

A SKEW-DUOIDAL ECKMANN-HILTON ARGUMENT AND QUANTUM CATEGORIES

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Dedicated to George Janelidze on his sixtieth birthday

ABSTRACT. A general result relating skew monoidal structures and monads is proved. This is applied to quantum categories and bialgebroids. Ordinary categories are monads in the bicategory whose morphisms are spans between sets. Quantum categories were originally defined as monoidal comonads on endomorphism objects in a particular monoidal bicategory \mathcal{M} . Then they were shown also to be skew monoidal structures (with an appropriate unit) on objects in \mathcal{M} . Now we see in what kind of \mathcal{M} quantum categories are merely monads.

1. INTRODUCTION

The proof that higher homotopy groups are commutative was abstracted to the statement that monoids in the category of monoids are commutative monoids. This is known as the Eckmann-Hilton argument [8].

In a seminar talk [15], Bob Walters suggested looking at a 2-dimensional version of this argument where monoids are replaced by monoidal categories. Joyal-Street [9] showed that monoidales (= pseudomonoids) in the 2-category of monoidal categories and strong monoidal functors were braided monoidal categories. They also pointed out that, repeating the process, monoidales in the 2-category of braided monoidal categories and braided strong monoidal functors were symmetric monoidal categories. Also, stabilization occurs at that stage: it is symmetric monoidal categories from there onwards. These facts together constitute the Eckmann-Hilton argument for monoidal categories; here, we shall be particularly interested in the fact that a monoidale in the 2-category of braided monoidal categories is a symmetric monoidal category.

If in the above strong monoidal functors are replaced by mere (lax) monoidal functors, no such collapsing or stabilization occurs. Monoidales in the 2-category of monoidal categories and monoidal functors are called “2-monoidal categories” in [1] and “duoidal categories” in [12] and [4].

Recently Kornel Szlachányi [14] has excited our investigations [10] and [13] into *skew* monoidal categories. These are defined similarly to monoidal categories, except that the morphisms expressing the associativity and unit laws are not required to be invertible. The paper [14] explained the relationship between skew monoidal categories and bialgebroids; this was extended in [10] to the case of quantum categories in place of

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bialgebroids. The question therefore arises as to whether there might be an Eckmann-Hilton-like argument in the skew context. Given the title of the paper, it will come as no surprise that this is the case; equally, given the non-invertibility inherent in the notion of skew monoidal category it should come as no surprise that what we have found is rather less tight than is the case for monoidal categories. Our result is Theorem 2.1 below; see also Remark 2.2 for a discussion of the sense in which it should be seen as an Eckmann-Hilton result. Then in Section 3 we generalize this to the case of internal structures in a symmetric monoidal bicategory \mathcal{M} .

Not that we were led to the above considerations directly! We began with our main application to quantum categories. Since [3], we have known that ordinary categories are monads in the bicategory Span whose morphisms are spans between sets. Quantum categories were originally defined in [7] as monoidal comonads on endomorphism objects in a particular monoidal bicategory \mathcal{M} of comonoids and comodules. When \mathcal{M} is Span , these are equivalent to ordinary categories. As mentioned in the previous paragraph, quantum categories in \mathcal{M} were shown in [10] also to be equivalent to skew monoidal structures (with an appropriate unit) on objects in \mathcal{M} .

The starting point of the present paper was a question by George Janelidze at the Category Theory Conference CT2009 in Calais, France. At the end of the second author's lecture, George asked why the definition of quantum category was so complicated. In his own lecture, George suggested studying monads in the bicategory of comonoids and comodules. This naturally leads to the question: in what kind of \mathcal{M} are quantum categories merely monads? We shall answer this in Section 4.

2. THE CATEGORICAL LEVEL

As mentioned in the introduction, a duoidal (or 2-monoidal) category is a monoidal in the monoidal 2-category of monoidal categories, strong monoidal functors, and monoidal natural transformations. See any of [1, 2, 4, 12] for a more explicit definition.

Our notation for skew monoidales is to write (A, i, p) , where A is the underlying object, p is the multiplication $A \otimes A \rightarrow A$, and i is the unit $I \rightarrow A$. In the case of skew monoidales in \mathbf{Cat} — that is, of skew monoidal categories — the domain of p is the product $A \times A \rightarrow A$, and p gives the tensor product of A ; while the domain of i is the terminal category 1, and we may identify i with its image, the unit object of A . The structure morphisms are invariably called α , λ , and ρ , and are omitted from the notation (A, i, p) .

