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q-Deformed Quantum Mechanical Evolution Equation via q-Deformed Schrödinger Algebra

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Abstract

We demonstrate how the 1-dimensional Schrödinger equation can be constructed using lowest weight modules and singular vectors of the (centrally extended) Schrödinger algebra in 1+1 spacetime dimension. After introducing a realization of a q-deformed Schrödinger algebra in terms of q-difference operators we derive a q-difference analogue of the 1-dimensional Schrödinger equation which is invariant under the action of this q-algebra.

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1 Introduction

The non-relativistic quantum mechanical evolution equation in $\mathbb{R}^n$ is the Schrödinger equation:

$$i\hbar \frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m} \triangle\psi + V(x)\psi \qquad (1)$$

which is a partial differential equation with solutions in the Hilbert space $L^2(\mathbb{R}^n, d^n x)$ and can be obtained via different quantization procedures e.g. canonical quantization or Borel quantization [1]. This equation is in many ways connected with and can be derived form certain Lie algebras. Some of these methods will briefly be mentioned below.

The theory of quantum groups that started in the mid 80's [2] renewed the interest in q-difference operators which where introduced as discrete analogues of differential operators already about 100 years ago by Jackson and others. The reason is that they proved to be a useful tool for representations of q-deformed Lie algebras where they replace the differential operators occuring in the representation theory of the undeformed algebras.

In the discussions about physical applications of deformed symmetries i.e. deformed Lie algebras it is interesting to see which deformations of (1) are possible and then to discuss solutions of the deformed equations and possibly check them against experiments. For the construction of q-deformed Schrödinger equations one often uses "q-analogues" of Lie algebraic methods connected with (1) in which the corresponding Lie algebras are replaced by their q-deformations. In the - rather frequent - case that the representations of the q-algebras contain q-difference operators one normally ends up with a q-difference equation which in the "classical limit" $q \to 1$ goes into the ordinary Schrödinger equation (1).

An early approach towards a q-deformed evolution equation was performed by Minahan [3] who used a special realization of the q-deformed canonical commutation relations $aa^+ - q^{-1}a^+a = q^N$ with q-difference operators to derive a q-difference analogue of the *stationary* (i.e. time independent) Schrödinger equation first for the harmonic oscillator ($V(x) = \omega x^2$) and then rather directly also for the free case ($V(x) \equiv 0$). For some more recent results connected with this method see also [4]. The combination of techniques from Lie theory and differential calculus provides another possible approach by constructing a q-Laplace operator $\triangle_q$ emerging from a $SO_q(N)$ covariant differential calculus. The corresponding stationary Schrödinger equations describe a quantum system on a noncommutative configuration with a harmonic oscillator potential [5] or without potential [6]. In [7] it was shown that the evolution of certain discrete spin chains can be descibed using a q-deformed Galilei group. However this approach does not explicitly work with q-differnce operators of the type to be used in this article. In the literature many other methods or different results based on the methods mentioned above can be found.

In this article we present a method to derive a q-difference analogue of the Schrödinger equation that uses a q-deformation of the (centrally extended) Schrödinger algebra which was introduced in [8] as the algebra of generators of the maximal group of transformations leaving invariant the space of solutions of the free Schrödinger equation. We give the resulting q-deformed Schrödinger equation explicitly.

Notation:

We will use the shorthand notation $\partial_y \equiv \frac{\partial}{\partial y}$ for partial derivatives and the q-numbers:

$$[a] \;\equiv\; \frac{q^a - q^{-a}}{q - q^{-1}}, \qquad [a]'_q \;\equiv\; \frac{q^{\frac{a}{2}} - q^{-\frac{a}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$$

which will also be used for diagonal operators instead of a. The q-binominal coefficient is given by:

$$\begin{pmatrix} a \\ b \end{pmatrix}_q \;\equiv\; \frac{[a]_q!}{[a-b]_q!\;[b]_q!}, \qquad [a]_q! \;\equiv\; [1]_q[2]_q \cdots [a]_q.$$

