On a \textit{a posteriori} error estimator
for the discontinuous Galerkin method

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\textbf{Abstract}

We present in this paper a new \textit{a posteriori} error estimator for the Baumann-Oden version of
the Discontinuous Galerkin Method. The error estimator is based on the residual of the partial
differential equation. In the case of the reaction-diffusion equation, the norm of the residual is
shown to be equivalent to the error in some specific energy-type norms. We propose here a method
to efficiently calculate the norm of the residual and present some numerical experiments which
demonstrate the reliability of the methodology.

\textbf{1 Introduction}

In the present paper, we report on new results dealing with \textit{a posteriori} error estimation for \textit{hp}
approximations obtained by a slightly modified (stabilized) version of the
Baumann-Oden Discontinuous Galerkin finite element method \cite{1, 2}. For theoretical
purposes, we consider here the Poisson problem as the model problem. To the best
knowledge of the authors, there is to date no published work on this subject, except
maybe the experimental work of Rivière \textit{et al.} \cite{4, 5}, which introduces an \textit{a posteriori}
error estimator for the same class of problems, but lacks a complete theoretical basis.
We also show some preliminary numerical results obtained on a two-point boundary
value problem to demonstrate the effectivity of our methodology.

The paper is organized as follows. Section 2 introduces some notations as well as the
model problem and the discontinuous Galerkin formulation. In Section 3, we present
the theoretical basis to derive the \textit{a posteriori} error estimator. Numerical experiments
are described in Section 4, and some concluding remarks are given in Section 5.

\textbf{2 Notations and preliminaries}

Let $\Omega$ denote a bounded open set in $\mathbb{R}^d$, $d = 1$ or 2, with Lipschitz continuous
boundary $\partial \Omega$. The parts of the boundary on which Dirichlet and Neumann conditions
are prescribed are respectively denoted $\Gamma_D$ and $\Gamma_N$, such that $\Gamma_D \cup \Gamma_N = \partial \Omega$, and
$\Gamma_D \cap \Gamma_N = \emptyset$. 
2.1 Finite element partition

A finite element partition $\mathcal{P}_h$ of $\Omega$ consists of a finite collection of $N_e$ open elements $K_i$, $i = 1, 2, \ldots, N_e$, such that:

$$\overline{\Omega} = \bigcup_{K_i \in \mathcal{P}_h} \overline{K_i}, \quad \text{and} \quad K_i \cap K_j = \emptyset \quad \text{for} \quad i \neq j.$$ 

Introducing the size $h_K$ of an element $K$ as the diameter of $K$, we associate with each partition the parameter $h$: $h = \max_{K \in \mathcal{P}_h} h_K$. In addition, the boundary of an element $K$ is denoted $\partial K$, while the unit normal vector outward from $K$ is denoted by $\mathbf{n}$.

Given a partition $\mathcal{P}_h$, the collection of element interfaces (points, edges, and faces, in one, two, and three dimensions respectively) is denoted by the set $\mathcal{E}_h = \{\gamma_l\}$, $l = 1, \ldots, N_{\gamma}$. The set $\Gamma_{\text{int}}$ of interior interfaces is defined as

$$\Gamma_{\text{int}} = \left( \bigcup_{l=1}^{N_{\gamma}} \gamma_l \right) \setminus \partial \Omega \quad (2.1)$$

so that:

$$\bigcup_{l=1}^{N_{\gamma}} \gamma_l = \overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}_{\text{int}}.$$ 

For each $\gamma_l$, we also associate a unit normal vector $\mathbf{n}$. In the case $\gamma$ is an interface associated with an element $K$ adjacent to $\partial \Omega$, the unit normal vector is simply defined as $\mathbf{n}$. For an interior edge $\gamma_{ij}$ in two dimensions for example, shared by two elements $K_i$ and $K_j$, where by convention $i > j$, $\mathbf{n}$ is chosen as the unit normal vector outward from $K_i$. Unless specified otherwise, we shall assume in the following analysis that the space dimension is equal to two.