A *skew duoidal category* (A, k, m, i, p) is a skew monoidal in the 2-category of skew monoidal categories, opmonoidal functors, and opmonoidal natural transformations. So we have two skew monoidal categories (A, i, p) and (A, k, m) such that $k: 1 \rightarrow A$ and $m: A \times A \rightarrow A$ and the constraints are opmonoidal with respect to (A, i, p) . Apart from the two skew monoidal categories, the extra data involved are four natural transformations

$$\begin{array}{ccccc}
 \begin{array}{ccc}
 AAAA & \xrightarrow{mm} & AA \\
 \downarrow 1c1 & & \uparrow m_2 \\
 AAAA & & \\
 \downarrow pp & & \\
 AA & \xrightarrow{m} & A
 \end{array} &
 \begin{array}{ccc}
 11 & \xrightarrow{!} & 1 \\
 \downarrow ii & \uparrow m_0 & \downarrow i \\
 AA & \xrightarrow{m} & A
 \end{array} &
 \begin{array}{ccc}
 11 & \xrightarrow{kk} & AA \\
 \downarrow ! & \uparrow k_2 & \downarrow p \\
 1 & \xrightarrow{k} & A
 \end{array} &
 \begin{array}{ccc}
 1 & \xrightarrow{=} & 1 \\
 \parallel & \uparrow k_0 & \downarrow i \\
 1 & \xrightarrow{k} & A
 \end{array}
 \end{array}$$

where we have omitted the tensor product symbol \otimes to save space. These natural transformations are subject to a long list of conditions which we shall not write out in full, but describe as follows:

- (1) there is an associativity condition for m_2 which involves the map α associated to (A, i, p) ;
- (2) two conditions stating that m_0 is a unit for m_2 , and involving the λ and ρ for (A, i, p) ;
- (3) an associativity condition for k_2 , once again involving the α for (A, i, p) ;
- (4) two unit conditions for k_0 involving the λ and ρ for (A, i, p) ;
- (5) two conditions stating that the α for (A, k, m) is opmonoidal, one of which involves m_2 and the other m_0 ;
- (6) two conditions stating that the λ for (A, k, m) is opmonoidal, one of which involves m_2 and k_2 , the other m_0 and k_0 ;
- (7) and two similar conditions stating that the ρ for (A, k, m) is opmonoidal.

An *opmonoidal monad* is a monad in the 2-category of monoidal categories, opmonoidal functors, and opmonoidal natural transformations. We typically write η for the unit and μ for the multiplication of a monad T , and we write T_2 and T_0 for the opmonoidal structure: here T_0 consists of a single map $TI \rightarrow I$, while T_2 consists of a natural family of morphisms $T(A \otimes B) \rightarrow TA \otimes TB$.

We saw in [10] that such an opmonoidal monad (T, η, μ, T_0, T_2) determines a skew monoidal category $(\mathcal{A}, I, *)$, with the same unit I , via the formulas

$$A * B = TA \otimes B ,$$

$$\begin{array}{ccc} (A * B) * C & \xrightarrow{\alpha_{A,B,C}} & A * (B * C) \\ \parallel & & \parallel \\ T(TA \otimes B) \otimes C & \xrightarrow{v_{A,B} \otimes 1} (TA \otimes TB) \otimes C \xrightarrow{\alpha_{TA,TB,C}} & TA \otimes (TB \otimes C) \end{array}$$

where $v_{A,B}$ is the ‘‘fusion operator’’

$$T(TA \otimes B) \xrightarrow{T_2} TTA \otimes TB \xrightarrow{\mu_A \otimes 1} TA \otimes TB$$

and the unit constraints $\lambda_A: I * A \rightarrow A$ and $\rho_A: A \rightarrow A * I$ are given by the composites

$$\begin{array}{c} I * A \equiv TI \otimes A \xrightarrow{T_0 \otimes 1} I \otimes A \xrightarrow{\lambda_A} A \\ A \xrightarrow{\eta_A} TA \xrightarrow{\rho_{TA}} TA \otimes I \equiv A * I. \end{array}$$

The extra point to be made here is that, if $(\mathcal{A}, I, \otimes)$ is lax braided, we obtain a skew duoidal category via the product and unit maps

$$\begin{array}{c} (A, I, \otimes) \times (A, I, \otimes) \xrightarrow{(*, \gamma)} (A, I, \otimes) \\ 1 \xrightarrow{(I, \mu)} (A, I, \otimes) \end{array}$$

in which the middle-of-four morphism γ is given by

$$\begin{array}{ccc} (A \otimes C) * (D \otimes B) & \xrightarrow{\gamma_{A,C,B,D}} & (A * D) \otimes (C * B) \\ \parallel & & \parallel \\ T(A \otimes C) \otimes (D \otimes B) & \xrightarrow{T_2 \otimes 1} (TA \otimes TC) \otimes (D \otimes B) \xrightarrow{\gamma} & (TA \otimes D) \otimes (TC \otimes B) \end{array}$$

and $\mu: I * I \rightarrow I$ is given by T_0 . Here the γ appearing at the bottom of the diagram is the middle-of-four morphism arising from the lax braiding on $(\mathcal{A}, I, \otimes)$.