We use the following q-difference operators:

$$\left.\begin{array}{rcl} \mathcal{D}_y\; f(y) & \equiv & \dfrac{f(qy) \;-\; f(q^{-1}y)}{y\;(q - q^{-1})} \\[2ex] \mathcal{D}'_y\; f(y) & \equiv & \dfrac{f(q^{\frac{1}{2}}y) \;-\; f(q^{-\frac{1}{2}}y)}{y\;(q^{\frac{1}{2}} - q^{-\frac{1}{2}})} \end{array}\right\} \tag{2}$$

for $q \to 1$ one has: $\mathcal{D}_y, \mathcal{D}'_y \;\to\; \partial_y$. With the "number" operator N_y for the coordinate y i.e. $N_y y^k \;\equiv\; k y^k$ we have for suitable (e.g. analytical) functions:

$$q^{aN_y}\; f(y) \;=\; f(q^a y).$$

2 q-deformed Schrödinger algebra and q-deformed Schrödinger equation

The Schrödinger equation is a partial differntial equation which is - in a sense to be explained below - invariant under the action of (a certain representation of) the (centrally extended) Schrödinger Lie algebra (resp. under its corresponding Lie group). We will derive a q-deformed Schrödinger equation as a q-difference equation that is (in analogy) invariant under the action of a q-deformed Schrödinger algebra. In order to motivate this construction we show how the usual free Schrödinger equation can be achieved using the (undeformed) Schrödinger algebra whose basic properties will be reviewed first.

2.1 The Schrödinger algebra

The $n+1$-dimensional Schrödinger algebra $\mathcal{S}(n)$ is the Lie algebra generated by $P_t,\ P_i,\ D,\ J_{ik}(= -J_{ki}),\ D,\ G_i$ and K, where $(i,j,k,l = 1,\ldots,n)$ with the following nonvanishing commutation relations:

$$\left.\begin{array}{l} [P_t, D] \;=\; 2P_t, \quad [P_t, G_i] \;=\; P_i, \quad [P_t, K] \;=\; D, \\ [P_i, J_{kl}] \;=\; \delta_{il}P_k \;-\; \delta_{ik}P_l, \quad [P_i, D] \;=\; P_i \quad [P_i, K] \;=\; G_i, \\ [D, G_i] \;=\; G_i, \quad [D, K] \;=\; 2K, \quad [G_i, J_{ik}] \;=\; \delta_{ik}G_j \;-\; \delta_{jk}G_i, \\ [J_{ij}, J_{kl}] \;=\; \delta_{ik}J_{jl} \;+\; \delta_{jl}J_{ik} \;-\; \delta_{il}J_{ik} \;-\; \delta_{jk}J_{il} \end{array}\right\} \tag{3}$$

it is not semisimple and allows for any n exactly one central extension $\hat{\mathcal{S}}(n)$ realized by the additional relation:

$$[P_i, G_j] \;=\; \mu\delta_{ij}, \quad \mu \in \mathbb{C}\,. \tag{4}$$

The algebra $\hat{\mathcal{S}}(n)$ has the following vector field realization:

$$\left.\begin{aligned}
P_t &= \partial_t, \\
P_i &= \partial_i, \\
D &= 2t\partial_t + x_i\partial_i - d, \\
J_{ik} &= x_k\partial_i - x_i\partial_k \\
G_i &= t\partial_i + \mu x_i \\
K &= t^2\partial_t + tx_i\partial_i + \frac{\mu}{2}x_ix_i - td
\end{aligned}\right\} \tag{5}$$

where $\partial_t \equiv \frac{\partial}{\partial_t}$ and $\partial_i \equiv \frac{\partial}{\partial_{x_i}}$.

The Schrödinger algebra was first introduced in [8] for $n = 3$ with $\mu = \frac{m}{i\hbar}$, $d = -\frac{3}{2}$ as the maximal (kinematical) symmetry algebra of the 3-dimesional free Schrödinger equation which means that its solutions are transformed into solutions by the action of the given representation of $\hat{\mathcal{S}}(n)$. Furthermore one can show that it is a (kinematical) symmetry of the n-dimensional Schrödinger equation for $d = -\frac{n}{2}$. Therefore $\hat{\mathcal{S}}(n)$ is a subalgebra of the Lie symmetry algebra of the free Schrödinger equation (see e.g. [9]). For $\mu = m$ solutions of the heat equation are invariant under the corresponding representations.

