FINITE PSEUODOCOMPLEMENTED LATTICES and "PERMUTOEDRE"

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Abstract
We study finite pseudocomplemented lattices and especially those that are also complemented. With regard to the classical results on arbitrary or distributive pseudocomplemented lattices (Glivenko, Stone, Birkhoff, Frink, Grätzer, Balbes, Horn, Varlet...), the finiteness property allows to bring significant more precise details on the structural properties of such lattices. These results can especially be applied to the lattices defined by the "weak Bruhat order" on a Coxeter group (and for instance to the lattice of permutations, called in French "le treillis permutoèdre") and to the lattice of binary bracketings.

Résumé
Soit $T$ un treillis avec plus petit élément noté $0$ ; l'élément $t$ de $T$ a un inf-pseudocomplément, noté $g(t)$, si $g(t)$ est le plus grand élément de l'ensemble des $x$ de $T$ tels que $x \land t = 0$ ; $T$ est inf-pseudocomplémenté (IPC.) si tout élément de $T$ a un inf-pseudocomplément. On définit dualement la notion de sup-pseudocomplément $f(t)$ de l'élément $t$ et de treillis sup-pseudocomplémenté (SPC) et par conjonction des deux propriétés IPC et SPC celle de treillis pseudocomplémenté. Ces treillis ont surtout été étudiés dans des cas où ils sont distributifs et infinis (treillis de Brouwer ou d'Heyting, treillis de Stone...). Notre intérêt pour le cas fini provient-entre autres-de ce que le "treillis permutoèdre"(Guilbaud et Rosenstiehl 1971), i.e. l'ensemble des permutations d'un ensemble fini muni de l'"ordre faible de Bruhat", est un treillis pseudo complémenté. Dans le cas d'un treillis IPC la classique correspondance de Galois associée à l'application $g$ d'inf-pseudocomplémentation permet de montrer que l'ensemble des inf-pseudo compléments a une structure de treillis booléen (Frink 1962). Dans le cas fini, nous donnons d'abord une caractérisation constructive des inf-pseudocompléments permettant de retrouver ce résultat. Nous montrons ensuite que pour un treillis complémenté les propriétés d'être IPC ou SPC sont équivalentes. Nous décrivons ensuite de façon approfondie la structure des treillis complémentés pseudocomplémentés, ces derniers résultats s'appliquant au treillis permutoèdre et au treillis des parenthésages.
INTRODUCTION

Let \( L \) be a lattice with a least element denoted \( 0 \); \( g(t) \in L \) is a meet-pseudocomplement of \( t \in L \), if \( [x \land t = 0 \iff x \leq g(t)] \). \( L \) is meet-pseudocomplemented if every element of \( L \) has a meet-pseudocomplement. One defines dually the notion of a join-pseudocomplement \( f(t) \) of \( t \) and of a join-pseudocomplemented lattice. A lattice is pseudocomplemented if it is meet- and join-pseudocomplemented (beware! often "pseudocomplemented" means only "meet-pseudocomplemented" and a join-pseudocomplemented lattice is sometimes called a "dual pseudocomplemented" lattice). Two classes of meet-pseudocomplemented lattices have been intensively studied. First the Brouwerian (called also Heyting or implicatice) lattices. They are the "relatively meet-pseudocomplemented" lattices what imply they are distributive (Glivenko 1929, Birkhoff 1940, 1948, etc....). Second, the Stone lattices which are distributive meet-pseudocomplemented lattices satisfying an additional condition (Stone 1937, Varlet 1963, Balbes and Horn 1970, etc.....). Grätzer (1978) and Varlet (1963, 1974-75) provide excellent accounts of the results known on arbitrary meet-pseudocomplemented lattices or on the above special classes. Observe that in most of the studied cases the considered lattices are infinite. We are interested here by the specific properties of the class of finite (meet- or/and join-) pseudocomplemented lattices. Indeed, the lattice of permutations (called in french, le "treillis permutotèdre", Guilbaud and Rosenstiehl 1971) and, more generally, the lattices defined by the "weak Bruhat order" on a Coxeter group (see Björner 1984) are (meet and join) pseudocomplemented lattices (Le Conte de Poly-Barbut 1990). It is also the case of the lattice of the binary bracketings (see Huang and Tamari 1972, and Lakser 1978); all these particular lattices are also complemented.

