Optimal Economic Dispatch By Fast Distributed Gradient

Chaojie Li
School of Electrical and
Computer Engineering
RMIT University
VIC 3000, Australia
email: cjlee.cqu@163.com

Xinghuo Yu
School of Electrical and
Computer Engineering
RMIT University
VIC 3000, Australia
email: x.yu@rmit.edu.au

Wenwu Yu
Department of Mathematics
Southeast University
Nanjing 210096, PR China
email:wenwuyu@gmail.com

Abstract—Concerning on optimal economic dispatch, interior point method via $\theta$-logarithmic barrier is employed to reformulate the cost function of power generation. Fully distributed technology-enabled algorithm is developed to solve the economic dispatch. More specifically, the minimum connected dominating set based distributed algorithm aims at efficiently allocating the task of supply-demand balance for the whole power grid. A fast gradient based distributed optimization method is designed to fast converge to optimal solution. The simulations illustrate the effectiveness and good performance of our algorithms.

I. INTRODUCTION

Until recently, smart grid technique enables power network operators and electric utilities to dispatch generation resources in the most economical way. The goal of economic dispatch is to optimize the consumption of electric energy under security constraints, which implies saving monetary resources and potentially avoiding to build expensive power infrastructure, such as new transmission networks and power plants.

The economic dispatch is formulated as an optimization problem with constraints which could be either convex or nonconvex. Consequently, various methods are developed to solve the problem in different circumstances. Convex optimization techniques including Newton method and dual variable maximization are studied in [1] and [2]. Heuristic algorithms involving genetic algorithm and particle swarm optimization are successfully designed to solve nonconvex cases. However, these two categories of methodologies involve in an intense centralized computation that requires global information over the whole power grid. Undoubtedly, robustness and scalability of centralized methods would fail to meet the increasing demand in our modern society. It is necessary to develop new methodology to handle this problem in a distributed way which only utilizes local information to update itself.

The concept of consensus has been understood with a more complex perspective. Consensus based distributed approaches are investigated in [3] and [4]. The authors of [3] formulate economic dispatch as an incremental cost consensus problem, where a penalty-like method is used to update Lambda. Obviously, the updating rule need a global information about each generation. In [4], similar upgrade rule is developed for economic dispatch with transmission losses. It is worth to point out that, after the projection operation on box constraints, Lambda may converge to a stable state which does not yield an optimal solution.

Regarding to weighted gradient method, the authors of [5] develop a distributed gradient method to tackle the updating rule. By assuming that the initial condition satisfies the demand of power network, the distributed gradient could converge to desired solution. Nonetheless, projection operator is employed to handle box constraints which may destroy the supply-demand balance at some time slots. Therefore, the method is unable to solve the problem for most cases.

To address economic dispatch in a completely distributed way, we reformulate the problem via $\theta$-logarithmic barrier which is able to guarantee all the outputs under their capacity. Furthermore, a minimum connected dominating set based distributed algorithm is applied to allocate task at the initial stage. In addition, fast gradient based method is designed to accelerate the convergence rate of distributed algorithm.

This paper is organized as follows. In Section II, the problem of economic dispatch is presented. Section III introduces a completely distributed method to solve the problem. Case study about IEEE 30-bus is given in Section IV. Conclusions are drawn in the last section.

A. Problem Formulation

In this section, some preliminaries about algebraic graph theory are introduced in advance. Afterward, optimal economic dispatch problem is described. A fully distributed computational model regarding to this problem is reformulated.

B. Algebraic Graph Theory

Denote a weighted undirected network by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with the set of vertices $\mathcal{V} = v_1, v_2, \cdots, v_N$ and the set of undirected edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. If there is a communication link between $v_i$ and $v_j (i, j \in N)$, an undirected edge $\mathcal{E}_{i,j}$ is defined by the element of adjacency matrices $\mathcal{A}$, i.e. $a_{ij} = a_{ji} > 0$. The degree of vertex $v_i$ is $d_i = \sum_{j=1,j\neq i}^{N} a_{ij}$, which is the total weights between this vertex and all the other vertices. An undirected graph $\mathcal{G}$ is connected if an undirected path between vertices $v_i$ and $v_j (i, j \in N)$ exists.