2.2 Function spaces

Given a positive integer $s$ and an open set $S$ of $\mathbb{R}^2$ ($S$ may define the whole domain $\Omega$, an element $K$ of $\mathcal{P}_h$, or an edge $\gamma$ of $\mathcal{E}_h$), $H^s(S)$ stands for the usual Sobolev space with norm $\| \cdot \|_{s,S}$. When $S$ represents $\Omega$, the norm will simply be written $\| \cdot \|_s$. The so-called broken space $H^s(\mathcal{P}_h)$ associated with the partition $\mathcal{P}_h$ is defined as:

$$H^s(\mathcal{P}_h) = \{ v \in L^2(\Omega); \ v|_K \in H^s(K), \ \forall K \in \mathcal{P}_h \}.$$ 

Here, we will consider finite element subspaces $V^{hp}$ of $H^s(\mathcal{P}_h)$ of polynomial functions, possibly discontinuous at the element interfaces, such as:

$$V^{hp} = \{ v \in L^2(\Omega); \ v|_K = \hat{v} \circ F_K^{-1}, \hat{v} \in P_{p_K}(\overline{K}), \ \forall K \in \mathcal{P}_h \} \quad (2.2)$$
where $F_K$ is the affine mapping from the master element $\hat{K}$ to the element $K$ in the partition, and $P_{p_K}(\hat{K})$ is the space of all polynomial functions of degree at most $p_K$ on $\hat{K}$. Denoting $p_K$ the polynomial degree associated with the element $K$, we define the global parameter $p$ for the partition $\mathcal{P}_h$ as: $p = \min_{K \in \mathcal{P}_h} p_K$. One advantage of DGMs over conventional $hp$ finite element methods is that the polynomial degrees $p_K$ do not necessarily match at the interfaces of the elements.

2.3 Model problem and weak forms

We consider the following model problem: find the scalar function $u$ such that

$$-\Delta u + cu = f, \quad \text{in } \Omega,$$

which satisfies the boundary conditions:

$$u = u_0, \quad \text{on } \Gamma_D,$$
$$\mathbf{n} \cdot \nabla u = g, \quad \text{on } \Gamma_N.$$  

Here, $f \in L^2(\Omega)$ represents a load scalar and $c$ a positive constant on $\Omega$.

We introduce the bilinear form $B(\cdot, \cdot)$ defined on $H^2(\mathcal{P}_h) \times H^2(\mathcal{P}_h)$ and the linear form $F(\cdot)$ on $H^2(\mathcal{P}_h)$ such as:

$$B(u, v) = \sum_{K \in \mathcal{P}_h} \int_K (\nabla u \cdot \nabla v + cuv) \, dx,$$

$$F(v) = \sum_{K \in \mathcal{P}_h} \int_K f v \, dx + \int_{\Gamma_N} g v \, ds.$$  

We also consider the bilinear form $J(\cdot, \cdot)$ on $H^2(\mathcal{P}_h) \times H^2(\mathcal{P}_h)$ which involve all boundary integrals on $\Gamma_{int}$ and $\Gamma_D$:

$$J(u, v) = \sum_{\gamma \subset \Gamma_{int} \cup \Gamma_D} \int_{\gamma} \langle \mathbf{n} \cdot \nabla u \rangle \, [v] \, ds = \int_{\Gamma_{int} \cup \Gamma_D} \langle \mathbf{n} \cdot \nabla u \rangle \, [v] \, ds.$$  

as well as the linear form on $H^2(\mathcal{P}_h)$ associated with the boundary $\Gamma_D$:

$$J_0(v) = \int_{\Gamma_D} u_0 (\mathbf{n} \cdot \nabla v) \, ds.$$  

The quantities $[v]$ and $\langle v \rangle$ denote the jump and average of any function $v \in L^2(K_i \cup K_j)$ on a interior edge $\gamma_{ij} \subset \Gamma_{int}$, $i > j$. By extension, $[v]$ and $\langle v \rangle$ on an boundary edge $\Gamma_D$ stands for the function itself

$$[v] = \begin{cases} v_i - v_j & \gamma \subset \Gamma_{int} \\ v & \gamma \subset \Gamma_D \end{cases} \quad \langle v \rangle = \begin{cases} \frac{1}{2} (v_i + v_j) & \gamma \subset \Gamma_{int} \\ v & \gamma \subset \Gamma_D \end{cases}$$