In this paper we give a summary of our results on the structure of finite meet-pseudocomplemented, (meet and join) pseudocomplemented, and pseudocomplemented and complemented lattices. For a detailed account with the proofs of these results see Chameni and Monjardet 1992. The specific theory of finite meet-pseudocomplemented lattices begins with the easy but crucial observation that a (finite) lattice is meet-pseudocomplemented if and only if each of its atoms has a meet-pseudocomplement. So, in such lattices the meet-pseudocomplements can be expressed by means of the meet-pseudocomplements of the atoms (Theorem 1). Then one shows (Theorem 2) that the joins of atoms define a Boolean lattice isomorphic with the lattice of the meet-pseudocomplements, thus reobtaining the classical result (Frink 1962) that this last lattice is Boolean. Proposition 4 and 5 study the properties of an element in a meet-pseudocomplemented lattice and especially when this element has a complement. Theorem 6 characterizes the meet-pseudocomplemented lattices which are complemented, such lattices being the same that the complemented join-pseudocomplemented lattices and thus that the
complemented (and) pseudocomplemented lattices (Theorem 7). Theorem 8 gives other characterizations of such complemented and pseudocomplemented lattices and Theorem 9 summarizes all our results on the structure of such lattices. For instance, we show that the Glivenko congruence "to have the same meet-pseudocomplement" is the same that "to have the same join-pseudocomplement" or that "to have the same complements" or, etc.... (see 9.4 ). The classes of this congruence are the $2^n$ intervals $[\lor A(x), \land C(x)]$ with $A(x) = \{\text{atoms } a : a \leq x\}$, $C(x) = \{\text{coatoms } c : c \geq x\}$, and $n$ the number of atoms -or of coatoms- of the lattice $L$. These results can be applied to the "concrete" lattices mentioned above. For instance, the figure below shows the lattice of permutations on 4 elements with two classes of the Glivenko congruence.

\[ g(1423) = 3421 \quad \text{Glivenko class of } 1423 = \{1243, 1423, 4123\} \]
\[ f(1423) = 3214 \quad \{\text{complements of } 1423\} = \{3214, 3241, 3421\} \]

Two (complementary) Glivenko classes of the lattice of permutations of \{1,2,3,4\}

RESULTS
In this paper $L$ denotes a finite lattice; 0 (respectively, 1) denotes the least (respectively, the greatest) element of $L$; $\preceq$ (respectively, $\prec$, $\land$ and $\lor$) denotes the order relation (respectively, the covering relation, the infimum - or meet - operation, the supremum - or join - operation) defined on $L$.

An element $t_*$ (respectively, $t^*$) of $L$ is a meee(move-pseudocomplement) (respectively, a join-pseudocomplement) of an element $t$ of $L$, if $x \land t = 0 \iff x \preceq t_*$ (respectively, $x \lor t = 1 \iff x \geq t^*$).

Otherwise said, $t$ has a meet-pseudocomplement $t_*$ (respectively, a join-pseudocomplement $t^*$) if $t_*$ (respectively, $t^*$) is the greatest element (respectively, the least element) of the set of all elements $x$ such that $x \land t = 0$ (respectively, such that $x \lor t = 1$) .
A lattice $L$ is *meet-pseudocomplemented* (respectively, *join-pseudocomplemented*) if each element of $L$ has a meet-pseudocomplement (respectively, a join-pseudocomplement). $L$ is *pseudocomplemented* if it is both meet and join-pseudocomplemented (take care: often "pseudocomplemented" means only "meet-pseudocomplemented"). Then, we denote by $g$ (respectively, $f$) the map $t \rightarrow gt = t^*$ (respectively, $t \rightarrow ft = t^*$).1

