A STUDY OF HIGH FREQUENCY TRADING IN LIMIT ORDER BOOKS

BY

YUAN JIANG

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ABSTRACT

In the thesis we study the high frequency trading and its applications in limit order books. We discuss the basic concepts and review the models in the limit order books. The review section focuses on the queues in the limit order books, optimal trading strategies, short-term volatilities and multi-agent problems in the scenario of limit order markets. Discussions on the shortage of some prevalent models of limit order books are addressed thereafter. For the main results of the thesis, market data are calibrated to facilitate the comparison between a theoretical model and the empirical behaviors in terms of order flows, price changes and diffusion limit of prices.

CHAPTER 1
INTRODUCTION

1.1 History of the High Frequency Trading

When trading in financial markets first came into existence, the trading process was artificially made, with the market information poorly gathered and utilized. As time goes, the utilize of advanced technological instruments and computer-based algorithms to trade financial securities emerged rapidly. That is what the term High-Frequency-Trading (HFT) means. To put it concisely, tiny fractions of money accumulate fast to produce significantly positive results at the end of every day.

Historically, HFT began its mission in the middle-to-late-1990s when electronic exchanges were accepted to do their jobs. The electronic behavior of execution venues enabled market participants (e.g. banks, brokers and their institutional and retail clients) to transmit orders electronically as opposed to via telephone or mail, which is referred to as electronic trading. HFT is a subset of electronic trading (ET). ET has a great deal of orders (usually with fairly small sizes) being sent into the financial market fastly, with execution times measured in microseconds.

In its early years, HFT accounted for a mere single-digit-percentage proportion in the major equity markets in the globe. However, this market grew with considerably high speed ever since. According to data from the New York Stocks Exchange (NYSE), the volume of high frequency trading grew at the rate of 164% between 2005 and 2009. As regards from January to March in 2009, high-frequency trading strategies contributed to $141 billion among the whole instruments traded in hedge funds. Set the United States as example, enterprises herein using HFT account for 2% out of the whole within their types operating today, yet account for more than 70% of all trading volumes. As of the emerging markets like China, India and Malaysia, HFT is also expected with great potential for rapid growth. With respect to the trading value, this kind of trading was estimated by the consultant firm Tabb Group to constitute more than 55% of equities traded in North American and 35% in European countries. (See [7] for further reference.)

Parallel to the market impact HFT has exerted, research in high-frequency market has been evolving. Back to the early 1980s, when exchanges were converted into automated trading platforms, in [5] the authors made an assupmtion that liquid markets display more continuity character of trading prices, which took place as long as trading is exhibit via a large volume being traded yet small size per trading individuals. Black also stated that despite how technology progressing with respect to empirical construction, market prices is mainly and largely impacted by large orders being executed.

At those days in stock markets, for instance, market makers are usually financial experts who provided liquidity. Just like in the trading arena of the derivative markets, quite a few floor traders were well-identifiable market makers. Both these experts from stock markets (or bond markets) and those floor traders from derivative markets prevail a a privilege over floor traders mentioned above, see [2].

Such a facilitated pattern allowed both financial experts and floor traders responding more rapidly into incoming limit/market order flows than traders without this property. Trading surroundings through which market makers are distinct from other traders are studied in the theoretical models of [16].

As markets became electronic, the difference between market makers and other traders faded. Equity exchanges increasingly took a limit order market construction. At the mean time, traders submit orders directly into the automated systems run by these advanced exchanges, surplus designated market makers. Nowadays in the electronic markets, HFT has a time-lag advantage over non-HFT traders. It allows them to get more quick response to the converts with respect to order flows. This occurred owing to the progress in technology, as well as requirements regarding laws and regulations. Theoretical models of limit order markets include, among others, [7], [13], [12] and [23]. More state-of-art research would be mentioned and developed in detail in the thesis.

A natural question comes into being: how to get more lucre by taking advantage of this 'high frequency' scenario. Technically saying, the higher frequency is not a strategy itself but rather a technically advanced method of *practising* certain strategies, like, providing liquidity or pursuing arbitrage (between cross-border markets or domestic ones) or to detect liquidity. One interesting issue is that though beaten by the advanced electronic trading system, we human kinds have not changed the pattern of trading itself that much, which is as a matter of fact built by us. Take for example the Citadel trading system in the International Securities Exchange (in New York City), the only exchange before the year 2000 that allowed electronic quotes to be made. (Also see [7] for reference.)