2.2 Construction of invariant partial differential equations

We will now show how the 1-dimensional Schrödinger equation can be constructed as a partial differential equation invariant under $\hat{\mathcal{S}}(1)$ using a (partial) generalization of a method introduced for semisimple Lie algebras in [10]. For details of the following construction we refer to [11].

The key observation of our approach is that we can introduce a *grading* for $\hat{\mathcal{S}}(n)$ by setting:

$$\deg K = 2, \quad \deg G_i = 1, \quad \deg D = \deg \mu = \deg J_{ik} = 0, \quad \deg P_i = -1, \quad \deg P_t = -2. \tag{6}$$

This allows us to define Verma modules V^d in the same way as for semisimple Lie algebras (see, e.g., [12]) with a lowest weight vector v_0 defined (for $n = 1$) by:

$$Dv_0 \;=\; -d\, v_0, \quad P_tv_0 \;=\; P_xv_0 \;=\; 0.$$

A basis of V^d is then given by $v_{kl} = G^k K^l v_0$ and by vitue of (3) we have as module relations:

$$\left.\begin{aligned} K\ v_{kl} &= v_{k,l+1}, \quad G\ v_{kl} \ = \ v_{k+1,l} \\ D\ v_{kl} &= (k+2l-d)\ v_{kl} \\ P_x\ v_{kl} &= l\ v_{k+1,l-1}\ +\ \mu k\ v_{k-1,l} \\ P_t\ v_{kl} &= l(k-l-1-d)\ v_{k,l-1}\ +\ \mu\frac{k(k-1)}{2}\ v_{k-2,l}. \end{aligned}\right\} \tag{7}$$

In analogy to the semisimple case we call $v_s \in V^d$ *singular* if $v_s \notin \mathbb{C} v_0$, it is an eigenvector of D and $P_t v_s = P_x v_s = 0$. If such a vector exists V^d is reducible (i.e. contains an invariant submodule). In [10] it was shown for semisimple Lie algebras g that if a singular vector exists it is given by Πv_0, where Π is a polynomial in the positively graded generators of the algebra. Moreover it was demonstrated there that if one inserts a vector field realization of g into Π the differential equation $\Pi f = 0$ is invariant under the action of g. We will use analogous arguments here.

For $\hat{\mathcal{S}}(1)$ we showed in [11] that V^d has a singular vector (for $\mu \neq 0$) iff $d = \frac{2r-3}{2}$ $(r \in \mathbb{N})$. In these cases we have:

$$v_s(p) \ = \ \left(G^2 \ - \ 2m\ K\right)^r\ v_0 \ \equiv \ \Pi_r(G,K)\ v_0 \tag{8}$$

(The case $\mu = 0$ is treated in [11].) Inserting (5) into (8) we find

$$\Pi_r(G,K) \ = \ \left(\ t^2(\partial_x^2 - 2\mu\partial_t)\ +\ 2(r-1)\mu t\ \right)^r \ = \ t^{2r}\ \left(\partial_x^2 - 2\mu\partial_t\right)^r \tag{9}$$

and therefore obtain a class of differential equations:

$$\left(\partial_x^2 - 2\mu\partial_t\right)^r\ \psi \ = \ 0 \tag{10}$$

which are invariant under the action of the representation (5) of $\hat{\mathcal{S}}(1)$ with $\mu = \frac{m}{i\hbar}$ and $d = \frac{2r-3}{2}$. The free Schrödinger equation is the lowest member $(n=1)$ (for $\mu = m$ we recover the heat equation).

We can further extend [10] to our non-semisimple situation by considering equations with non-zero RHS. However, invariance w.r.t. the Schrödinger algebra requires that the RHS is an element associated with the Verma module $V^{-r-3/2}$, while the functions in the LHS are not restricted to the solution subspace. Thus, using the operator (9) we obtain instead of (10) the following hierarchy of equations :

$$t^{2r}\ \left(\partial_x^2 - 2\mu\partial_t\right)^r\ \psi \ = \ \psi' \tag{11}$$

where ψ, ψ', resp., are associated to the Verma modules $V^{r-3/2}$, $V^{-r-3/2}$, resp.