An obvious but significant observation made by Birkhoff (1948) is that in a meet-pseudocomplemented lattice $L$ the map $g$ of meet-pseudocomplementation is a "symmetric Galois connection" (for the definitions of a Galois connection and of other notions of lattice theory not defined here, see Birkhoff 1967 or Grätzer 1978). Then, $g^3 = g$, $g^2 = \phi$ is a closure operator on $L$ and the set $G = g(L)$ of all meet pseudocomplements is a lattice sub meet-semilattice of $L$. We recall also the following classical results:

An atom (respectively, a coatom) of $L$ is an element covering 0 (respectively, covered by 1). We denote by $A$ (respectively, $C$) the set of all atoms (respectively, coatoms) of $L$, and we use the following notations: for $x$ in $L$, $A(x) = \{a \in A : a \le x\}$, $C(x) = \{c \in C : c \ge x\}$; $A'(x) = A \wedge A(x)$; $C'(x) = C \setminus C(x)$.

**Theorem 1**

Let $L$ be a (finite) lattice; the two following conditions are equivalent:
1) $L$ is a meet-pseudocomplemented lattice,
2) each atom of $L$ has a meet-pseudocomplement (denoted by $ga$)

These conditions imply that for all $x, y$ in $L$,
3) $gx = \wedge g[A(x)] = g[\bigvee A(x)]$,
4) $qx = \phi[\bigvee A(x)] = \wedge (g[A(x)])$,
5) $A(x) = A(qx)$,
6) $gx = gy \iff qx = qy \iff A(x) = A(y)$.

We denote by $\Pi$ the equivalence defined on $L$ by the equalities in 6) above. It is easy to see that $\Pi$ is a congruence on $L$ called the Glivenko congruence. Then, the quotient lattice $L/\Pi$ is isomorphic to the lattice $G$ of the meet-pseudocomplements of $L$. We denote by $\Pi x$ the congruence class of $x$.

Let $A^\vee = \{\bigvee X, X \subseteq A\}$ be the set of all the join of sets of atoms of $L$. The set $A^\vee$ is a lattice for the order defined on $L$ (observe that $\bigvee \emptyset = 0$).

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1 The image $m(x)$ of an element $x$ by a map $m$ will be generally noted $mx$. 
Theorem 2
Let $L$ be a meet-pseudocomplemented lattice:
1) The lattice $A'$ of join of atoms of $L$ is a Boolean lattice, sub join-semilattice of $L$ and with same least element 0.
2) $g$ (respectively, $\varphi$) induces a dual isomorphism (respectively, an isomorphism) between $A'$ and the lattice $G$ of the meet-pseudocomplements of $L$:
   \[ A(gx) = A'(x), \quad g(x) = \varphi[V A'(x)]. \]
3) The classes of the Glivenko congruence $\Pi$ are the $2^{|A|}$ intervals defined for each $X \subseteq A$ by $[\vee X, \varphi(\vee X)]$:
   \[ \Pi x = [\vee A(x), \varphi[\vee A(x)], \quad \Pi(gx) = [\vee A'(x), \varphi[\vee A'(x)]. \]

Corollary 3
The lattice $G$ of all meet-pseudocomplements of a (finite) lattice $L$ is a Boolean lattice, sub meet-semilattice of $L$, with same least and greatest elements 0 and 1.

Remark. In fact, this last result is true for an arbitrary "(meet) pseudocomplemented meet-semilattice" (Frink 1962). Frink’s proof uses a concise axiomatic of a Boolean lattice, whereas Grätzer (1980) gives a direct proof of the distributivity of $G$.

Obviously there are dual results for the join-pseudocomplemented lattices with the coatoms of $L$ playing the role of atoms. We give now two propositions on the (meet and join) pseudocomplemented lattices preparing our results on the complemented pseudocomplemented lattices.

