Abstract—In this paper, we present new extractability conditions for blind source extraction of linear instantaneous mixtures. Two general conditions for source extraction of arbitrarily mixed nonzero sources are presented. A sufficient condition is provided to guarantee that sequential extraction can be continued. We also show an important property for inseparable mixtures; that is, any two extracted signals involving the same sources are proportional to each other. For sub-Gaussian or super-Gaussian source signals with only mutual independence, cost functions based on fourth-order cumulants are introduced to sequentially extract all separable single sources and all inseparable mixtures. By minimizing the cost functions, gradient-based methods are developed. Our algorithms are guaranteed to converge. Finally, simulation results show the operation characteristics and the effectiveness of our methods.

I. INTRODUCTION

In recent years, blind separation or extraction of independent source signals from their mixtures has received wide attention [2]–[10]. In this paper, we consider a general linear instantaneous mixture of time series described by the following mixing model

\[ y(t) = Ax(t) \]  

where \( x(t) = [x_1(t), \ldots, x_n(t)]^T \) is a vector of mutually independent unknown sources with zero means, \( y(t) = [y_1(t), \ldots, y_m(t)]^T \) is a vector of mixed signals (as a result, \( y(t) \) have zero means) and \( A = \{ a_{ij} \} \) is an \( m \times n \) unknown constant matrix referred to as the mixing matrix. As usual, it is assumed that at most one source signal has Gaussian distribution. \( A \) is said to be a regular case if \( A \) has full column rank (e.g., nonsingular matrix \( A \) \( (m = n) \) and overdetermined matrix \( A \) \( (m > n) \) where \( \text{rank}(A) = n \)). \( A \) is said to be an ill-conditioned case if \( A \) has no full column rank, i.e., \( \text{rank}(A) < n \) (e.g., singular matrix \( A \) \( (m = n) \) and underdetermined matrix \( A \) \( (m < n) \)). There are many existing algorithms dealing with the regular case such as simultaneous separation approach in which all separable sources are separated simultaneously (see, e.g., [3], [4], [5]) and extraction approach in which sources are extracted one by one (see, e.g., [8] and [10]).

In recent years, some results on the ill-conditioned case are presented; e.g., [8], [6], [7]. In fact, ill-conditioned matrix is more general. In the present paper, we deal with sequential blind source extraction in the regular case as well as the ill-conditioned case. We only assume the mutual independence of sources. To the best of our knowledge, blind extraction for the ill-conditioned case is still far from being completely solved.

In this paper, we use the following general blind extraction model:

\[ z = By = Cx \]  

where \( z \) is an \( L \)-dimensional output vector, \( B = \{ b_{ij} \} \) is an \( L \times m \) blind extraction matrix, where \( L \leq m \) and \( C = BA \). The task of blind source extraction is to determine \( B \) such that one component in \( z \) corresponds to either a source signal up to a scale or a small mixture of some source signals.

II. PROBLEM FORMULATION

Generally speaking, one source \( x_i \) can be extracted if and only if there exists a \( 1 \times m \) extraction matrix \( B \) such that \( z = By = BAx = [0, \ldots, 0, c_i, 0, \ldots, 0]x \), where \( c_i \neq 0 \). How to find such kind of extraction matrix \( B \) is a key issue. Most of existing extraction approaches focus on the mixing matrix \( A \) having full column rank since all independent sources in this case can be extracted by using the above extraction model. The extraction matrix \( B \) can usually be obtained by maximizing the absolute value of the fourth-order cumulant (e.g., kurtosis) of the output of the extraction model subject to certain constraints [10]. In the ill-conditioned case, however, a mixture \( z \) of \( p \) sources can usually be extracted, where \( p \geq 1 \). The mixture \( z \) of \( p \) sources, for example, the sources \( x_1, \ldots, x_p \), can be extracted if and only if there exists a \( 1 \times m \) extraction matrix \( B \) such that

\[ z = By = BAx = [c_1, \ldots, c_p, 0, \ldots, 0]x \]  

