g^{* \otimes (q - v - 1)} \) have a rank equal to \( P_g \text{Min}(v, 2q-v) \), \( 1 \leq g \leq G \)

**H3:** \( P(G, q, v) \triangleq \sum_{g=1}^{G} P_g \text{Min}(v, 2q-v) \leq N\text{Min}(v, 2q-v) \)

**H4:** \( \text{A}(v,q) \triangleq [A_g \otimes A_g^{* \otimes (v - q)}] \) and \( \text{A}(q,v) \triangleq [A_g \otimes A_g^{* \otimes (q - v)}] \)

**H4:** \( \text{A}(v,q) \) and \( \text{A}(q,v) \) have rank \( P(G, q, v) \)

In particular, for statistically independent sources \((G = P)\), \( P(G, q, v) = P \), \( A_g \) reduces to \( a_g \) and \( H1 \) to reduce to

**H1':** \( P \leq N\text{Min}(v, 2q-v) \)

**H2':** \( \text{A}(v,q) \triangleq [a_1 \otimes a_1^{* \otimes (v - q)}] \) and \( \text{A}(q,v) \triangleq [a_1 \otimes a_1^{* \otimes (q - v)}] \)

**H4':** \( \text{A}(v,q) \) and \( \text{A}(q,v) \) have full rank \( P \)

#### 4.2. Rectangular 2\( q \)-MUSIC algorithm

The matrix \( C_{2q,v} \) has full rank, \( P_g \text{Min}(v, 2q-v) \), in general since the components of \( s_g(t) \) are statistically dependent. We then deduce from \( H1 \) to \( H4 \) that, for \( q = 1 \), \( C_{2q,q} \), has a rank equal to \( P(G, q, v) \). To build a MUSIC-like algorithm
from $C_{2q,s}^v$ for $q \geq 1$, we first compute the singular value decomposition (SVD) of the latter, given by

$$C_{2q,s}^v = \begin{bmatrix} U_{2q,s}^v & U_{2q,n}^v \end{bmatrix} \begin{bmatrix} \Sigma_{2q,s}^v & 0 \\ 0 & \Sigma_{2q,n}^v \end{bmatrix} \begin{bmatrix} V_{2q,s}^v \\ V_{2q,n}^v \end{bmatrix}$$

where $\Sigma_{2q,s}^v$ is the $(P(G, q, v) \times P(G, q, v))$ diagonal matrix of the nonzero singular values of $C_{2q,s}^v$, and $U_{2q,s}^v$ and $U_{2q,n}^v$ are the $(N^q \times P(G, q, v))$ and $(N^{2q-v} \times P(G, q, v))$ unitary matrices of the left and right singular vectors of $C_{2q,s}^v$, respectively.

5.1. Hypotheses

As Span{$V_{2q,s}^v$, $V_{2q,n}^v$} coincides with the $2q$-th order circular steering vectors of the source array, we define all vectors $\{V_{2q,s}^v, V_{2q,n}^v\}$ associated with the $(N^q \times P(G, q, v))$ non-zero singular values, respectively. $\Sigma_{2q,s}^v$ and $\Sigma_{2q,n}^v$ are the $(N^q \times P(G, q, v))$ and $(N^{2q-v} \times P(G, q, v))$ diagonal matrices of the singular values of the C-matrices $C_{2q,s}^v$ and $C_{2q,n}^v$, such that $U_{2q,s}^v V_{2q,s}^v = \Sigma_{2q,s}^v$ and $U_{2q,n}^v V_{2q,n}^v = \Sigma_{2q,n}^v$. Hence, all vectors $\{V_{2q,s}^v, V_{2q,n}^v\}$ share the same unitary matrices of the left and right singular vectors of $C_{2q,s}^v$, associated with the zero singular values.

5.2. Performance of $2q$-MUSIC algorithms

We deduce from (6) that each source $p$ contributes to $C_{2q,s}^v$ through a rank one matrix $c_{2q,p}^v a_{q,p,1}^v a_{q,p,2}^v \Sigma_{2q}^v H$ whose left and right vectors, $a_{q,p,1}^v$ and $a_{q,p,2}^v$, correspond to the $(N^q \times 1)$ left and $(N^{2q-v} \times 1)$ right virtual steering vectors of source $p$ for the considered array of sensors. It has been shown in [4] that $a_{q,p,1}^v$ and $a_{q,p,2}^v$ can be considered as true steering vectors of the source $p$ but for two VA of $N^q$ and $N^{2q-v}$ virtual sensors (VS), called hereafter left and right VA respectively. The positions of the left and right VS are defined respectively by

$$\begin{array}{c}
\sum_{j=1}^{v+1} \sum_{u=1}^{q-v} p_{j} a_{q,p,1}^v \sum_{j=1}^{v+1} \sum_{u=1}^{q-v} p_{j} a_{q,p,2}^v \\
\sum_{j=1}^{v+1} \sum_{u=1}^{q-v} p_{j} a_{q,p,1}^v \sum_{j=1}^{v+1} \sum_{u=1}^{q-v} p_{j} a_{q,p,2}^v \\
\end{array}$$