Remark. It is interesting to note that (11) looks similar to an hierarchy of equations involving the d'Alembert operator and conditionally invariant w.r.t. conformal algebra $su(2,2)$:

$$\Box^{\,r}\ \varphi(\mathbf{x}) \ = \ \varphi'(\mathbf{x})\ , \qquad r \in \mathbb{N} \tag{12}$$

where φ, φ' are scalar fields of different fixed conformal weights depending on r, $\mathbf{x} = (x_0, x_1, x_2, x_3)$ denotes the Minkowski space-time coordinates, and $\Box$ is the d'Alembert operator: $\Box = \partial^\mu \partial_\mu = (\vec{\partial})^2 - (\partial_0)^2$, cf. [13] and references therein. $\Diamond$

We note that the construction given above is not limited to $n = 1$ but can be extended to higher dimensions. Work in this direction is in progress.

2.3 The q-deformed case

According to general philosophy mentioned in the introduction we try to replace the differential operators (5) by q-difference operators in such a way that in the $q \to 1$ limit one recovers (5) and hopefully obtain a q-deformed analogue of the Schrödinger equation that will be invariant under a q-deformed Schrödinger algebra $\hat{\mathcal{S}}_q(1)$. This analysis was carried out in [14] for $n = 1$ and we will now present the results.

If one replaces ∂_t and ∂_x in the nonzero graded generators of (5) by the q-difference operators (2) one finds that it is necessary to deform some commutation relations into q-commutation relations and to introduce additional factors of the type $q^{aN_t + bN_x}$. We find a "simple" possible realization of a q-deformed Schrödinger algebra $\hat{\mathcal{S}}_q(1)$:

$$\left.\begin{aligned}
P_t &= \mathcal{D}_t \; q^{N_t + N_x} \\
P_x &= \mathcal{D}'_x \; q^{\frac{1}{2} N_x} \\
D &= 2N_t + N_x - d \\
G &= t \; \mathcal{D}'_x \; q^{-\frac{1}{2} N_x} + q^{-\frac{1}{2}} \; \mu x \; q^{-N_x} \\
K &= q^d \; t^2 \; \mathcal{D}_t \; q^{-N_t} + q^d \; tx \; \mathcal{D}_x \; q^{-2N_t - N_x} \\
&\quad - q^{-1} \; [d]_q \; t + q^{-\frac{3}{2} + d} \; [\tfrac{1}{2}]_q \; \mu x^2 \; q^{2N_t - 2N_x}
\end{aligned}\right\} \qquad (13)$$

with nontrivial (q-)commutation relations:

$$\left.\begin{aligned}
P_t P_x - q^{-1} P_x P_t &= 0 & [P_t, D] &= 2P_t \\
P_t G - q^{-1} G P_t &= P_t & [P_x, D] &= P_x \\
P_x G - q^{-1} G P_x &= \mu & [P_x, K] &= G \; q^{-D} \\
[D, G] &= G & [D, K] &= 2K
\end{aligned}\right\} \qquad (14)$$

For $q \to 1$ we obtain $\hat{\mathcal{S}}(1)$ with the vector field representation (5). We remark that the subalgebra generated by D, K and P_t fullfills the usual (Drinfeld-Jimbo) relations of $U_q(sl(2, \mathbb{C}))$. In [14] a more general q-difference realization of (14) than (13) is considered. However the above example is sufficient to discuss the main properties of this approach.

From (14) we see that the $\hat{\mathcal{S}}(1)$ generators admit the same grading (6) as for $\hat{\mathcal{S}}(1)$. Therefore we can construct Verma-modules V_q^d in the same fashion as for the undeformed case. The corresponding module structure is (with $v_{kl} \equiv G^k K^l v_0$) given by:

$$\left.\begin{aligned} K\ v_{kl} &= v_{k,l+1}, \quad G\ v_{kl} \ = \ v_{k+1,l} \\ D\ v_{kl} &= (k+2l-d)\ v_{kl} \\ P_x\ v_{kl} &= q^{d+1-l-k}\ [l]_q\ v_{k+1,l-1}\ + q^{\frac{1-k}{2}}\ \mu[k]'_q\ v_{k-1,l} \\ P_t\ v_{kl} &= [l]_q[k-l-1-d]_q\ v_{k,l-1}\ +\ \mu\frac{[k]'_q[k-1]'_q}{[2]'_q}\ v_{k-2,l}. \end{aligned}\right\} \tag{15}$$