where \( c_i \neq 0, i = 1, \ldots, p \). Recently Li and Wang provide an extractability condition (Theorem 1 [6]) for extraction of source signals in the case of ill-conditioned mixing matrix \( A \) and point out that the condition is weaker than the simultaneous blind separation condition (Theorem 1 [2]) which is also suitable for the ill-conditioned matrix \( A \). Once the condition is satisfied, one can use the proposed Gaussian-Newton method to find an \( m \times m \) nonsingular extraction matrix \( B \) such that a mixture of \( p \) (\( \geq 1 \)) sources, for example, the sources \( x_1, \ldots, x_p \), is extracted in one component (for...
example $z_1$ of $z$, the other components do not contain these sources; that is,
\[
z = By = BAx = \begin{bmatrix}
c_{11} & \cdots & c_{1p} & 0 & \cdots & 0 \\
0 & \cdots & 0 & c_{2(p+1)} & \cdots & c_{mn} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & c_{m(p+1)} & \cdots & c_{m2} \\
\end{bmatrix} \begin{bmatrix} x \\
\end{bmatrix}
\]
where $c_{ij} \neq 0, i = 1, \cdots, p$. In this case, the first row of $B$ plays the role of extraction since it is such that $z_1 = [c_{11}, \cdots, c_{1p}, 0, \cdots, 0]x$. The remaining rows of $B$ play the role of deflation since they are such that $z_2, \cdots, z_m$ do not contain the sources $x_1, \cdots, x_p$. Hence, in the approach the extraction and deflation are carried out simultaneously.

### III. Extractability Analysis

In this section, we will establish our results on extractability.

**Theorem 1:** Let $A_1$ be an $m \times (n-p)$ submatrix in $A$ and $\bar{A}_1$ be the remaining $m \times (n-p)$ submatrix composed of $n-p$ columns in $A$. There exists an $L \times m$ matrix $B$ with $\text{rank}(B) = L = m - \text{rank}(A_1) + 1$ such that a mixture of $p$ sources can be extracted in component $z_1$ of $z = [z_1, \cdots, z_L]^T$, the other components do not contain these sources and there exists at least some nonzero component $z_j$ $(i \in \{2, \cdots, L\})$ if and only if there exist $A_1$ and $\bar{A}_1$ such that $\text{rank}(A_1) < \text{rank}(A)$, $\text{rank}(A_1) < \text{rank}(A)$, and $\text{rank}(A : \bar{A}_1) = \text{rank}(\bar{A}_1) + 1$ for any column vector $a$ of $A_1$.

Note that $\text{rank}(A_1) < \text{rank}(A) \leq m$ in Theorem 1, we have $L = m - \text{rank}(A) + 1 \geq 2$. When $\text{rank}(A_1) = \text{rank}(A)$, Theorem 1 actually implies the following result.

**Theorem 2:** Let $A_1$ be an $m \times p$ submatrix of $A$ and $\bar{A}_1$ be the remaining $m \times (n-p)$ submatrix composed of $n-p$ columns of $A$. If $A_1$ and $\bar{A}_1$ satisfy that $\text{rank}(A_1) = \text{rank}(A)$, $\text{rank}(A_1) < \text{rank}(A)$, and $\text{rank}(A : \bar{A}_1) = \text{rank}(\bar{A}_1) + 1$ for any column vector $a$ of $A_1$. Then, there exists $A_1 \times m$ row vector $B$ such that a mixture of $p$ sources can be extracted.

According to Theorem 1, if $A$ satisfies the conditions in Theorem 1, we can extract a single source or a mixture of several sources as the component $z_1$ of $z = [z_1, \cdots, z_L]^T$ by an $L \times m$ extraction matrix $B$. Now we use the extraction matrix $B$ to construct an $m \times m$ nonsingular matrix $B^*$ according to the following rule:

\[
B^* = \begin{bmatrix}
b_{(L+1)1} & \cdots & b_{(L+1)m} \\
\vdots & \ddots & \vdots \\
b_{m1} & \cdots & b_{mm} \\
\end{bmatrix}
\]

where $[b_{i1}, \cdots, b_{im}], i = L + 1, \cdots, m,$ are such that $\text{rank}(B^*) = m$. Let $z = [z_1, \cdots, z_L, z_{L+1}, \cdots, z_m]^T = B^* y = B^* Ax$. Then we can use the remaining $m-1$ components (i.e., $z_2, \cdots, z_m$) of $z$ as new mixtures and continue to perform sequential blind extraction.