Proposition 4
Let $x$ be an element of the pseudocomplemented lattice $L$ with $|A| = n$ and $|C| = p$. Let us write $f'x = \vee A'(x)$ and $g'x = \wedge C'(x)$. Then:
1) $x \vee y = 1 \Rightarrow y \geq f'x$.
2) $x \wedge y = 0 \Rightarrow y \leq g'x$.
3) $f'x \leq gx \wedge fx$, \hspace{1cm} $g'x \geq gx \vee fx$.
4) $C'(x) = g[A(x)] \Leftrightarrow gx = g'x \Rightarrow C(gx) = C(g'x) \Leftrightarrow C(gx) = C'(x) \Leftrightarrow C(gx) = C'(x) \Leftrightarrow C(gx) = C'(x) \Leftrightarrow C(gx) = C'(x)$
   \[ A'(x) = f[C(x)] \Leftrightarrow fx = f'x \Rightarrow A(fx) = A(f'x) \Leftrightarrow A(fx) = A'(x) \Leftrightarrow A(fx) = A'(x) \Leftrightarrow A(fx) = A'(x) \Leftrightarrow A(fx) = A'[x] \Leftrightarrow |A(x)| + |A(fx)| = n. \]

Proposition 5
Let $x$ be an element of the pseudocomplemented lattice $L$. The following conditions (1) or (2) imply the equivalent conditions (3) and (4):
1) $gx = g'x$,
2) $fx = f'x$. 
3) \( f x \leq g x \),
4) \( x \) has a complement in \( L \).

Distributive lattices are pseudocomplemented lattices since in such lattices, one easily checks that \( g x = \vee \{ t \in L : x \wedge t = 0 \} \) and \( f x = \wedge \{ t \in L : x v t = 1 \} \). On the contrary, (non distributive) upper locally distributive lattices (see Monjardet 1990, for a presentation of these lattices first studied by Dilworth) are meet-pseudocomplemented lattices not pseudocomplemented. As said in the introduction, the lattices defined by the "weak Bruhat order" on a (finite) Coxeter group and the lattice of binary bracketings are pseudocomplemented and complemented lattices. So, we come now to our results on complemented pseudocomplemented lattices. First, we give six characterizations of such lattices (see also Theorem 7).

**Theorem 6**
Let \( L \) be a meet-pseudocomplemented lattice. The following conditions are equivalent:
1) \( L \) is complemented,
2) \( \Pi 1 = \{1\} \),
3) \( A(x) = A \Rightarrow x = 1 \),
4) \( \forall A = 1 \),
5) \( g \) induces a bijection between \( A \) and \( C \),
6) \( \varphi \) is the identity map on \( C \),
7) \( L \) is strictly meet-semicomplemented (i.e., \( x \neq 1 \) implies that there exists \( y \neq 0 \) with \( x \wedge y = 0 \)).

**Remark.** In theorem 6, Condition (7) is due to Varlet (1963).

In fact the significant following result shows that the complemented meet-pseudocomplemented lattices are the complemented pseudocomplemented lattices (and also the complemented join-pseudocomplemented lattices).

**Theorem 7**
For a complemented lattice \( L \), the three following conditions are equivalent:
1) \( L \) is meet-pseudocomplemented,
2) \( L \) is join-pseudocomplemented,
3) \( L \) is pseudocomplemented.

We give now other characterizations of pseudocomplemented lattices \( L \) which are also complemented. In the following results, \( \Pi \) denotes the congruence on \( L \) defined by \( x \Pi y \Leftrightarrow f x = f y \Leftrightarrow C(x) = C(y) \).
**Theorem 8**

Let $L$ be a pseudocomplemented lattice with $n$ atoms and $p$ coatoms. The following conditions are equivalent:

1) $L$ is complemented,
2) For every $x \in L$, $gx = g'x$ (or $|C(x)| + |C(gx)| = p$),
3) For every $x \in L$, $fx = f'x$ (or $A(x) + |A(fx)| = n$),
4) For every $x \in L$, $fx \leq gx$,
5) For every $x \in L$, $\Pi(gx) = \Pi^*(fx)$,
6) $\Pi = \Pi^*$,
7) $\Pi0 = \Pi^00$,
8) $\Pi1 = \Pi^1$.

The following theorem summarizes all our results on the structure of complemented pseudocomplemented lattices; there $\Psi = f^\circ$ is the dual closure operator defined on a (join-) pseudocomplemented lattice $L$, $F = f(L) = \Psi L$, and $C^\lor$ is the lattice formed by all the meet of coatoms.