3
The weak formulation of (2.3) as established in the Baumann-Oden version of the Discontinuous Galerkin Method (see [1, 2]) consists in finding $u$ such that

$$B(u, v) = F(v), \quad \forall v \in H^2(\mathcal{P}_h),$$

(2.9)

where

$$B(u, v) = B(u, v) - J(u, v) + J(v, u),$$

$$F(v) = J_0(v).$$

(2.10)

In the present paper, we add a stabilization term of the form:

$$J^\sigma(u, v) = \int_{\Gamma_{\text{int}}} \frac{h_K}{4} [n \cdot \nabla u][n \cdot \nabla v] \, ds$$

(2.11)

so that the new bilinear form reads:

$$B(u, v) = B(u, v) - J(u, v) + J(v, u) + J^\sigma(u, v)$$

(2.12)

The associated finite element version consists in finding $u_h \in \mathcal{V}^{hp}$ such that

$$B(u_h, v) = F(v), \quad \forall v \in \mathcal{V}^{hp}.$$ 

(2.13)

It can be shown that a solution $u \in C^2(\overline{\Omega})$ of Problem (2.3)-(2.4) satisfies the weak formulation (2.9). Conversely, if $u \in H^1(\Omega) \cap H^2(\mathcal{P}_h)$ is a solution of (2.9), then $u$ satisfies the partial differential equation (2.3) and boundary conditions (2.4). Although such a result ensures the existence of solutions of (2.9), it does not imply that these solutions are unique.

### 2.4 Norm and continuity of the linear forms

We will consider in this paper the norm:

$$\|v\|_{\mathcal{P}_h}^2 = \sum_{K \in \mathcal{P}_h} \int_K |\nabla v|^2 + c|v|^2 \, dx$$

$$+ \sum_{K \in \mathcal{P}_h} \int_{\partial K \setminus \Gamma_N} h_K (n \cdot \nabla v)^2 \, ds + \int_{\Gamma_{\text{int}} \cup \Gamma_D} \frac{1}{h_K} |v|^2 \, ds$$

(2.14)

Introducing the new bilinear forms $I_1(\cdot, \cdot)$ and $I_2(\cdot, \cdot)$ on $H^2(\mathcal{P}_h) \times H^2(\mathcal{P}_h)$ as:

$$I_1(u, v) = \sum_{K \in \mathcal{P}_h} \int_{\partial K \setminus \Gamma_N} h_K (n \cdot \nabla u)(n \cdot \nabla v) \, ds,$$

$$I_2(u, v) = \int_{\Gamma_{\text{int}} \cup \Gamma_D} \frac{1}{h_K} [u][v] \, ds,$$

we naturally have

$$\|v\|_{\mathcal{P}_h}^2 = B(v, v) + I_1(v, v) + I_2(v, v).$$
Lemma 2.1 Let $B(\cdot, \cdot)$ be the bilinear form defined in (2.12). Then $B(\cdot, \cdot)$ is continuous with respect to the norm $\| \cdot \|_{P_h}$, i.e.

$$|B(u, v)| \leq \|u\|_{P_h} \|v\|_{P_h}, \quad \forall u, v \in H^2(P_h).$$

(2.15)

Proof: From the definition of the bilinear form, we have for any $u$ and $v$ in $H^2(P_h)$:

$$|B(u, v)| \leq |B(u, v)| + |J(u, v)| + |J(v, u)| + |J''(u, v)|$$

Applying Cauchy-Schwarz, the first term is simply bounded by:

$$|B(u, v)| \leq \sqrt{\sum_{K \in P_h} \int_K |\nabla u|^2 + c|u|^2 \, dx} \sqrt{\sum_{K \in P_h} \int_K |\nabla v|^2 + c|v|^2 \, dx}$$

The second term gives:

$$|J(u, v)| \leq \sqrt{\int_{\Gamma_{int} \cup \Gamma_D} h_K (n \cdot \nabla u)^2 \, dx} \sqrt{\int_{\Gamma_{int} \cup \Gamma_D} \frac{1}{h_K} [v]^2 \, dx}.$$  