Next result guarantees that sequential blind extraction can be continued.

**Theorem 3:** If $A_1$ and $\bar{A}_1$ satisfy the conditions in Theorem 1, and $A_2$ and $\bar{A}_2$ satisfy the conditions in Theorem 1 where $A_1$ and $A_2$ have no common column, then, two kinds of mixtures can be extracted by the sequential blind extraction.

Theorem 1 provides a general condition for source extraction. The question now is whether or not the extracted signal $z_1$ is separable or under what condition the extracted signal $z_1$ is inseparable. For clarity, we define the concept of inseparability as follows: The mixture $z$ of $p$ $(p \geq 2)$ sources, for example, the sources $x_1, \cdots, x_p$, is said to be inseparable if there does not exist a $1 \times m$ extraction matrix $B$ such that $z = By = BAx = \begin{bmatrix} c_1, \cdots, c_p, 0, \cdots, 0 \end{bmatrix}x$ is a nonzero vector with $c_1 c_2 \cdots c_p \neq 0$.

The following theorem describes an important property of inseparable mixtures.

**Theorem 4:** Assume that there is an extraction vector $B = [b_1, \cdots, b_m]$ such that $z = B Ax = [c_1, \cdots, c_p, 0, \cdots, 0]x$ is inseparable where $c_i \neq 0, i = 1, \cdots, p$. If there is another extraction vector $B = [\tilde{b}_1, \cdots, \tilde{b}_m]$ such that $z = \tilde{B} Ax = [\tilde{c}_1, \cdots, \tilde{c}_p, 0, \cdots, 0]x$, then $z$ is proportional to $\tilde{z}$.

### IV. Cost Functions and Gradient-Based Methods

From Theorem 1, we can choose a proper extraction matrix $B$ such that the first component of the output is an extracted signal. Suppose that $x_1, \cdots, x_n$ are mutually independent and at most one of them is Gaussian. According to Theorem 1 in [6], if the first output $z_1$ is pairwise independent to others, then $c_{kj} = 0$ for all $k = 1, \cdots, i-1, i+1, \cdots, L$ when there is a $c_{ij} \neq 0$ where $j = 1, \cdots, n$. Hence, the pairwise independence of the first output with other $L - 1$ outputs is a basis of the blind extraction principle in this paper.

Based on the idea of [6] and model (2), next, a cost function for blind source extraction can be formulated as follows:

\[
J = \sum_{j=2}^{L} \text{cum}_2^2(z_1, z_j)
\]

where $\text{cum}_2^2(z_1, z_j)$ is defined as in [6]. Obviously, if $z_1$ and $z_j (j = 2, \cdots, L)$ are pairwise independent, then $J = 0$.

If source signals (and mixing matrices) are complex-valued, then we introduce the following cost function based on fourth-order cumulant:

\[
\tilde{J} = \sum_{j=2}^{L} \text{cum}_2^2(z_1^*, z_1^*, z_j, z_j^*)
\]

where $z_i^*$ denotes the complex conjugate of $z_i$. Under the assumption that all sources are mutually independent, one can see that $\tilde{J}$ is a real-valued function with real variables (real part variables and imaginary part variables of the extraction matrix). Thus, the optimization problem of $\tilde{J}$ is similar to the real-valued case.

Noting Theorem 4 in [6] and its proof, similarly we can have the following result.

**Theorem 5:** Suppose that $x_1(t), \cdots, x_n(t)$ are super-Gaussian (or sub-Gaussian) mutually independent stationary sources with zero means.
1) If there exists an $L \times m$ extraction matrix $B$ with $\text{rank}(B) = L$ such that $J = 0$ and $z_1 \neq 0$, then $z_1$ is an extracted source signal.