Assume that the network is always connected throughout the paper. The graph Laplacian $W$ is defined by

$$W_{ij} = -a_{ij}, \ i \neq j; W_{ii} = d_i,$$  \hspace{1cm} (1)
C. Optimal Economic Dispatch Problem

In a power grid, the cost function of power generation is modeled by

\[ c_i(P_i) = \alpha_i P_i^2 + \beta_i P_i + \gamma_i, \quad (2) \]

where \( P_i \) is the output of the \( i \)-th power generation, \( \alpha_i, \beta_i \) and \( \gamma_i \) are parameters for the \( i \)-th power generation. By assuming a \( N \)-generation power grid, the goal of optimal power flow dispatch is to minimize the entire cost of power grid, such that

\[
\min_{P_i} \quad \sum_{i=1}^{N} c_i(P_i), \\
\text{s.t.} \quad \sum_{i=1}^{N} P_i = P_D, \quad (3) \\
P_{i,MIN} \leq P_i \leq P_{i,MAX},
\]

where \( P_D \) is the total power demand of load in this power grid. \( P_{i,MIN} \) and \( P_{i,MAX} \) correspond to the minimal and maximal capacity of the \( i \)-th power generation. Intrinsically, each power generation should be operated under its state constraint, i.e. \( P_i \in \mathcal{P}_i \) denoted by \([P_{i,MIN}, P_{i,MAX}]\).

In order to develop a fully distributed optimization algorithm, it is reformulated by \( \theta \)-logarithmic barrier. Specifically, for a parameter \( \theta > 0 \), we consider the primal problem as follow

\[
\min_{P_i \in \mathcal{P}_i} \quad \sum_{i=1}^{N} f_i(P_i), \\
\text{s.t.} \quad \sum_{i=1}^{N} P_i = P_D, \quad (4) 
\]

where

\[
f_i(P_i) = c_i(P_i) - \theta \left( \ln(P_i - P_{i,MIN}) + \ln(P_{i,MAX} - P_i) \right). \quad (5) 
\]

Note that this function is available only for the box constraint strictly satisfied, which implies the trajectory of each \( P_i \) lays in feasible domain and refers as the interior point method. The concept of interior point methods is initialized within a feasible domain and a sufficient large parameter \( \theta \).

The corresponding Lagrange dual function is

\[
g_\theta(\mu) = -\mu P_D + \sum_{i=1}^{N} \inf_{P_i \in \mathcal{P}_i} \left\{ f_i(P_i) + \mu P_i \right\}. \quad (6) 
\]

If initial condition

\[
\sum_{i=1}^{N} P_i - P_D = 0 
\]

holds, by the derivative of Lagrange dual function with respect to dual variable \( \mu \), it will not update. At each subsequent iteration, \( \theta \) adaptively decreases in a way that the resultant problem is readily to solve if the global minimum of its immediate predecessor can be available as the new initial point. Mathematically, the sequence of solution \( P_i \) will converge to the global minimum of the problem in (3).

Let Hessian Matrix of the objective function denoted by

\[
H = \begin{pmatrix} 2\alpha_1 + M_1 & 0 & 0 & 0 \\ 0 & 2\alpha_2 + M_2 & 0 & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 2\alpha_N + M_N \end{pmatrix}
\]

where \( 2\alpha_i + M_i (i = 1, 2, ..., N) \) is the second order derivative of \( f_i(P_i) \) with respect to \( P_i \), such that

\[
M_i = \theta \left\{ \frac{1}{(P_i - P_{i,MIN})^2} + \frac{1}{(P_{i,MAX} - P_i)^2} \right\}.
\]

Because \( P_i \) is an interior point of feasible domain, \( H \) is always positive and \( M_i \) is upper bounded by a positive number \( M \) that depends on computational accuracy.

II. A FULLY DISTRIBUTED OPTIMIZATION APPROACH

To develop a fully distributed optimization approach, we divide this problem into two parts. The one is to initialize the start point of each power generation in a distributed way, which has to guarantee the equality constraint satisfied at beginning. The another is to design a iterative algorithm that enables locally information exchange amongst neighbors. At the meantime, the accumulative sequence \( P_i(k)(k = 1, 2, ...) \) generated by this algorithm will converge to optimal solution of the problem.