**Theorem 9**

1) $f = f^\circ = f^\Psi = \Psi f = \Psi g = f^\varphi \leq g = g^3 = g\varphi = \varphi g = \varphi f = g\Psi$,
2) $g = gL = qL = C^\lor$ is a Boolean lattice, sub meet-semilattice of $L$ with same 0 and 1. $F = fL = \Psi L = A'$ is a Boolean lattice, sub join-semilattice of $L$, with same 0 and 1. The maps $f$ and $g$ (respectively, $\varphi$ and $\Psi$) induce two inverse antiisomorphisms (respectively, isomorphisms) between $G$ and $F$; the map $g$ on $G$ (respectively, $f$ on $F$) is the complementation in this lattice; $gA = C = \varphi C$, $fC = A = \Psi A$.
3) $g$ (respectively, $f$) is a morphism of $L$ on the dual of $G$ (respectively, $F$), $\varphi$ (respectively, $\Psi$) is a morphism of $L$ on $G$ (respectively, $F$).
4) For $x, y \in L$,

$$gx = gy \Leftrightarrow \varphi x = \varphi y \Leftrightarrow A(x) = A(y) \Leftrightarrow V A(x) = V A(y) \Leftrightarrow fx = fy \Leftrightarrow$$

$$\Psi x = \Psi y \Leftrightarrow C(x) = C(y) \Leftrightarrow \Lambda C(x) = \Lambda C(y) \Leftrightarrow [fx, gx] = [fy, gy] \Leftrightarrow$$

$$[\Psi x, \varphi x] = [\Psi y, \varphi y].$$

The relation $\Pi$ defined on $L$ by these equivalent equalities is a congruence; the congruence classes of $\Pi$ are the $2^n$ intervals $[V A(x), \Lambda C(x)]$ (with $n = |A| = |C|$). The maps $g$ and $f$ (respectively, $\varphi$ and $\Psi$) are isomorphisms (respectively, involutive dual isomorphisms) between $L/\Pi$, $F$ and $G$. Let $(\Pi x)'$ be the class complement of the class $\Pi x$ in the Boolean lattice $L/\Pi$. Then,

$$gx = \max([\Pi x]'), \quad fx = \min([\Pi x]').$$
\[
\phi x = \max(\Pi x), \quad \Psi x = \min(\Pi x),
\]

\{complements of \( x \) in \( L \}\} = (\Pi x)' = [fx, gx].

5) \( \Pi 1 = \{1\} = \forall A \), \( \Pi 0 = \{0\} = \forall C \).

Note added in proof} Several results of this paper were first published in the Chameni-Nembua thesis (1989, see reference below) and in a CAMS report (C. Chameni-Nembua and B. Monjardet, Les treillis pseudocomplémenté finis, Rapport CAMS P O61, Paris, 1990). Since 1991 we become aware of a M.K. Bennett and G. Birkhoff preprint (1991, to appear in Algebra Universalis) Two families of Newman lattices, of several G. Markowsky reports on the permutation (or "permutoèdre") lattice, beginning in 1990, (the last one being Permutation lattices revisited, August 1992, University of Maine), and of a annex (Retracts and Glivenko intervals) written with G. Markowsky in 1992 to a Birkhoff paper (to appear in the Proceedings of the 1991 Darmstadt Conference on lattice theory); the Bennett and G. Birkhoff paper studies a class of lattices containing both the permutation lattice and the binary bracketings lattice (especially it contains new results on this last one); the Markowsky reports contains old and new results on the permutation lattice (some of them have been independently got by V. Duquenne and A. Cherfouh, On the permutation lattice, Rapport CAMS P O77, Paris, 1991); these reports and especially the above quoted annex contain also several results which have been obtained but not published by C. Le Conte de Poly-Barbut (see reference below) and which are special cases of theorems of this paper; indeed the C. Le Conte de Poly-Barbut results on the permutation lattice were our main motivation to study the more general class of finite pseudocomplemented (and possibly complemented) lattices. The reader will be able to find in the Chameni-Nembua and Monjardet paper Les treillis pseudocomplémentés finis (reference below) a much more complete bibliography on the permutation lattice and related topics (we just add here that the order dimension of the "multinomial lattices" and especially of the permutation lattice has been determined by S. Flath, preprint 1492, Darmstadt, 1992).