As in the undeformed case we have a singular vector iff $d = \frac{2r-3}{2}$ $(r \in I\!N)$. It is determined by

$$v_s^q(r) \ = \ \sum_{l=0}^{r} (-[2]'_q\mu)^l \binom{r}{l}_q v_{2(r-l),l} \ = \ \left(\ G \ - \ [2]'_q \mu\ K\ \right)_q^r\ v_0 \ \equiv \ \Pi_r^q(G,K)\ v_0. \tag{16}$$

The q-difference equations $\Pi_r^q(G,K)\psi = 0$ are invariant under the action of $\hat{\mathcal{S}}_q(1)$İn the limit $q \to 1$ we recover equations (10). By analogy to the results of the last subsection we interpret the lowest member of the above family ($r = 1$) with $\mu = \frac{m}{i\hbar}$ as *q-deformed free Schrödinger equation* which is different from the ones that have already been presented in the literature. It is explicitly given as

$$S_q\psi = 0 \tag{17}$$

with:

$$\left.\begin{aligned} S_q \ &= q^{\frac{1}{2}}\ \left(\frac{\hbar^2}{[2]'_q\ m}\ \mathcal{D}'^2_x q^{-N_x}\ +\ q^{-2}\ i\hbar\ \mathcal{D}_t q^{-N_t}\right) \\ &-\ \frac{i\hbar}{[2]'_q}\ t^{-1}x\ \mathcal{D}'_x \left([2]'_q\ -\ (1+q^{N_x})\ q^{-\frac{3}{2}-2N_t}\right) q^{-\frac{3}{2}N_x} \\ &+\ (1-q^{-2})\ \frac{i\hbar}{[2]'_q}\ t^{-1}x\ \mathcal{D}_x\ q^{-N_x} \\ &-\ (q^{-1}-q^{-3})\ \frac{m}{[2]'_q}\ t^{-1}x^2\mathcal{D}_t\ q^{-N_t-2N_x}. \end{aligned}\right\} \tag{18}$$

As expected we recover for $q \to 1$ the ordinary free Schrödinger equation. (The above equation has been rescaled in comparison to the one given in [14] so that this limit becomes more transparent). Analogously to (11) we can introduce also a Schrödinger equation with nontrivial RHS.

We remark that the q-Schrödinger algebra (14) and its realization (13) differs from the results given in [15] by Floreanini and Vinet who considered symmetries for solutions of a q-deformed heat equation different from (18). The main difference is that their algebra closes only on the space of solutions of that equation whereas ours is q-algebraically closed without reference to any special function space.

3 Conclusion

We presented a q-deformed Schrödinger equation $S_q\psi = 0$ which is justified by the following properties:

1. For $q \to 1$ we have the usual Schrödinger operator as the limit of S_q
2. $S_q\psi = 0$ is invariant under the action of a q-deformed Schrödinger algebra $\hat{\mathcal{S}}_q(1)$ which for $q \to 1$ goes into the ordinary Schrödinger algebra $\hat{\mathcal{S}}(1)$.
3. The "classical" counterpart of its algebraic construction using the Schrödinger algebra leads to the usual Schrödinger equation.

The observation that (18) differs from other results obtained by other methods (which are also different compared to each other) shows that the q-deformation "removes a degeneracy of the *classical constructions*" in the sense that for $q = 1$ different approaches always lead to the same result - the usual Schrödinger equation - whereas for $q \neq 1$ one ends up with quite distinct q-difference equations sharing only the property that they have the same $q \to 1$ limit.

The question whether one should consider q-deformed quantum theories to describe physical systems and - if positively answered - which deformation is really appropriate should be answered experimentally. However the fact that the undeformed Schrödinger equation provides a sensible explaination of - non-relativistic - quantum mechanical experimental data has the consequence that any possible deformation can only differ from $q = 1$ by very small amount which means that only quantum mechanical precission measurements can possibly give an answer to this question.

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