2) All local minima of $J$ with $z_1 \neq 0$ are global ones.

According to Theorem 5, if $J = 0$ with $\text{rank}(B) = L$ and $z_1 \neq 0$, then $z_1$ is an extracted signal. Now construct an $m \times m$ nonsingular matrix $B^*$ according to the rule (4) and write $z = B^*Ax = [z_1, \ldots, z_L, z_{L+1}, \ldots, z_m]^T$. We can use $z_2, \ldots, z_m$ as new mixtures to extract the remaining sources sequentially. However, when $A_1$ in Theorem 1 satisfies $\text{rank}(A_1) \geq 2$, the resulting mixtures $z_{L+1}, \ldots, z_m$ still involve the sources in the extracted signal $z_1$. Thus, to extract the remaining sources from new mixtures $z_2, \ldots, z_m$, we have to take deflation process again. Let us redefine $\tilde{y} = [z_2, \ldots, z_m]^T = Ax$ and assume that two submatrices $\tilde{A}_1$ and $\tilde{A}_2$ satisfy the conditions in Theorem 1. By Theorem 1, there exists an $L \times (m-1)$ extraction matrix $B$ such that the component $\tilde{z}_1$ of $\tilde{z} = [\tilde{z}_1, \ldots, \tilde{z}_L]^T = B\tilde{y} = Cx$ is an extracted signal. Therefore, to perform source extraction, we propose the following cost function:

$$ J = \sum_{j=2}^{L} \text{cum}^2_{2,2}(\tilde{z}_1, \tilde{z}_j) + \text{cum}^2_{2,2}(\tilde{z}_1, z_1). $$

(7)

where $z_1$ is the extracted signal in the previous extraction.

To extract more source signals, the above process can be continued until no source signal can be extracted any more.

In view of Theorem 5, blind source extraction using (2) is converted into solving the following constrained minimization:

$$ \min_{\text{subject to } \text{rank}(B)=L} J. $$

(8)

Let $B = [b_{ij}]_{L \times m}$ and $b = [b_1, \ldots, b_1, b_2, \ldots, b_{2m}, \ldots, b_{L,1}, \ldots, b_{L,m}]$. $J$ can be rewritten as $J(b)$. Letting the time derivative of the variable $b$ be directly proportional to the negative gradient of $J(b)$ with respect to the variable $b$, we have the following dynamic equation

$$ \frac{db}{dt} = -\mu \frac{\partial J(b)}{\partial b}, \quad b(0) = b_0 $$

(9)

where $\mu$ is a positive scaling constant.

Equation (9) is the gradient-based method for the minimization of cost function $J$. Its convergence can easily be seen. By Theorem 5, if the point of convergence $b^*$ is such that $J(b^*) = 0$ with $\text{rank}(B) = L$ and $z_1 \neq 0$, then $z_1$ is an extracted signal. We can propose similar gradient-based method for (7).

V. SIMULATION RESULTS

**Example 1:** Consider the mixing model (1) with six sources and four mixtures where

$$ A = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 2 & 1 & 1 \\ 1 & -1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 1 & 0 \end{bmatrix}. $$

Six speech sources (3500 samples) from the 1st to 6th signals of the file Speech20.mat downloaded from [1] are used in this example. The kurtosis of $x_1, x_2, x_3, x_4, x_5, x_6$ are 0.9428, 1.2226, 2.6526, 1.4799, 0.5880, 2.0333. Thus, all six sources are super-Gaussian. Six sources and four observable mixtures are plotted in the first, second and third rows, of Figure 1. For the above mixing matrix $A$, there are several kinds of submatrix pairs $(A_1, A_2)$ satisfying Theorem 1, e.g.,

$$ A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ -1 & 0 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad \tilde{A}_1 = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}. $$

(10)

