Algorithmic Correspondence and Canonicity for Distributive Modal Logic

Willem Conradie        Alessandra Palmigiano

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10 July 2009
Sahlqvist theory

Modal formulas define classes of Kripke frames through the notion of validity. Via validity, every modal formula semantically corresponds to a monadic second order formula. Some modal formulas semantically correspond to first order formulas (undecidable property of modal formulas [Chagrova]).

Sahlqvist theory gives syntactic conditions on modal formulas that are guaranteed a local first order correspondent (Sahlqvist formulas). Effectively computes their first order correspondents (Reduction strategies). Sahlqvist formulas are canonical (proof via correspondence). Sahlqvist formulas generate logics that are strongly complete w.r.t. first-order definable classes of frames.
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Two generalizations of Sahlqvist theory: algebraically

Algebraic perspective on the classical setting

From Kripke frames to their algebraic duals:
Perfect BAOs.

Sahlqvist reduction strategies rephrased in perfect BAOs.

From perfect BAs to perfect DLs

Sahlqvist formulas for $L^K$ generalize to Sahlqvist inequalities for Distributive Modal Logic [Gehrke Nagahashi Venema].

$\phi ::= p \in \text{AtProp} | \top | \bot | \phi \land \psi | \phi \lor \psi | \Box \phi | \neg \phi | \Diamond \phi | \Leftarrow \phi$

Sahlqvist reduction strategies are essentially the same as in the Boolean setting!

Benefits

Sahlqvist theory available to e.g. PML, intuitionistic modal logics.

Canonicity treated independently of (global) correspondence.

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Model-theoretic generalization of Sahlqvist theory

Inductive formulas \[\text{[Goranko Vakarelov]}\] are syntactically defined, proper extension of Sahlqvist formulas. These are in general a proper semantic extension of Sahlqvist formulas.

SQEMA-algorithm \[\text{[Conradie Goranko Vakarelov]}\] based on the Ackermann's lemma. This generates local first-order correspondents of input modal formulas. SQEMA-formulas properly cover all inductive formulas. SQEMA-formulas are canonical. SQEMA-formulas still lack a syntactic characterization.
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Merging paths: main contributions [Conradie P.]

Inductive inequalities are a syntactically defined, proper extension of the Sahlqvist inequalities for Distributive Modal Logic, as to their restriction to classical inductive formulas:

- Classical
- Distributive
- Recursive definition
- Forbidden combination
- Sahlqvist fm's
- Van Benthem fm's
- Inductive fm's [GV]
- Inductive fm's [CP]
- Sahlqvist ineq's [GNV]
- Inductive ineq's [CP]

ALBA-algorithm generates local first-order corrected input DM inequalities. Properly covers all inductive (hence all Sahlqvist) inequalities. Consequence: Sahlqvist inequalities have local first-order corrected. ALBA-inequalities are canonical (proof via correspondence).
Inductive inequalities

The slide discusses the main contributions by Conradie P. on inductive inequalities. It mentions a syntactically defined, proper extension of the Sahlqvist inequalities for Distributive Modal Logic. The focus is on how these inequalities apply to classical inductive formulas, contrasting with forbidden combinations in Sahlqvist inequalities and van Benthem inequalities. It also mentions the generation of local first-order corrections by the ALBA-algorithm and the consequence that Sahlqvist inequalities have local first-order corrections. The ALBA-inequalities are noted to be canonical, with a proof via correspondence.
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An informal description of ALBA

**Based Algorithm**

Its core is a DL version of Ackermann's lemma.

Three stages: preprocessing, reduction rules and Ackermann elimination step.

Reduction rules: residuation, approximation.

Crucial use of the perfect DL environment: approximation: both $\lor$-generated by the c. $\lor$-primes and $\land$-gen. by the c. $\land$-primes; residuation: by completeness, all the operations are either right- or left-adjoints.

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**Ackermann Lemma Based Algorithm**

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  j \leq \Box (\Diamond q \lor p) \land \Box q, \\
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    \Diamond j - \Diamond q \leq p, \quad \Diamond (p \land q) \leq m \\
    \Diamond j \leq q
\end{array} \right. \lor \text{ residuation}
\]
\( \Box (\Diamond q \lor p) \land \Box q \leq \Diamond (p \land q) \)

\[
\begin{align*}
\{ \ j \leq \Box (\Diamond q \lor p) \land \Box q, \quad \Diamond (p \land q) \leq m \ \} & \text{ first approx.} \\
\{ \ j \leq \Box q, \quad \Diamond (p \land q) \leq m \ \} & \text{ splitting} \\
\{ \ \Diamond j \leq \Diamond q \lor p, \quad \Diamond (p \land q) \leq m \ \} & \Box \text{residuation} \times 2 \\
\{ \ \Diamond i \leq q \ \} & \\
\{ \ \Diamond j \land \Diamond q \leq \Diamond j, \quad \Diamond (p \land q) \leq m \ \} & \lor \text{residuation} \\
\{ \ \Diamond j \leq q \ \} & \\
\{ \ \Diamond j \leq q, \quad \Diamond ((\Diamond j \land \Diamond q) \land q) \leq m \ \} & \text{Ackermann elim. of } p
\end{align*}
\]
\(\square(\triangleleft q \lor p) \land \square q \leq \lozenge(p \land q)\)

\[
\begin{cases}
  j \leq \square(\triangleleft q \lor p) \land \square q,
  \lozenge(p \land q) \leq m \\
  \end{cases}
\text{first approx.}
\]

\[
\begin{cases}
  j \leq \square(\triangleleft q \lor p),
  \lozenge(p \land q) \leq m \\
  j \leq \square q
\end{cases}
\text{splitting}
\]

\[
\begin{cases}
  \lozenge j \leq \triangleleft q \lor p,
  \lozenge(p \land q) \leq m \\
  \lozenge i \leq q
\end{cases}
\square\text{residuation} \times 2
\]

\[
\begin{cases}
  \lozenge j - \triangleleft q \leq p,
  \lozenge(p \land q) \leq m \\
  \lozenge j \leq q
\end{cases}
\lor\text{residuation}
\]

\[
\begin{cases}
  \lozenge j \leq q,
  \lozenge((\lozenge j - \triangleleft q) \land q) \leq m
\end{cases}
\text{Ackermann elim. of } p
\]

\[
\begin{cases}
  \lozenge((\lozenge j - \triangleleft \lozenge j) \land \lozenge j) \leq m
\end{cases}
\text{Ackermann elim. of } q
\]
Unfinished business

Expanding the signature \([CP]\).

Algorithmic proof of canonicity for linear inductive formulas \([CP]\) and for inductive inequalities \([S. van Gool]\).

Algorithmic correspondence in the non-distributive setting \([CP]\).

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