Since on each interior edge, the flux average is bounded by:

$$\langle n \cdot \nabla u \rangle = \frac{1}{4} n \cdot (\nabla u|_K + \nabla v|_L)^2 = \frac{1}{4} ((n \cdot \nabla u)_K - (n \cdot \nabla v)_L)^2$$

$$\leq \frac{1}{2} ((n \cdot \nabla u)_K^2 + (n \cdot \nabla v)_L^2),$$

it follows that

$$|J(u, v)| \leq \frac{1}{\sqrt{2}} \sqrt{\sum_{K \in P_h} \int_{\partial K \setminus \Gamma_N} h_K (n \cdot \nabla u)^2 \, ds} \sqrt{\int_{\Gamma_{int} \cup \Gamma_D} \frac{1}{h_K} [v]^2 \, dx}.$$  

Similarly, we get:

$$|J(v, u)| \leq \frac{1}{\sqrt{2}} \sqrt{\sum_{K \in P_h} \int_{\partial K \setminus \Gamma_N} h_K (n \cdot \nabla v)^2 \, ds} \sqrt{\int_{\Gamma_{int} \cup \Gamma_D} \frac{1}{h_K} [u]^2 \, dx}.$$  

Finally, we have for the stabilization term:

$$|J''(u, v)| \leq \frac{1}{4} \sqrt{\int_{\Gamma_{int}} h_K [n \cdot \nabla u]^2 \, ds} \sqrt{\int_{\Gamma_{int}} h_K [n \cdot \nabla v]^2 \, ds}.$$  

Again, since

$$[n \cdot \nabla u]^2 = n \cdot (\nabla u|_K - \nabla v|_L)^2 = ((n \cdot \nabla u)_K + (n \cdot \nabla v)_L)^2$$

$$\leq 2((n \cdot \nabla u)_K^2 + (n \cdot \nabla v)_L^2)$$
we obtain
\[ |J^\sigma(u,v)| \leq \frac{1}{2} \sqrt{\sum_{K \in \mathcal{P}_h} \int_{\partial K \setminus \Gamma_N} h_K (n \cdot \nabla u)^2 \, ds} \sqrt{\sum_{K \in \mathcal{P}_h} \int_{\partial K \setminus \Gamma_N} h_K (n \cdot \nabla v)^2 \, ds}. \]

Combining the above inequalities using the discrete Schwarz inequality, we obtain that \(|\mathcal{B}(u,v)| \leq \|u\|_{\mathcal{P}_h} \|v\|_{\mathcal{P}_h} \).

**Lemma 2.2** Let \( \mathcal{F}(\cdot) \) be the linear form defined in (2.10). Then \( \mathcal{F}(\cdot) \) is continuous with respect to the norm \( \| \cdot \|_{\mathcal{P}_h} \), i.e. there exist a positive constant \( C \) such that
\[ |\mathcal{F}(v)| \leq C \|v\|_{\mathcal{P}_h}, \quad \forall v \in H^2(\mathcal{P}_h), \] (2.16)
where \( C \) depends on the data \( f, g, u_0, c \) and \( h \).

**Proof:** We first observe that:
\[ |\mathcal{F}(v)| \leq |F(v)| + |J_0(v)|. \]

Using Cauchy-Schwarz, the first term on the right-hand side gives:
\[ |F(v)| \leq \sum_{K \in \mathcal{P}_h} \|f\|_{0,K} \|v\|_{0,K} + \sum_{\gamma \subset \Gamma_N} \|g\|_{0,\gamma} \|v\|_{0,\gamma} \]
\[ \leq \sqrt{\sum_{K \in \mathcal{P}_h} \|f\|_{0,K}^2} \sqrt{\sum_{K \in \mathcal{P}_h} \|v\|_{0,K}^2} + \sqrt{\sum_{\gamma \subset \Gamma_N} \|g\|_{0,\gamma}^2} \sqrt{\sum_{\gamma \subset \Gamma_N} \|v\|_{0,\gamma}^2}. \]

Since \( \|v\|_{0,\gamma} \leq \|v\|_{0,\partial K} \leq C(K) \|v\|_{1,K} \) (Trace Theorem, see Schwab [6]), where \( C(K) \) is a constant depending on the size of the element, we then have
\[ |F(v)| \leq \min\{1, c^{-1}\} \left( \|f\|_0 + \max_{K \in \mathcal{P}_h} \{C(K)\} \|g\|_{0,\Gamma_N} \right) \|v\|_{\mathcal{P}_h}. \]