1.2 A Glimpse of Research on High Frequency Trading

For research interest, we are interested in how to deal with the bid-ask spread and its relationship with market liquidity, the time duration between the consecutive arrival of orders, the shape of the order book, market resilience of a big (one-sided) order, time-scale analysis as of the resemblance of long/short time effect of the market for dominant orders, the market volatility pertaining to HFT, the diffusion limit of the price behavior, and so forth. It is with the development of HFT that these problems came into the academic sight.

As a literature review on HFT, this thesis is organized as follows. In the next chapter, some fundamental concepts and descriptions are introduced and discussed; Chapter 3 studies several mathematical topics of the limit-order markets, covering a large part of the contemporary research on HFT in limit order books and optimal strategies for market participants. Most of the review is done mathematically, with backgrounds, approaches, mathematical models, the main thoughts behind the curve of what is reviewed; Chapter 4 studies the pros and cons of the topics in Limit Order Books. Some of them contain the interdisciplinary within mathematical finance; Chapter 5 discusses a typical model on the limit order markets. In this last chapter we do the data calibration and give some comparison between theoretical and empirical results, before conclusion.

CHAPTER 2
CONCEPTS FROM HIGH FREQUENCY TRADING

2.1 Limit Order Books

In the literature of high frequency trading, the most significant conception here is the Limit Order Book (LOB). In the highly competitive and fast-paced financial world nowadays, LOB are employed in no less than a half of the major financial markets worldwide.

The term 'order book' is generally meant to depict the pairs of two-sided prices, namely bid prices and ask prices, and trading volumes corresponding the the prices. Historically and broadly speaking, there had been mainly two types of financial equity markets, order-driven markets and quote-driven markets. The order-driven markets require that all buyers and sellers suggest their prices of their willing to trade in a particular equity as well as the volumes to be bought or sold. The quote-driven markets, on the other hand, only has the large-sized orders to be displayed and bid for execution. To put it another way, the orders are centralized therein.

Transparency through liquidity is, as it were, one of the biggest strengths for order-driven markets. In this type of markets the prices-volumes pairs constitute the order book, which is displayed for investors willing and wishing to access this information. To get such precious information, most exchanges do not provide free meals though. By contrast, a quote-driven market provides smaller liquidity, in that the market makers and financial experts have got to transact business at their disposal with regards to prices and trading volumes.

- Limit-Order Markets

Compared to both market types from above, a *limit-order market* has much more flexibility since the orders are free to be either quoted or executed for each and every market participants, regardless of market makers and investors or their trading volumes or the price they quote.

Definition 2.1.1. *An order* $O_x = (q_x, s_x)$ *with the pair of* q_x *price and* $s_x > 0$ *trading volume (respectively,* $w_x < 0$*) is a commitment to sell (respectively, buy)* $\leq |s_x|$ *units of the equity traded, at a price* $\geq$ *(respectively* $\leq$*)* q_x*.*

Empirically an order will be existing for some time till its either being executed (matched objectively) or canceled (quit subjectively).

Definition 2.1.2. *The Limit Order Book (LOB)* $L(t) = \{O_x\}_x$ *for* $s \leq t$ *i.e. the set of all orders that exist within the trading scenario that is neither matched nor canceled.*

Definition 2.1.3. *The bid price* $b(t)$ *in* $L(t)$ *is the highest (i.e. the best) price among which the buy orders are proposed. The ask price* $a(t)$ *in* $L(t)$ *is the lowest (i.e. the best) price among which the sell orders are proposed.*

Once submitted, the order in the LOB may either get traded immediately or go into a queue of unfilled (still existing) ones, which is referred to as the limit order book. Upon entry the $L(t)$, either may eventually occur to an order: being executed or being canceled.

There would be deep relationship/copula between $a(t)$ and $b(t)$. As seen, $b(t)$ is crucial in limit order placement, because it decides the marginal (boundary) condition for which the whole order book shape will be accounted. at or below $b(t)$ will at least partially match immediately. A similar role is played by $a(t)$ for buy orders. There is a strong coupling between $b(t)$ and $a(t)$. A simple example even not necessarily including LOB scenario is that when an order (e.g. a buy order) comes into a stock market, the sell price and the buy price would both move, and both interactive. With LOB on stage, the case would be more complicated.

Definition 2.1.4. *The bid-ask spread at time t is $s(t)$ such that: $s(t) = a(t) - b(t)$.*

The spread is the absolute value of the bid and ask prices, yet the signal is non-negative.