Theorem 2.1. *Let $(\mathcal{A}, I, \otimes)$ be a lax-braided monoidal category. The assignment just described is an equivalence between opmonoidal monads (T, η, μ, T_0, T_2) on $(\mathcal{A}, I, \otimes)$ and those skew duoidal structures $(\mathcal{A}, I, *, I, \otimes)$ with $(\mathcal{A}, I, \otimes)$ as the second of the two monoidal structures, for which the following composite is invertible.*

$$A * B \xrightarrow{\rho_A * \lambda_B^{-1}} (A \otimes I) * (I \otimes B) \xrightarrow{\gamma} (A * I) \otimes (I * B) \xrightarrow{1 \otimes \lambda} (A * I) \otimes B \quad (2.1)$$

Proof. Given a skew duoidal category of the form $(\mathcal{A}, I, *, I, \otimes)$ with (2.1) invertible, define an endofunctor $T: \mathcal{A} \rightarrow \mathcal{A}$ by $TA = A * I$. Put η_A equal to $\rho_A: A \rightarrow A * I = TA$, and put $\mu_A: TTA \rightarrow TA$ equal to the composite

$$(A * I) * I \xrightarrow{\alpha} A * (I * I) \xrightarrow{1 * \lambda_I} A * I.$$

This defines a monad (T, η, μ) on \mathcal{A} . The opmonoidal structure is given by

$$\begin{array}{ccc} T(A \otimes B) & \xrightarrow{T_2} & TA \otimes TB \\ \parallel & & \parallel \\ (A \otimes B) * I & \xrightarrow{1 * \rho_I} (A \otimes B) * (I \otimes I) \xrightarrow{\gamma} & (A * I) \otimes (B * I) \\ & & \\ TI & \xrightarrow{T_0} & I \\ & \searrow & \nearrow \\ & I * I & \end{array}$$

□

Remark 2.2. As was mentioned in the introduction, one version of the Eckmann-Hilton argument states that to give to a braided monoidal category a further monoidal structure (in the 2-category of braided monoidal categories and braided strong monoidal functors) is actually not further structure, but just the requirement that the braided monoidal category be symmetric. We regard Theorem 2.1 as a generalization of this fact. Start with a lax-braided monoidal category in place of a braided one, and then consider a further skew monoidal structure on it. This time this *does* give further structure, but provided that we require the composite (2.1) to be invertible, this further structure reduces to an opmonoidal monad on the lax-braided monoidal category. In the non-skew case, this opmonoidal monad would be the identity.

3. THE SYMMETRIC MONOIDAL BICATEGORY CONTEXT

In this section we internalize the results of the previous section, working in a braided monoidal bicategory \mathcal{M} in the sense of [6]. We write as if \mathcal{M} were in fact a 2-category. The braiding is denoted by $c_{A,B}: A \otimes B \rightarrow B \otimes A$.

We write $\text{Mnd}(\mathcal{M})$ for the 2-category of monads in \mathcal{M} , and $\text{Mnd}^*(\mathcal{M})$ for the bicategory $\text{Mnd}(\mathcal{M}^{\text{op}})^{\text{op}}$; the objects of $\text{Mnd}^*(\mathcal{M})$ are still just the monads in \mathcal{M} , but the 1-cells are the opmorphisms of monads: these are similar to morphisms of monads except that the direction of the 2-cell involved in the definition is reversed [11]. (The definition of $\text{Mnd}^*(\mathcal{M})$ does not use the monoidal structure of \mathcal{M} .)

We also write $\text{Skew}(\mathcal{M})$ for the 2-category of skew monoidales, opmonoidal morphisms, and monoidal natural transformations. (This uses the monoidal structure of \mathcal{M} , but not the braiding.)

If \mathcal{M} is in fact braided, then $\text{Skew}(\mathcal{M})$ is also monoidal, and so we can define monoidales and skew monoidales there. A skew monoidale in $\text{Skew}(\mathcal{M})$ consists of skew monoidales (A, i, p) and (A, k, m) such that k, m , and the structure 2-cells α, λ , and ρ for (A, k, m) are opmonoidal with respect to (A, i, p) ; such a structure (A, k, m, i, p) is what we call a skew duoidale in \mathcal{M} .