A. Distributed Task Allocation Toward Initialization and Adjustment

Technically, fully distributed algorithm for economic dispatch requires that technology-enabled strategy provide a distributed way matching supply to demand over the whole time horizon in the power network, such that

\[
\sum_{i=1}^{N} P_i(k) = P_D \quad \forall k \geq 1. \quad (7) 
\]

As known, the minimum connected dominating set of a graph provides a least cost for information exchange and routing in a network environment. We design a minimum connected dominating set based algorithm to allocate the original output of each generation as following.

- **Step 1.** Given a connected topology of power network \( \mathcal{G} \), it can be recursively divided into different connected dominating sets \( \mathcal{D} \) in terms of different levels \( \mathcal{L} \).

- **Step 2.** From top level to bottom one, task allocation of each power generation is executed among adjacent
that set seeking algorithm in [6] as a part of our algorithm, such as removing the gray nodes, a new minimum connected dominating set is given by [12]. More specifically, there are 19 genera-
tions of minimal and that of maximal output capacity at different levels and the graph of a power network.

Fig. 1: (a) The minimal connected dominating set $\mathcal{D}_G$ in $\mathcal{L}_1$, (b) The minimal connected dominating set $\mathcal{D}_B$ in $\mathcal{L}_2$, (c) The minimal connected dominating set $\mathcal{D}_R$ in $\mathcal{L}_3$, (d) The whole topology of different levels.

Algorithm 1 Distributed Task Allocation by minimal CDS

1: procedure DISTRIBUTEDTASKALLOCATION($\mathcal{G}, P_D$)
2: Initialize $P_i = P_i,MIN, \forall i \in \mathcal{G}$,
3: $(N_G, D) = \text{DistrMiniCDS}(\mathcal{G})$;
4: $k = N_G$;
5: $P^D = P^D - \{\text{Sum}L_{1,MIN} + ... \text{Sum}L_{k,MIN}\}$;
6: repeat
7:   $t = P^D - \{\text{Sum}L_{k,MAX} - \text{Sum}L_{k,MIN}\}$;
8:   if $t > 0$ then
9:     $P_i = P_i,MAX \ \forall i \in \mathcal{D}_k$;
10:    $P^D = t$;
11:    $k = k - 1$;
12:    else
13:       for $i \in \mathcal{D}_k$ do
14:          if $t > P_i,MAX - P_i,MIN$ then
15:            $P_i = P_i + (P_i,MAX - P_i,MIN)$;
16:            $t = t - (P_i,MAX - P_i,MIN)$;
17:          else
18:            $P_i = P_i + t$;
19:            Break;
20:        end if
21:       end for
22:    end if
23: until $k > 0$
24: return $P$;
25: end procedure

nodes at the same level, where messages of information exchange are delivered by immediate nodes in the predecessor level.

- **Step 3.** If there were some new nodes of power generation that join into the network, they would be connected in bottom level which does not have any impact on the previous steps. However, if some nodes in a higher level want to quit from the network, it is necessary to restart the Step 1 and 2 for reconstructing all minimum connected dominating sets and corresponding levels.

We employ the distributed minimum connected dominating set seeking algorithm in [6] as a part of our algorithm, such that

$$(N_G, D) = \text{DistrMiniCDS}(\mathcal{G})$$

where $D, \mathcal{G}$ are all minimum connected dominating sets in terms of different levels and the graph of a power network. Denote $N_G$ is the number of total levels in a power network. $\text{Sum}L_{j,MIN}$ and $\text{Sum}L_{j,MAX}(j=1,2,...,N_G)$ are the summation of minimal and that of maximal output capacity at $j$ level, respectively, which can be easily obtained by message routing. Now, we are ready to introduce this algorithm.