5.3. Identifiability of $2q$-MUSIC algorithms

Denoting by $c_{2q,s}^v$, the maximal rank of $C_{2q,s}^v$, in the absence of noise, it is straightforward to show that the maximal number of sources which may be processed by $2q$-MUSIC ($v \times q$) is equal to $c_{2q,s}^v$. Denoting by $N_1^{2q,s}$ and $N_2^{2q,s}$ the number of different VS of the left and right $2q$th
order VA associated with $C_{v,l}^{(v,l)}$ respectively, it is obvious from (6) that $r_{2q,x}^{(v,l)} = \min(N_{2q,l}^{(v,l)}, N_{2q,2}^{(v,l)})$. For given values of $(q, l)$, $r_{2q,x}^{(v,l)}$ is then maximized for $v = q$, i.e., by the square arrangement $C_{2v,q}^{(v,l)}$ for which $r_{2q,x}^{(v,l)} = N_{2q,q}^{(v,l)}$, denoted by $N_{2q,2}^{(v,l)}$ in [4]. Consequently, for given values of $q$ and $l$, as the performance of $2q$-MUSIC$(v, l)$ for multiple sources increases with $v$, as shown in section 5.2, in practice $v$ should be chosen such that $q \leq v \leq 2q$ to optimize the performance for a given number of sources (greater than one) to be processed. This means that $C_{2v,x}^{(v,l)}$ must be either square or tall and in this case, $r_{2q,x}^{(v,l)} = N_{2q,2}^{(v,l)}$. As $N_{2q,2}^{(v,l)}$ decreases with $v$ while the performance improves, there is a trade-off between the number of sources to be processed and the performance for multiple sources and a compromise has to be found in practice. The possibility to adjust the compromise between the number of sources to be processed by 2$q$-MUSIC$(v, l)$ and its performance for multiple sources corresponds to one of the main interests of rectangular arrangements with respect to square ones.

5.4. Optimal arrangement index $l$

Finally, for given values of $q$ and $v \geq q$, and from an asymptotical (as $K \rightarrow \infty$) performance point of view, the index $l$ is of no importance for the left VA associated with $C_{2v,q}^{(v,l)}$ as shown in [4]. Thus, as $K$ becomes large, the optimal index $l$ is the one which maximizes the number of sources to be processed, i.e. $N_{2q,2}^{(v,l)}$. Using the results of [4], we deduce that $l_{opt} = v/2$ if $v$ is even and $l_{opt} = (v-1)/2$ if $v$ is odd.

5.5. Computer simulations

The results of this paper are illustrated in this section by computer simulations. Two performance criteria presented in [3] are considered for each source. The first one is the probability of non-aberrant results, i.e., the probability that the estimated left hand side of (5) is lower than a threshold $\eta$. The second one is the averaged root mean square error (RMSE), computed from the non-aberrant results. We assume that 2 synchronized statistically independent QPSK sources, sampled at the symbol rate, are received by a Uniform Circular Array (UCA) of $N = 3$ omnidirectional sensors with a radius $r$ such that $r = 0.3 \lambda$. The 2 QPSK sources have the same input SNR equal to 10 dB and a direction of arrival equal to $\theta_1 = 90^\circ$ and $\theta_2$ respectively. Under these assumptions, Figure 1 shows the variations, as a function of the number of snapshots $K$, of the RMSE for the source 1. RMSE$_1$, (we obtain similar results for the source 2), estimated from $M = 500$ realizations, at the output of 2-MUSIC(1, 1), 4-MUSIC(2, 1), 4-MUSIC(2, 2), 4-MUSIC(3, 1), 4-MUSIC(3, 2), 6-MUSIC(3, 1) and 6-MUSIC(3, 2). For these figures, $\theta_2 = 105^\circ$ and the steering vectors $a_p$ ($1 \leq p \leq 2$) are corrupted by a zero-mean circular Gaussian modeling error vector $e_p$, such that $\mathbb{E}[e_p e_p^H] = \sigma^2 \delta_{p5} \delta_{N,0}$ where $\sigma = 0.03$. Note that beyond $K = 100$ snapshots, the probability of non-aberrant results with $\eta = 0.1$ is equal to 1 for all the methods. Note, above a few hundred of snapshots, the better performance of the rectangular 4-MUSIC algorithms, 4-MUSIC(3, 1) and 4-MUSIC(3, 2), with respect to the square ones, 4-MUSIC(2, 1) and 4-MUSIC(2, 2). Note also in this case the almost same performance of rectangular 4-MUSIC algorithms with the square 6-MUSIC algorithms. To complete the previous results, we consider again the previous scenario with modelling errors but with now an infinite number of snapshots $K$ and an arbitrary value of $\Delta \theta$. Under these assumptions, Figure 2 shows the RMSE$_1$ as a function of $\Delta \theta$ at the output of the previous methods. We note that rectangular 4-MUSIC(3, 1) and 4-MUSIC(3, 2) outperform square 4-MUSIC(2, 1) and 4-MUSIC(2, 2) especially for close sources and allow to obtain the performance of square 6-MUSIC(3, 1) and 6-MUSIC(3, 2) using $4^th$ order statistics only.

6. CONCLUSION

In this paper, rectangular arrangements of the HO data statistics have been considered for direction finding with the proposed rectangular 2$q$-MUSIC algorithms. It has been shown that these rectangular arrangements allow to optimize the compromise between performance and maximal number of sources to be processed. Besides, they also allow a complexity reduction for a given level of performance. These results, completely new, should allow the development of new methods for HO array processing.
6. REFERENCES


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