We also use the full braided monoidal structure of \mathcal{M} when we define $\text{LBrMon}(\mathcal{M})$ to be the monoidal 2-category of lax braided monoidales in \mathcal{M} , with opmonoidal morphisms. For an object $A \in \text{LBrMon}(\mathcal{M})$, we write $\nabla: A \otimes A \rightarrow A$ for the multiplication, $j: I \rightarrow A$ for the unit, and γ for the 2-cell, defined using the lax braiding, which expresses the fact that ∇ is itself opmonoidal. (The remaining structure is generally not mentioned explicitly.)

A lax braided monoidale (A, i, p) determines a skew duoidale (A, i, p, i, p) .

A morphism in $\text{LBrMon}(\mathcal{M})$ from A to B involves a 1-cell $f: A \rightarrow B$ and 2-cells

$$\begin{array}{ccc}
 A \otimes A & \xrightarrow{f \otimes f} & B \otimes B \\
 \nabla \downarrow & \uparrow f_2 & \downarrow \nabla \\
 A & \xrightarrow{f} & B,
 \end{array}
 \qquad
 \begin{array}{ccc}
 & I & \\
 j \swarrow & \xrightarrow{f_0} & \searrow j \\
 A & \xrightarrow{f} & B.
 \end{array}$$

There is a 2-functor $R: \text{Skew}(\mathcal{M}) \rightarrow \text{Mnd}(\mathcal{M})$ sending a skew monoidale (A, i, m) to the monad

$$A \xrightarrow{1 \otimes i} A \otimes A \xrightarrow{m} A$$

with multiplication

$$\begin{array}{ccccc}
 A & \xrightarrow{1i} & AA & & \\
 \downarrow 1i & & \downarrow 1i1 & \xrightarrow{1\lambda} & \\
 AA & \xrightarrow{11i} & AAA & \xrightarrow{1m} & AA \\
 \downarrow m & & \downarrow m1 & \xrightarrow{\alpha} & \downarrow m \\
 A & \xrightarrow{1i} & AA & \xrightarrow{m} & A
 \end{array}$$

and with unit ρ .

In the diagram above we have omitted the tensor products to save space; we have also not explicitly named the invertible 2-cells coming from pseudofunctoriality of the tensor product on \mathcal{M} . We shall continue to follow this practice throughout the paper, also not naming certain isomorphisms which form part of the ‘‘ambient structure’’ in \mathcal{M} or $\text{LBrMon}(\mathcal{M})$, such as the associativity isomorphisms $\nabla \cdot \nabla 1 \cong \nabla \cdot 1 \nabla$ for a lax braided monoidale.

Since $\text{LBrMon}(\mathcal{M})$ is a monoidal bicategory, there is a corresponding 2-functor

$$R: \text{Skew}(\text{LBrMon}(\mathcal{M})) \rightarrow \text{Mnd}^*(\text{LBrMon}(\mathcal{M})) .$$

On the other hand there is a 2-functor

$$T: \text{Mnd}^*(\text{LBrMon}(\mathcal{M})) \rightarrow \text{Skew}(\text{LBrMon}(\mathcal{M}))$$

sending a monad (A, t) to the skew monoidale with multiplication

$$A \otimes A \xrightarrow{t \otimes 1} A \otimes A \xrightarrow{\nabla} A$$

with unit $j: 1 \rightarrow A$, with associativity constraint α given by

$$\begin{array}{ccccc} AAA & \xrightarrow{1t1} & AAA & \xrightarrow{1\nabla} & AA \\ t11 \downarrow & \uparrow \mu t1 & \downarrow t11 & & \downarrow t1 \\ AAA & \xrightarrow{t11} & AAA & \xrightarrow{1\nabla} & AA \\ \nabla 1 \downarrow & \uparrow t_2 1 & \downarrow \nabla 1 & & \downarrow \nabla \\ AA & \xrightarrow{t1} & AA & \xrightarrow{\nabla} & A \end{array}$$

with right unit constraint ρ given by

$$\begin{array}{ccc} A & \xrightarrow{1j} & AA \\ \left(\begin{array}{c} \eta \\ \Rightarrow \end{array} \right) t & & \downarrow t1 \\ A & \xrightarrow{1j} & AA \\ & \searrow 1 & \downarrow \nabla \\ & & A \end{array}$$

and with left unit constraint λ given by

$$\begin{array}{ccc} AA & \xleftarrow{j1} & A \\ t1 \downarrow & \xrightarrow{t_0 1} & \downarrow t1 \\ AA & \xleftarrow{j1} & A \\ \nabla \downarrow & & \downarrow \nabla \\ A & & A \end{array}$$