Since $\text{rank}(A_1) = 2$ (as a result, Theorem 1 [6] and [7] cannot be used), according to Theorem 1 we have $L = 3$ and there exists a $3 \times 4$ matrix $B$ such that a mixture $z_1$ of sources $x_2$ and $x_3$ can be extracted. Moreover, $z_1$ is inseparable. When we do simulation, in order to keep all rows of $B$ away from zero vectors, we let

$$ B = \begin{bmatrix} 1 & b_{12} & b_{13} & b_{14} \\ b_{21} & 1 & b_{23} & b_{24} \\ b_{31} & b_{32} & 1 & b_{34} \end{bmatrix}. $$

Let $\mu = 1 \times 10^7$ and use the gradient-based method. The initial value of $(b_{12}, \ldots, b_{34})^T$ is chosen randomly as $(1.5742, -1.9031, 1.8170, 1.7862, -1.2296, 0.2754, -1.7349, -0.8605, 1.5767)^T$. We obtain the blind extraction matrix $B$ and, in turn, the matrix $C = BA$ as follows:

$$ B = \begin{bmatrix} 1 & 0.5004 & -1.5004 & 0.4996 \\ 1.4096 & 1 & -0.2048 & -1.2048 \\ -2.3401 & -0.3401 & 1 & 1.3401 \end{bmatrix}, $$

$$ C = \begin{bmatrix} -0.001 & 2.500 & 2.499 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.590 & 2.205 & -1.614 \\ 0.000 & 0.000 & 0.000 & 1.659 & -1.680 & 3.340 \end{bmatrix}. $$

In Figure 1, the extracted signal $z_1$ is shown in the first subplot of the fourth row. The mixtures $z_2$ and $z_3$ of the sources $x_4, x_5, x_6$ are shown in the second and third subplots, respectively, of the fourth row. The last subplot in the fourth row shows the calibrated deviation $z_1 - (2.5004x_2 + 2.4996x_3)$, which shows that the mixture of the sources $x_2$ and $x_3$ is extracted. It is noted that the source $x_1$ are almost not involved in any one of $z_1, z_2, z_3$.

To continue to the second step blind extraction, let

$$ B^* = \begin{bmatrix} 1 & 0.5004 & -1.5004 & 0.4996 \\ 1.4096 & 1 & -0.2048 & -1.2048 \\ -2.3401 & -0.3401 & 1 & 1.3401 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

which is nonsingular. Let \( \tilde{y} = (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)^T = (z_2, z_3, z_4)^T \). From \( z = (z_1, z_2, z_3, z_4)^T = B^*Ax \) we have

$$ \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{y}_3 \end{bmatrix} = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.5903 & 2.2048 & -1.6145 \\ 0.0 & 0.0 & 0.0 & 1.6599 & -1.6802 & 3.3401 \end{bmatrix} x $$

\( \tilde{A}x \)
In the second row of Figure 2 shows that the extracted signal $\tilde{z}_{11}$ is the mixture of sources $x_4$ and $x_5$.

To verify Theorem 4, we choose another random initial condition as $(0.9902, 0.5465, 0.9427, 0.1089, 0.1680, 0.3705)^T$ and obtain the blind extraction matrix $\mathbf{B}_2$ and, in turn, the matrix $\mathbf{C}_2 = \mathbf{B}_2^T \mathbf{A}$ as follows:

$$
\mathbf{B}_2 = \begin{bmatrix}
0.9737 & 0.4706 & 0.0000 \\
-0.0009 & -0.0799 & 0.1685 \\
0.168 & 0.168 & 0.337 & 0.000 & 0.000 & 0.000
\end{bmatrix},
$$

$$
\mathbf{C}_2 = \begin{bmatrix}
0.000 & 0.000 & 0.000 & 1.356 & 1.356 & -0.000 \\
0.168 & 0.168 & 0.337 & 0.000 & 0.000 & 0.000
\end{bmatrix}.
$$

The simulation result is presented in the third row of Figure 2. From the last subplot in the third row of Figure 2 one can see that the extracted signal $\tilde{z}_{11}$ is the mixture of the sources $x_4$ and $x_5$ and is very similar to $\tilde{z}_{11}$.

**ACKNOWLEDGMENT**

This work was supported by the Hong Kong Research Grants Council under Grant CUHK4203/04E.

**REFERENCES**