The second term is bounded by:
\[ |J_0(v)| \leq \int_{\Gamma_D} h_K^{-1} |u_0|^2 \, ds \sqrt{\int_{\Gamma_D} h_K |n \cdot \nabla v|^2 \, ds} \]
\[ \leq \min_{K \in \mathcal{P}_h} \{h_K^{-1/2}\} \|u_0\|_{0,\Gamma_D} \|v\|_{\mathcal{P}_h}. \]

Combining the above results yields:
\[ |\mathcal{F}(v)| \leq \min\{1, \frac{1}{c}\} \left( \|f\|_0 + \max_{K \in \mathcal{P}_h} \{C(K)\} \|g\|_{0,\Gamma_N} + \min_{K \in \mathcal{P}_h} \frac{1}{\sqrt{h_K}} \|u_0\|_{0,\Gamma_D} \right) \|v\|_{\mathcal{P}_h}. \]

In other words, \(|\mathcal{F}(v)| \leq C \|v\|_{\mathcal{P}_h} \) where \( C \) depends on \( f, g, u_0, c \) and \( h \). \qed
3 A posteriori error estimation

It is assumed in the following that there exists a unique solution $u$ to (2.9) and that $u_h$ is the finite element solution to (2.13).

3.1 Error equation and residual functional

Since $u \in H^1(\Omega) \cap H^2(P_h)$ and $u_h \in V^h \subset H^2(P_h)$, the error in the approximation $u_h$ of $u$, namely $e = u - u_h$, belongs to the space $H^2(P_h)$ as well. In replacing $u$ by $u_h + e$ in (2.9), keeping the term involving the unknown error $e$ on the left-hand side, passing all the terms involving $u_h$ to the right-hand side of (2.9), the error is then governed by the residual equation:

$$ B(e, v) = \mathcal{R}^h(v), \quad \forall v \in H^2(P_h), \quad (3.1) $$

where the linear functional $\mathcal{R}^h(\cdot)$, the residual, is given by

$$ \mathcal{R}^h(v) = \mathcal{F}(v) - B(u_h, v). \quad (3.2) $$

It is straightforward to deduce from the discrete problem (2.13) that the residual vanishes on $V^h$, i.e.

$$ \mathcal{R}^h(v) = 0, \quad \text{or} \quad B(e, v) = 0, \quad \forall v \in V^h. \quad (3.3) $$

Lemma 3.1 The residual functional is bounded on $H^2(P_h)$, i.e. there exists a positive constant $C$ such that

$$ |\mathcal{R}^h(v)| \leq C \|v\|_{P_h}, \quad \forall v \in H^2(P_h). \quad (3.4) $$

The constant $C$ depends here on the data $f$, $g$, $w_0$, $c$, and $h$ and on the finite element solution $u_h$.

Proof: From the definition of the residual, we have:

$$ |\mathcal{R}^h(v)| \leq |\mathcal{F}(v)| + |B(u_h, v)|. $$

Using the continuity of $B(\cdot, \cdot)$ and $\mathcal{F}(\cdot)$, there exists a positive constant $C$ such that:

$$ |B(u_h, v)| \leq \|u_h\|_{P_h} \|v\|_{P_h} \quad \text{and} \quad |\mathcal{F}(v)| \leq C \|v\|_{P_h}. $$

In consequence, we have:

$$ |\mathcal{R}^h(v)| \leq (\|u_h\|_{P_h} + C) \|v\|_{P_h} $$

which shows that the residual is bounded. \hfill \Box
3.2 Relationship between the residual and the error

The objective here is to relate the error to the residual in some appropriate norms. Since we have shown that $R^h$ is bounded with respect to the norm $\| \cdot \|_{P_h}$, an obvious choice for the norm of $R^h$ is:

$$\|R^h\|_* = \sup_{v \in H^2(P_h) \setminus \{0\}} \frac{|R^h(v)|}{\|v\|_{P_h}}. \quad (3.5)$$

Unlike the classical finite element formulation of the same problem (see [3] for details), the norm of the residual is not equal here to the norm of the error. However, we can show that these quantities are equivalent.