Now consider the composite RT . This sends a monad t on A to a monad on A whose underlying 1-cell is the right hand composite in the diagram

$$\begin{array}{ccc} A & \xrightarrow{1j} & AA \\ t \downarrow & & \downarrow t1 \\ A & \xrightarrow{1j} & AA \\ & \searrow 1 & \downarrow \nabla \\ & & A \end{array}$$

(in which the two regions commute up to isomorphisms coming from pseudofunctoriality of the tensor in $\text{LBrMon}(\mathcal{M})$, and the right unit constraint for the lax braided

monoidal structure on A). Compatibility of this isomorphism with the units for the monads holds by definition of the monad on the right, and a straightforward calculation gives compatibility with the multiplications for the monads as well.

Thus we have an isomorphism $RT \cong 1$, whose component at an object (A, t) of $\text{Mnd}(\text{LBrMon}(\mathcal{M}))$ is the morphism $(A, t) \rightarrow RT(A, t)$ of monads which is the identity $A \rightarrow A$ equipped with the isomorphism of monads described above.

Now consider the other composite TR . Suppose that $A = (A, i, m)$ is a skew monoidal in $\text{LBrMon}(\mathcal{M})$, for which $i: 1 \rightarrow A$ is strong (op)monoidal, as will always be the case for an object in the image of T . In particular, we have $i \cong j$, so we may as well take i to be j itself.

For such an A , we have a 2-cell

$$\begin{array}{ccc} AAA & \xrightarrow{m_1} & AA \\ 1\nabla \downarrow & \psi \uparrow\uparrow & \downarrow \nabla \\ AA & \xrightarrow{m} & A \end{array}$$

given by the composite

$$\begin{array}{ccc} AAA & \xrightarrow{m_1} & AA \\ \downarrow 1\nabla & \uparrow m\lambda & \downarrow \nabla \\ AAA & \xrightarrow{mm} & AA \\ \downarrow \nabla^2 & m_2 \uparrow\uparrow & \downarrow \nabla \\ AA & \xrightarrow{m} & A \end{array}$$

where $\nabla^2 = (A^4 \xrightarrow{1c_{A,A^1}} A^4 \xrightarrow{\nabla\nabla} A^2)$ is the multiplication on A^2 .

Proposition 3.1. *The 2-cell ψ satisfies*

$$\begin{array}{ccc} & A^3 \xrightarrow{1\nabla} A^2 & \\ m_{11} \nearrow & & \searrow \nabla \\ A^4 & \psi \uparrow\uparrow & A^2 \xrightarrow{\nabla} A \\ 1\nabla \searrow & m_1 \nearrow & \uparrow \psi \\ & A^3 \xrightarrow{1\nabla} A^2 & \end{array} = \begin{array}{ccc} & A^3 \xrightarrow{1\nabla} A^2 & \\ m_{11} \nearrow & m_1 \nearrow & \searrow \nabla \\ A^4 & \xrightarrow{1\nabla} A^3 & \psi \uparrow\uparrow & A \\ 1\nabla \searrow & 1\nabla \searrow & \uparrow \psi & \nearrow m \\ & A^3 \xrightarrow{1\nabla} A^2 & \end{array}$$

Proof. Use naturality, coassociativity of m_2 , monoidal axioms for (A, ∇, j) , and op-monoidality of λ . □

Proposition 3.2. *The 2-cell ψ satisfies*

$$\begin{array}{ccc} & A^3 \xrightarrow{m_1} A^2 & \\ 1m_1 \nearrow & & \searrow \nabla \\ A^4 & 1\psi \uparrow\uparrow & A^2 \xrightarrow{m} A \\ 1\nabla \searrow & 1m \nearrow & \uparrow \psi \\ & A^3 \xrightarrow{m_1} A^2 & \end{array} = \begin{array}{ccc} & A^3 \xrightarrow{m_1} A^2 & \\ 1m_1 \nearrow & \alpha \uparrow\uparrow & \searrow \nabla \\ A^4 & \xrightarrow{m_{11}} A^3 & m_1 \uparrow\uparrow & A \\ 1\nabla \searrow & 1\nabla \searrow & \uparrow \psi & \nearrow m \\ & A^3 \xrightarrow{m_1} A^2 & \end{array}$$

Proof. Rewrite m_2 in terms of $(1m)_2$, then use naturality and opmonoidality of α , and the skew monoidale axioms for (A, m, j) . \square

Restricting ψ along $1j1: AA \rightarrow AAA$ and using the isomorphism $1\nabla.1j1 \cong 1$ gives a 2-cell

$$\begin{array}{ccc} AA & \xrightarrow{t1} & AA \\ \parallel & \chi \Uparrow & \Downarrow \nabla \\ AA & \xrightarrow{m} & A \end{array}$$

where t is the induced monad, given by $m.1j$.