The topology of IEEE 118-bus test system, for instance, is given by [12]. More specifically, there are 19 generations spatially connected into 118-bus. First, we use the minimum connected dominating set searching algorithm to obtain $\mathcal{L}_1$, as shown in Figure (1 a), where green nodes are connected into a minimal connected dominating set $\mathcal{D}_G = \{10, 25, 26, 49, 54, 61, 65, 66, 80, 100\}$. Note that gray nodes in $\mathcal{L}_1$ can communicate with green nodes directly. Second, by removing the gray nodes, a new minimum connected dominating set $\mathcal{D}_B = \{10, 26, 65, 80\}$ is organized by blue nodes into $\mathcal{L}_2$, see Figure (1 b). Likewise, by taking away green nodes, the set $\mathcal{D}_R = \{10\}$ is found as $\mathcal{L}_3$ in Figure (1 c).

Each minimal connected dominating set supplies computational service in each level. At beginning, each node of power generation $P_i$ is initialized by its minimal power capacity $P_i,MIN$. From $\mathcal{L}_3$, Node 10 of $\mathcal{D}_R$ is able to check the rest of mismatched output. If the demand could be satisfied and completed in this level, Node 10 would communicate with the other nodes except in $\mathcal{D}_R$ and allocate the task for them. Otherwise, all nodes in $\mathcal{L}_3$ take their maximal capacity and pass the rest task to $\mathcal{L}_2$. Recursively, the task would be completed by some levels in this distributed way. It is worth to point out that if the demand is overloading, the task could not be executed and the warning message will return.

B. Fast Distributed Gradient Toward Optimal Dispatch

In order to achieve a global optimal dispatch, the distributed gradient algorithm is developed to locally update the state in [7], such that

$$P(k + 1) = P(k) - \zeta W \nabla F(P),$$

where $P = [P_1, P_2, \cdots, P_N]^T$ and $\nabla F(P) = [f_1(P_1), f_2(P_2), \cdots, f_N(P_N)]^T$. The weight matrix $W$ corresponds to the topology of the underlying information exchange graph which is frequently modeled by the graph Laplacian $W$. 


One obtains

\[ W1 = 0, \]
\[ 1^T W = 0. \]

Given certain stepsize \( \zeta \), it is readily to check that the sequence of updating rule (8) converges to a fixed point that satisfies the optimal conditions of economic dispatch problem.

When the networked system is large, the convergence rate of this algorithm tends to be slow. Note that the objective function is quadratic with twice differentiable gradient. Intrinsically, a fast distributed gradient method enabling a better convergence rate is desired in Algorithm 2.

**Algorithm 2 Optimal Dispatch by Fast Distributed Gradient**

1. **procedure** FastDistributedGradient\((P_0)\)
2. Initialize \( \xi \in [0, 1], \zeta \in (0, \frac{\max_{i \in N} \lambda_i(W'T)}{2(1+\xi)}); \)
3. \( P(1) = P_0, \theta > 0, \nu \in [0, 1]; \)
4. \( k = 1; \)
5. repeat
6. repeat
7. \( Q = P(k) - P(k - 1); \)
8. \( P(k + 1) = P(k) + \zeta Q - \zeta W'VF(P); \)
9. \( k = k + 1; \)
10. until \( \| P(k) - P(k - 1) \| \leq \epsilon \)
11. \( \theta = \nu \theta; \)
12. until \( \theta \leq \epsilon \)
13. return \( P; \)
14. end procedure

\( Q \) is a temporally variable for recording the difference of states between previous step and current step. If initial values \( P_0 \) satisfy the equality constrain of (3), it gives

\[ 1^T P(k + 1) = 1^T P(k) + \zeta 1^T Q - \zeta 1^T W'V F(P). \]

By the fact of \( P(1) = P_0 \), we have

\[ 1^T P(k + 1) = 1^T P(k), \text{ } \forall k \geq 1, \]

which implies equality constrain of (3) will always hold during the evolution of Algorithm 2. The prototype of accelerated gradient method was firstly developed by the author of [8] in 1983. In each step, not only does the updating rule depend on current gradient information but also previous one. More specifically, the previous gradient information can be considered as a momentum that is able to accelerate the update with convergence rate \( O(\frac{1}{k^\frac{3}{2}}) \).

One observes that

\[ \| P(k + 1) - P^\ast \| \leq r\| P(k) - P^\ast \|, \]

where \( r \) is the linear convergence rate of inner-loop in Algorithm 2.