**Theorem 3.1** Let the numerical error $e$ and the residual $R^h$ be as defined above. Then, there exist positive constants $C_1$ and $C_u$ such that

$$C_1 \|e\|_{P_h} \leq \|R^h\|_* \leq C_u \|e\|_{P_h}. \quad (3.6)$$

**Proof:** From the definition of the norm of the residual and the continuity of the bilinear form, we have:

$$\|R^h\|_* = \sup_{v \in H^2(P_h) \setminus \{0\}} \frac{|R^h(v)|}{\|v\|_{P_h}} = \sup_{v \in H^2(P_h) \setminus \{0\}} \frac{|B(e,v)|}{\|v\|_{P_h}} \leq \sup_{v \in H^2(P_h) \setminus \{0\}} \frac{\|e\|_{P_h} \|v\|_{P_h}}{\|v\|_{P_h}}$$

which implies that $\|R^h\|_* \leq \|e\|_{P_h}$. In other words, $C_u$ is equal to unity.

Observing from the residual equation that $B(e,e) = R^h(e) \leq \|R^h\|_* \|e\|_{P_h}$, we then have:

$$\|e\|_{P_h}^2 = B(e,e) + I_1(e,e) + I_2(e,e)$$

$$= B(e,e) - J^\sigma(e,e) + I_1(e,e) + I_2(e,e)$$

$$\leq \|R^h\|_* \|e\|_{P_h} + I_1(e,e) + I_2(e,e) - J^\sigma(e,e).$$

Since $B(e,e)$ is a positive quantity (which is equal to zero if and only if $e = 0$), it is clear that:

$$I_1(e,e) + I_2(e,e) \leq \|e\|_{P_h}^2.$$

Subtracting the stabilization term $J^\sigma(e,e)$ from the inequality above, we may show that there exists a positive number $\alpha$, $0 \leq \alpha < 1$, such that

$$I_1(e,e) + I_2(e,e) - J^\sigma(e,e) \leq \alpha \|e\|_{P_h}^2.$$
Then, we have:

\[ \| e \|_{L_{\infty}^2} \leq \| R^h \|_* \| e \|_{P_h} + \alpha \| e \|_{P_h}^2, \]

that is

\[ (1 - \alpha) \| e \|_{P_h} \leq \| R^h \|_* \]

The constant \( C_1 \) is equal to \( (1 - \alpha) \). Note that \( \alpha \) will get closer to zero as the stabilization term \( J^s(\cdot, \cdot) \) gets larger. Finally, we also remark that the value of \( \alpha \) depends on the error \( e \) and thus on the mesh size \( h \). \( \square \)

### 3.3 Estimation of the norm of the residual

Because the space \( H^2(P_h) \) with the norm \( \| \cdot \|_{P_h} \) does not satisfy the properties of a Hilbert space, it is not guaranteed that there exists a Riesz representer in \( H^2(P_h) \) of the residual functional \( R^h \). Let \( W \) be a finite element subspace of \( H^2(P_h) \), whose dimension is significantly larger than the one of \( V^h \). Then, we can establish the following theorem:

**Theorem 3.2** Let \( R^h \) be a bounded linear functional with norm as defined in (3.5). Then there exists a unique function \( \phi \) in \( W \) such that

\[ B(\phi, v) + I_1(\phi, v) + I_2(\phi, v) = R^h(v), \quad \forall v \in W, \tag{3.7} \]

and

\[ \| \phi \|_{P_h} = \sup_{v \in W, v \neq 0} \frac{|R^h(v)|}{\| v \|_{P_h}} \leq \| R^h \|_* \tag{3.8} \]

**Proof:** The proof is straightforward using the Riesz Representation Theorem and the fact that \( W \) is a finite subspace of \( H^2(P_h) \), hence a Hilbert space. \( \square \)

Equation (3.7) provides a global problem for \( \phi \). In order to decouple the equation into elementwise local problems, let us first consider the test function \( v \) which is zero everywhere except in \( K \). Equation (3.7) then yields:

\[ B_K(\phi, v) + \int_{\partial K \setminus \Gamma_{N}} h_K(n \cdot \nabla \phi)(n \cdot \nabla v) \, ds + \int_{\partial K \setminus \Gamma_{N}} \frac{1}{h_K} [\phi] v \, ds = R^h(v) \]