Proposition 3.3. *The 2-cells ψ and χ are linked via the equation*

Proof. Take the equality in Proposition 3.1 and restrict along the arrow $1j11: A^3 \rightarrow A^4$. \square

We shall show that χ is compatible with the associativity and unit constraints and so makes the identity morphism $1: A \rightarrow A$ into a morphism of skew monoidales from (A, m) to $TR(A, m)$.

Restricting λ along $j: 1 \rightarrow A$ gives $m_0: m.jj \rightarrow j$; it follows that χ is compatible with the right unit constraints. Compatibility with the left unit constraints once again uses the fact that $\lambda.j = m_0$, along with the fact that λ is opmonoidal.

It remains to check that χ is compatible with the associativity constraints. This says that the composites

are equal, where α' is the associativity constraint for $TR(A, m, i)$, given by

$$\begin{array}{ccccccc}
 A^3 & \xrightarrow{A^2jA} & A^4 & \xrightarrow{AmA} & A^3 & \xrightarrow{A\nabla} & A^2 \\
 \downarrow AjA^2 & & \downarrow AjjA^3 & \uparrow Am_0mA & \downarrow AjA^2 & & \downarrow AjA \\
 A^4 & \xrightarrow{A^2jAjA} & A^6 & \xrightarrow{AmmA} & A^4 & \xrightarrow{A^2\nabla} & A^3 \\
 \downarrow mA^2 & & \downarrow mA^3 & \uparrow \alpha mA & \downarrow mA^2 & & \downarrow mA \\
 A^3 & \xrightarrow{AjAjA} & A^5 & \xrightarrow{mmA} & A^3 & \xrightarrow{A\nabla} & A^2 \\
 \downarrow \nabla A & & \downarrow \nabla^2 A & \uparrow m_2A & \downarrow \nabla A & & \downarrow \nabla \\
 A^2 & \xrightarrow{AjA} & A^3 & \xrightarrow{mA} & A^2 & \xrightarrow{\nabla} & A
 \end{array}$$

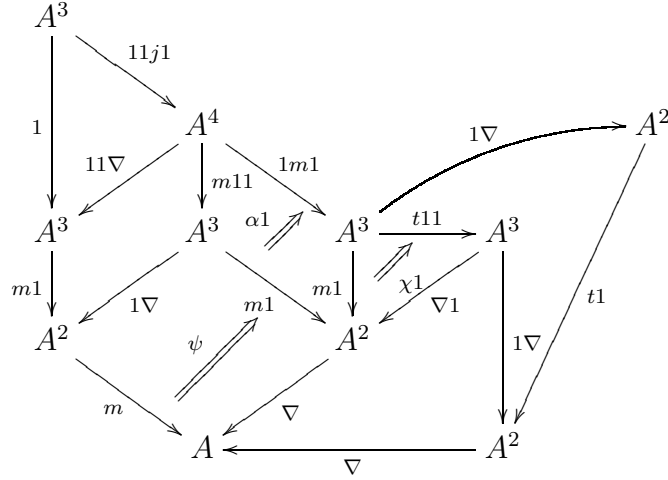
where $\nabla^2: A^4 \rightarrow A^2$ denotes the multiplication on AA , defined using ∇ and the braiding. We can rewrite this as

$$\begin{array}{ccccccc}
 A^3 & \xrightarrow{A^2jA} & A^4 & \xrightarrow{AmA} & A^3 & \xrightarrow{A\nabla} & A^2 \\
 \downarrow AjA^2 & & \downarrow AjjA^3 & \uparrow Am_0mA & \downarrow AjA^2 & & \downarrow AjA \\
 A^4 & \xrightarrow{A^3jA} & A^5 & \xrightarrow{A^2jA^3} & A^6 & \xrightarrow{AmmA} & A^4 & \xrightarrow{A^2\nabla} & A^3 \\
 \downarrow mA^2 & & \downarrow mA^3 & & \downarrow mA^4 & \uparrow \alpha mA & \downarrow mA^2 & & \downarrow mA \\
 A^3 & \xrightarrow{A^2jA} & A^4 & \xrightarrow{AjA^3} & A^5 & \xrightarrow{mmA} & A^3 & \xrightarrow{A\nabla} & A^2 \\
 \downarrow \nabla A & & \searrow \nabla A^2 & \downarrow \nabla^2 A & \downarrow \nabla A & & \downarrow \nabla A & & \downarrow \nabla \\
 A^2 & \xrightarrow{AjA} & A^3 & \xrightarrow{mA} & A^2 & \xrightarrow{\nabla} & A
 \end{array}$$