**Theorem 1. If the stepsize**

\[
\zeta = \left( \frac{\sqrt{\max_{i \in N} \lambda_i(W'H^\ast) - \sqrt{\lambda_2(W'H^\ast)}}}{\max_{i \in N} \lambda_i(W'H^\ast) + \sqrt{\lambda_2(W'H^\ast)}} \right)^2, \tag{11}
\]

\[
\zeta = \left( \frac{2}{\max_{i \in N} \lambda_i(W'H^\ast) + \sqrt{\lambda_2(W'H^\ast)}} \right)^2, \tag{12}
\]

**then the convergence rate**

\[
r = \frac{\max_{i \in N} \lambda_i(W'H^\ast) - \sqrt{\lambda_2(W'H^\ast)}}{\max_{i \in N} \lambda_i(W'H^\ast) + \sqrt{\lambda_2(W'H^\ast)}}, \]

where \( H^\ast \) is Hessian matrix of the objective function with \( \theta \)-logarithmic barrier at optimal solution.

To illustrate the performance of our algorithm, we consider a simple example in which there is an objective functions involving six variables with constraints.

\[
\min_{x \in R^6} \sum_{i=1}^{6} (x_i - i)^2, \quad \text{s.t.} \quad \sum_{i=1}^{6} x_i = 20, \quad 2 \leq x_1 \leq 3, \quad 1 \leq x_2 \leq 2, \quad 0 \leq x_3 \leq 1, \quad 3 \leq x_4 \leq 5, \quad 4 \leq x_5 \leq 6, \quad 6 \leq x_6 \leq 7.
\]

The optimal solution \( x^\ast = \{2, 2, 1, 4, 5, 6\} \) and the corresponding minimal value of the objective function is 5. The simulation is presented in Figure 2. We quote our method and the one in [7] as Fast Distributed Gradient (FDG) method and Distributed Gradient (DG) method, respectively. In the figure, solid line and dash line are used to depict our method and the counterpart one. It is shown that our method has a better performance in the sense of convergence rate, see details in Figure (2 c).

**III. Case Study**

In this section, we will illustrate our algorithm via IEEE 30-bus test case. The parameters of cost function are given in Table I that was presented in [13]. We assume the demand \( P_D = 295.36 \). Therefore, by Algorithm 1, the initial output of generations are \( P_0 = \{133.36, 80, 50, 10, 10, 12\} \). Finally, the optimal solution is obtained at 808.5279 and \( P_\ast = \)

<table>
<thead>
<tr>
<th>Generator No.</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( \gamma )</th>
<th>( P_{min} )</th>
<th>( P_{max} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.00375</td>
<td>2</td>
<td>0</td>
<td>50</td>
<td>200</td>
</tr>
<tr>
<td>2</td>
<td>0.0175</td>
<td>1.75</td>
<td>0</td>
<td>20</td>
<td>80</td>
</tr>
<tr>
<td>3</td>
<td>0.0625</td>
<td>1.0</td>
<td>0</td>
<td>15</td>
<td>50</td>
</tr>
<tr>
<td>8</td>
<td>0.00834</td>
<td>3.25</td>
<td>0</td>
<td>10</td>
<td>35</td>
</tr>
<tr>
<td>11</td>
<td>0.025</td>
<td>3.0</td>
<td>0</td>
<td>10</td>
<td>30</td>
</tr>
<tr>
<td>15</td>
<td>0.025</td>
<td>3.0</td>
<td>0</td>
<td>12</td>
<td>40</td>
</tr>
</tbody>
</table>
IV. CONCLUSION

In this paper, the problem of optimal economic dispatch has been investigated. Interior point method is employed to solve this problem where \( \theta \)-logarithmic barrier is used to reformulate the cost function. A distributed algorithm is designed to allocate the task of supply-demand balance for the whole power networks in advance. Moreover, fast gradient method is utilized to accelerate the convergence rate. This method provides a safety technique that all the generations are running under their capacity. Meanwhile, perfect supply-demand balance is always guaranteed over entire time horizon in the electricity market.

REFERENCES