We observe that a global coupling still exists because of the term [\( \phi \)] in the last integral on the left-hand side. However, the above equation suggests us the following decoupling of the problem.
Let \( \mathcal{W}_K \) be the space of functions in \( \mathcal{W} \) which are restricted to the element \( K \). On each element \( K \) of the partition, we compute the function \( \psi_K \in \mathcal{W}_K \) such that for all \( v \in \mathcal{W}_K \)

\[
B_K(\psi_K, v) + \int_{\partial K \setminus \Gamma_N} h_K (\mathbf{n} \cdot \nabla \psi_K) (\mathbf{n} \cdot \nabla v) \, ds + \int_{\partial K \setminus \Gamma_N} \frac{1}{h_K} \psi_K v \, ds = \mathcal{R}_K^h(v),
\]

(3.9)

where \( \alpha = 2 \) if \( \gamma \subset \Gamma_{int} \) and \( \alpha = 1 \) if \( \gamma \subset \Gamma_D \). The coefficient \( \alpha \) introduced above ensures that the functions \( \psi_K \) will provide us with a lower bound on the residual, as shown below.

**Theorem 3.3** Let \( \psi \) be the function in \( \mathcal{W} \) such that \( \psi |_K = \psi_K, \forall K \in \mathcal{P}_h \), where \( \psi_K \) is the solution of (3.9). Then,

\[
\| \psi \|_{\mathcal{P}_h} \leq \| \mathcal{R}^h \|_*.
\]

(3.10)

**Proof:** Starting from the norm of \( \psi \), we have:

\[
\| \psi \|_{\mathcal{P}_h}^2 = B(\psi, \psi) + I_1(\psi, \psi) + I_2(\psi, \psi).
\]

We note from (3.9) that

\[
B(\psi, \psi) = \sum_{K \in \mathcal{P}_h} B_K(\psi_K, \psi_K)
= \sum_{K \in \mathcal{P}_h} \left\{ \mathcal{R}_K^h(\psi_K) - \int_{\partial K \setminus \Gamma_N} h_K (\mathbf{n} \cdot \nabla \psi_K)^2 \, ds - \int_{\partial K \setminus \Gamma_N} \frac{1}{h_K} \psi_K^2 \, ds. \right\}
\]

Then, by substitution,

\[
\| \psi \|_{\mathcal{P}_h}^2 = \sum_{K \in \mathcal{P}_h} \mathcal{R}_K^h(\psi_K) + I_2(\psi, \psi) - \sum_{K \in \mathcal{P}_h} \int_{\partial K \setminus \Gamma_N} \alpha \frac{1}{h_K} \psi_K^2 \, ds.
\]

We now derive some upper bounds on the term \( I_2(\psi, \psi) \). Note that on an edge \( \gamma \subset \Gamma_D \), we have

\[
[\psi]^2 = \psi^2,
\]

whereas for an interior edge \( \gamma_{ij} \), we have

\[
[\psi]^2 = (\psi_i - \psi_j)^2 \leq 2(\psi_i^2 + \psi_j^2).
\]

It follows that:

\[
I_2(\psi, \psi) \leq \sum_{K \in \mathcal{P}_h} \int_{\partial K \setminus \Gamma_N} \alpha \frac{1}{h_K} \psi_K^2 \, ds.
\]

Therefore, the bound on \( \| \psi \|_{\mathcal{P}_h}^2 \) reduces to:

\[
\| \psi \|_{\mathcal{P}_h}^2 \leq \mathcal{R}_h^h(\psi) \leq \| \mathcal{R}^h \|_* \| \psi \|_{\mathcal{P}_h}
\]

which yields the expected bound on \( \| \psi \|_{\mathcal{P}_h} \).
3.4 An a posteriori error estimator

We propose the global error estimator $\eta$ which is given here as:

$$
\eta^2 = \sum_{K \in \mathcal{P}_h} \int_K \left| \nabla \psi_K \right|^2 + c|\psi_K|^2 \, dx + \sum_{K \in \mathcal{P}_h} \int_{\partial K \setminus \Gamma_N} h_K (n \cdot \nabla \psi_K)^2 \, ds + \int_{\Gamma_{in} \cup \Gamma_D} \frac{1}{h_K} [\psi]^2 \, ds
$$

where $\psi_K$ is the solution of the local problems (3.9) and $\psi$ is such that $\psi|_K = \psi_K$ on each element $K$. The estimator $\eta$ was shown to be a guaranteed lower bound on the norm of the residual and of the error. We show in the next section some numerical experiments to demonstrate the effectivity of such an estimator. Indeed, although the estimator is shown to be a lower bound, it still provides accurate estimates of the error.