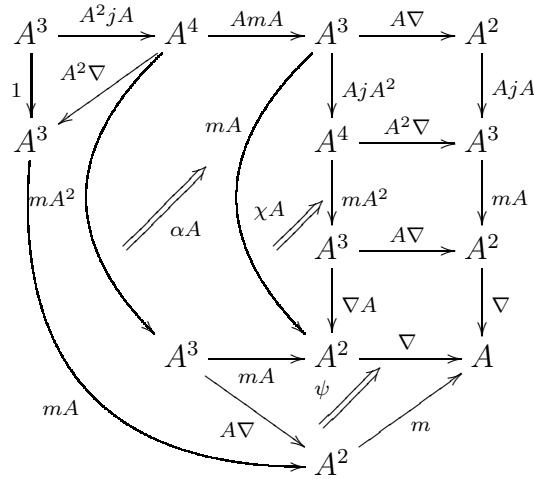
and now the left hand side of (3.2) becomes

$$\begin{array}{ccccccc}
 A^3 & \xrightarrow{A^2jA} & A^4 & \xrightarrow{AmA} & A^3 & \xrightarrow{A\nabla} & A^2 \\
 \downarrow AjA^2 & & \downarrow AjjA^3 & \uparrow Am_0mA & \downarrow AjA^2 & & \downarrow AjA \\
 A^4 & \xrightarrow{A^3jA} & A^5 & \xrightarrow{A^2jA^3} & A^6 & \xrightarrow{AmmA} & A^4 & \xrightarrow{A^2\nabla} & A^3 \\
 \downarrow mA^2 & & \downarrow mA^3 & & \downarrow mA^4 & \uparrow \alpha mA & \downarrow mA^2 & & \downarrow mA \\
 A^3 & \xrightarrow{A^2jA} & A^4 & \xrightarrow{AjA^3} & A^5 & \xrightarrow{mmA} & A^3 & \xrightarrow{A\nabla} & A^2 \\
 \downarrow \nabla A & & \searrow \nabla A^2 & \downarrow \nabla^2 A & \downarrow \nabla A & & \downarrow \nabla A & & \downarrow \nabla \\
 A^2 & \xrightarrow{AjA} & A^3 & \xrightarrow{mA} & A^2 & \xrightarrow{\nabla} & A \\
 \swarrow 1 & \swarrow A\nabla A & \swarrow \psi & \swarrow 1 & \swarrow A\nabla & \swarrow \psi & \swarrow m & & \\
 A^3 & & A^3 & & A^2 & & A^2 & &
 \end{array}$$

which by Proposition 3.2 is the same as



which we can rewrite as



and now (3.2) will follow if we can prove

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 A^2 & \xrightarrow{A^2j} & A^3 & \xrightarrow{Am} & A^2 \\
 \downarrow AjAj & & \downarrow AjjA^2 & \uparrow Am_0m & \downarrow AjA \\
 A^4 & \xrightarrow{A^2jA^2} & A^5 & \xrightarrow{Amm} & A^3 \\
 \downarrow A^2j & \searrow mA^2 & \downarrow mA^3 & \uparrow \alpha m & \downarrow mA \\
 A^3 & \xrightarrow{AjA^2} & A^4 & \xrightarrow{mm} & A^2 \\
 \downarrow A\bar{\nabla}A & \uparrow \psi A & \downarrow \bar{\nabla}A & \uparrow m_2 & \downarrow \bar{\nabla} \\
 A^3 & \xrightarrow{mA} & A^2 & \xrightarrow{m} & A
 \end{array} & = & \begin{array}{ccc}
 A^2 & \xrightarrow{A^2j} & A^3 & \xrightarrow{Am} & A^2 \\
 \downarrow AjA & & \downarrow AjA & & \downarrow AjA \\
 A^3 & \xrightarrow{mA} & A^2 & \xrightarrow{m} & A
 \end{array} \\
 & & \begin{array}{ccc}
 & \xrightarrow{\alpha} & \\
 & \xrightarrow{\chi} & \\
 & \xrightarrow{m} &
 \end{array}
 \end{array} \tag{3.3}$$

4. QUANTUM CATEGORIES IN THE CARTESIAN CONTEXT

In this final section we turn to the question of which monoidal bicategories \mathcal{M} have the property that quantum categories in \mathcal{M} are just monads.