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In 1963 Guilbaud and Rosenstiehhl show that a partial order defined on the set of all linear orders on a finite set of size $n$ (and so, equivalently on the set $S_n$ of permutations of this set) is a lattice. Their proof –made more understandable- in the 1971 version of their paper – is taken again in Berge 1968 and in Knuth 1973 (Exercises 11 & 12, page 18-19 in the last edition). The same result also appears in the framework of rank statistics' theory. Indeed, for the need of this theory, Savage defines several orders on the set of permutations (or more generally on some sets of words). Under some probabilistic assumptions, the order defined, for example, between two permutations of the $n$ first integers induces an order between the probabilities that these permutations be the rank vectors of $n$ ordinal random variables. In 1964 Savage studies the possibility that these partial orders be lattices and raises several questions. Answering one of them, Yanagimoto and Okamoto (1969) define on $S_n$ the same partial order that Guilbaud and Rosenstiehhl and claim that it is a lattice (Theorem 2.1). But one must observe that they evade the only difficult point to prove by writing "it can be shown that" (!!). One can also note that their preliminary result (Proposition 2.2) amounts to showing that a partial order $O$ contained in a linear order $L$ has dimension 2 if and only if $L$ is a "non separating linear extension" of $O$, an obvious corollary of Dushnik and Miller’ 1942 characterization of partial orders of dimension 2. The partial order on $S_n$ considered by Guilbaud and Rosenstiehhl, as well as by Yanagimoto and Okamoto, corresponds to what is often now called the "weak Bruhat order" on the symmetric group $S_n$ (see Björner, 1984 ; note that one of the partial orders considered by Savage was the strong Bruhat order on $S_n$).

Some simple properties of the lattice $S_n$ are in Barbut and Monjardet (1970) and Monjardet (1971). The characterization of the irreducible elements of the lattice $S_n$ appears in Chameni-Nembua (1989), whereas the
characterization of the order between the join- and the meet- irreducible elements (the so-called lattice table in "formal concept analysis") is in Duquenne and Cherfouh (1991) and in Markowsky (1991). Other properties of the lattice $S_n$, or/and, more generally, of Bruhat weak orders on a finite Coxeter group, have been obtained by Björner (1984) who especially shows that they are orthocomplemented lattices, Le Conte de Poly-Barbut (1986-1990, 1994) and Leclerc (1991). Since the lattice $S_n$ is pseudocomplemented and complemented, it also satisfies properties given in Chameni-Nembua and Monjardet (1992, 1993). A significant property is that $S_n$ is a bounded lattice (Caspard 2000), a result generalized to any finite Coxeter lattice by Caspard, Le Conte de Poly-Barbut and Morvan (2004).

On the other hand, Stanley (1984) and Edelman and Greene (1984,1987) enumerate maximal chains of the lattice $S_n$ (or of intervals of this lattice) and show correspondences between these maximal chains and some Young tableaux.


In 1951, Tamari defines an order on the set of all possible binary bracketings on a sequence of $n+1$ letters. Later, he shows with Friedman (1967) and Huang (1972), that this order is a lattice $T_n$ (see also Grätzer, 1978, Exercises 26 to 36, and Huguet 1975 for a geometrical proof of this result). These lattices $T_n$ called Tamari lattices (and also associahedra) are shown to be pseudocomplemented and complemented by Lakser (1978).

On the other hand, in 1991 Bennett and Birkhoff define a Newman (or multinomial lattice) as a lattice formed by a set of words (sequences of an alphabet) ordered from "positive elementary transformations". This class of lattices include the lattices $S_n$, its generalizations obtained when the words can contain more than one instance of each letter and the lattices $T_n$ (in which the transformation corresponds to an associativity rule). Bennett and Birkhoff’s paper contains the characterization of the ordered set of the irreducible elements of the lattice $T_n$.

Since then (or sometimes before) many classes of lattices which are restrictions or extensions of Tamari lattices -like Kreweras lattices and Stanley lattices or/and more general classes of lattices like Cambrian lattices- have been investigated. They are sets of combinatorial or geometrical objects like noncrossing partitions, bracketings, Dick words, binary trees,

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