The estimator can also be used as an indicator for mesh adaptation. Indeed, we can decompose $\eta$ into elementwise contributions $\eta_K$ such as:

$$
\eta_K = \int_K \left| \nabla \psi_K \right|^2 + c|\psi_K|^2 \, dx + \int_{\partial K \setminus \Gamma_N} h_K (n \cdot \nabla \psi_K)^2 \, ds + \int_{\partial K \setminus \Gamma_N} \frac{1}{h_K} h_K [\psi]^2 \, ds
$$

where, as before, $\alpha = 2$ if $\gamma \subset \Gamma_{in}$ and $\alpha = 1$ if $\gamma \subset \Gamma_D$. An adaptive scheme would then consist in refining the elements $K$ for which the contributions $\eta_K$ are the largest.

4 Numerical experiments

We now present our preliminary results obtained on two-point boundary-value problems.

4.1 Example 1

In the first example, we consider the model problem:

$$
- \frac{d^2u}{dx^2} + cu = f, \quad \text{in } \Omega = (0, 1) \tag{4.11}
$$

where $u$ satisfies the boundary conditions:

$$
u(0) = 0 \quad \text{and} \quad u(1) = 0. \tag{4.12}
$$

In this example, the coefficient $c$ is set to unity and $f$ is chosen such that the solution $u$ is smooth, i.e.

$$
u(x) = 1 + [(e - 1)e^x - (e^2 - e)e^{-x}]/(1 - e^2). \tag{4.13}
$$
The domain $\Omega$ is partitioned into uniform meshes with various mesh sizes $h$. The approximations of $u$ are computed using polynomial functions of degree up to 2, 3, 4 or 5. We measure the quality of the error estimator $\eta$ in terms of the effectivity index $\gamma$, which is defined as the ratio of $\eta$ by the exact error $\|e\|_{P_h}$.

We show in Fig. 1 and Fig. 2 the evolution of the exact error $\|e\|_{P_h}$ and the associated rates of convergence computed on a sequence of uniformly refined meshes. The results confirm that the method is optimal with respect to the mesh size $h$ as the rate of convergence is found to be of the order $h^p$. We then compare in Fig. 3 and Fig. 4 the effectivity indices of both $\|\phi\|_{P_h}$ and $\eta$. In both cases, the effectivity indices are slightly smaller than one as expected. We also emphasize here that decoupling the global problem for $\phi$ only slightly affects the accuracy of the error estimator. Indeed, the difference in the effectivity indices of $\|\phi\|_{P_h}$ and $\eta$ is never larger than ten percent for this particular example.

### 4.2 Example 2

In the second example, we consider the model problem:

$$-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right) + cu = f, \quad \text{in } \Omega = (0, 1)$$

(4.14)

where the coefficient $a$ depends on $x$ as shown in Fig. 5. Here $c = 1$ as well. The problem is set up such that the solution $u$ is given by:

$$u(x) = \frac{27}{4}x(1-x)^2e^{(x-1/3)^2/0.04}.$$  

(4.15)

We then repeat the experiments described above and the results are shown in Fig. 6 and Fig. 7 for the exact error and convergence rates, and in Fig. 8 and Fig. 9 for the effectivity indices of $\|\phi\|_{P_h}$ and $\eta$, respectively. The same conclusions hold in this case.

### 5 Concluding remarks

We have presented in this work an \textit{a posteriori} error estimator for a stabilized version of the Baumann-Oden Discontinuous Galerkin method. A thorough theoretical analysis show that the estimator provide a guaranteed computable lower bound on the approximation error. Numerical experiments on a two-point boundary value problem actually confirm that this error estimator yield very accurate estimates of the error. The methodology seems very promising and we plan, in the near future, to test it on two-dimensional problems and to extend it to hyperbolic problems. We also plan to devise an adaptive strategy based on this error estimator in order to achieve better solution accuracy at a lesser cost.
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