The basic example of such an \mathcal{M} is the bicategory Span of sets and spans. The cartesian product of sets provides Span with a monoidal structure, although it is not a bicategorical product in Span. This bicategory has been studied from many points of view; the relevant one here is that it is a *cartesian bicategory* in the sense of [5].

The first property of cartesian bicategories that we use is that every left adjoint in a cartesian bicategory is opmonadic, and so in particular restriction along any left adjoint is conservative. Thus the hypotheses of Theorem 3.5 are satisfied.

The other key property of a cartesian bicategory \mathcal{M} is that every object has a canonical symmetric monoidal structure, with respect to which every morphism has symmetric opmonoidal structure, and with respect to these, every 2-cell is opmonoidal. It follows that the forgetful 2-functor $\text{LBrMon}(\mathcal{M}) \rightarrow \mathcal{M}$ from the 2-category of lax braided monoidales in \mathcal{M} is a biequivalence.

Combining the previous two theorems we now deduce:

Theorem 4.1. *For a (strict) cartesian bicategory \mathcal{M} , the 2-category $\text{Mnd}^*(\mathcal{M})$ of monads in \mathcal{M} is biequivalent to the 2-category $\text{Skew}(\mathcal{M})$ of left skew monoidales in \mathcal{M} with unit $I \rightarrow A$ given by the unique map.*

The bicategory Span of spans of sets can be generalized to a bicategory $\text{Span}(\mathcal{E})$ of spans in a finitely complete category \mathcal{E} ; taking \mathcal{E} to be the category of sets and functions, we recover Span itself. The bicategory $\text{Span}(\mathcal{E})$ is also cartesian, and so in $\text{Span}(\mathcal{E})$ once again quantum categories are just monads.

REFERENCES

- [1] Marcelo Aguiar and Swapneel Mahajan, *Monoidal Functors, Species and Hopf Algebras*, CRM Monograph Series **29** (American Mathematical Society, November 2010). [1](#), [2](#)
- [2] Michael Batanin and Martin Markl, *Centers and homotopy centers in enriched monoidal categories*, *Advances in Mathematics* **230(4–6)** (2012) 1811–1858. [2](#)
- [3] Jean Bénabou, *Introduction to bicategories*, *Lecture Notes in Mathematics* **47** (Springer-Verlag, 1967) 1–77. [2](#)
- [4] Thomas Booker and Ross Street, *Tannaka duality and convolution for duoidal categories*, arXiv:1111.5659 (2011). [1](#), [2](#)
- [5] Aurelio Carboni, G. Max Kelly, Robert F.C. Walters, and Richard J. Wood, *Cartesian bicategories II*, *Theory and Applications of Categories* **19(6)** (2008) 93–124. [14](#)
- [6] Brian J. Day and Ross Street, *Monoidal bicategories and Hopf algebroids*, *Advances in Math.* **129** (1997) 99–157. [4](#)
- [7] Brian J. Day and Ross Street, *Quantum categories, star autonomy, and quantum groupoids*, in “Galois Theory, Hopf Algebras, and Semiabelian Categories”, *Fields Institute Communications* **43** (American Math. Soc. 2004) 187–226. [2](#)
- [8] Beno Eckmann and Peter J. Hilton, *Group-like structures in general categories. I. Multiplications and comultiplications*, *Math. Ann.* **145** (1961/1962) 227–255. [1](#)
- [9] André Joyal and Ross Street, *Braided tensor categories*, *Advances in Mathematics* **102** (1993) 20–78. [1](#)
- [10] Stephen Lack and Ross Street, *Skew monoidales, skew warpings and quantum categories*, *Theory and Applications of Categories* **27(3)** (2012) 27–46. [1](#), [2](#), [3](#)
- [11] Ross Street, *The formal theory of monads*, *Journal of Pure and Applied Algebra* **2** (1972) 149–168. [5](#)

- [12] Ross Street, *Monoidal categories in, and linking, geometry and algebra*, Bulletin Belgian Math. Society — Simon Stevin **19(5)** (2012) 769–821. [1](#), [2](#)
- [13] Ross Street, *Skew-closed categories*, Journal of Pure and Applied Algebra **217** (2013) 973–988. [1](#)
- [14] Kornel Szlachányi, *Skew-monoidal categories and bialgebroids*, Advances in Mathematics **231(3–4)** (2012) 1694–1730. [1](#)
- [15] Robert F.C. Walters, *Bicategories with tensor products*, Report to the Sydney Category Seminar on a discussion of A. Carboni, F.W. Lawvere and R.F.C. Walters (Dec. 1984). [1](#)

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