Yet it is worthy to notice that the bid-ask spread $s(t)$ is not positive necessarily or even with positive probability to be zero, like we will see in the market models in Chapter 5. Such situations lead the orders to either be filled or be canceled.

Definition 2.1.5. *The mid price at time t is the arithmetic mean (average) of the ask price and the bid price at time t: $m(t) = \frac{a(t)+b(t)}{2}$.*

In the prevailing models of limit order markets, mid-prices often plays the role of true price (instantaneously), yet as we will see in the following sections of the thesis, that could lead to inaccuracy in comparison with other modeling of the price dynamics. Also refer to existing works like [34].

- Market Orders and Cancelation Orders

As mentioned, the order book consists of a list of all buy and sell limit orders, with corresponding prices and trading volumes. Essentially, other two types of orders could be submitted as well:

△ market orders: to immediately buy or sell a certain number of shares at the best available opposite quote;

△ cancelation orders: to cancel an existing limit order.

Practically, market orders can be executed at once, yet at the probably undesirable prices; limit orders reflect the market intent to trade at the satisfactory price but facing the delay or failure of being executed. There is a trade-off between the two types of orders. That is, if the market participants place a bid order at a price which prevails the ask price, then this bid order is immediately executed (/filled) by the second-best limit order at the ask price. To summarize, bid and ask limit orders are centralized in a LOB capable to the market participants while market orders are executed against the best available orders in the limit order book. See for reference [19].

- Order Flows and Arrival Rates

Both quantitative and financial research of the limit order markets yield the significant concepts of order flows and arrival rates. Orders come from both sides as time goes by, forming the chronical flow complied with the size and depth of the consecutive orders (whether modeled continuously or discretely).

Meanwhile the order arrival is a stuff dependent on the instantaneous market popularity or liquidity and as well the stochastic processes of limit/market/cancelation orders (dynamics of prices and trading volumes) preceded from just now. Thus the conception of the arrival rate is introduced which is in some cases modeled as a Poisson Process with the rate λ. However, as we will discuss in Chapter 5, no evidence shows that the empirical (limit/market order) arrival rates actually happen to be so.

Mathematically speaking, LOB has much to be explored, e.g. the dynamic and equilibrium of the order book, the order flow composition and the trading cost, the instability of limit order dynamics creating chaos (butterfly effect), the market resilience of the one-sided LOB or the optimal control problem in a limit order market. The above studies come more or less from the idealization of the market model into mathematically solvable cases, e.g. via Hamilton-Jacobi-Bellman Equation, or via the proposition of convex functions (their sub-derivatives, etc.) Yet to emphasize is the mathematical modeling of the market impact of HFT on it, structurally different from human (trading) behavior. For instance, the traditional interpretation of limit order traders as 'patient providers of liquidity' (in [12], for example) has to be rediscovered if limit-order lasts for a mere millisecond due to the impact of HFT.

2.2 Ultra-High Frequency Trading

As a subset of HFT, Ultra High-Frequency-Trading has a steady growth as of its research. Mentioned before in the thesis, HFT by its market behavior would endure the issue of being mis-priced where an security traded goes away from its true meaningful market value which is especially magnified among the cases when time-scale vanishes. Therefore one would like to try to predict this dislocated misleading coming from the drifts for the mean trading prices. On the other hand, most of the high frequency traders 'operate on the range of the milliseconds and mainly try to predict movements in prices on liquid stocks' ([18]) to merely do with the price issue and be returned with rebates from the corresponding exchanges.

Based on these facts, researchers assume (ideally) that market participants in high frequency markets would not be so keen on the meaningful price discovery yet they would firmly grasp the lucre via liquidity affording (just as we took the example of the ballpark scalpers). On such basis, a question rises that what if the indications from markets urge them forward to trade-in at a time-horizon below 10^{-6} seconds, all the way till the time they execute their trade? (See also for reference [18].)

The shortcoming of Ultra-HFT comes several ways. For example we would have the flooding arbitrage opportunities when the failure to price an equity fairly and properly due to the tiny time-scale issue. For deeper reasons, the allocations to the strategy would shrink, even when the dislocations grow ([18]). The arbitrage opportunities are further led by the neglects of high-frequency traders but not low-frequency traders, with the latter eyes the lament chances to make a accumulatively high profits. In the setting, low/high-frequency market participants would form counterparts with zero-sum gains. In summary, ultra-HFT is more of a choice for frequency-scale rather than a particular concept of what the scale is.

2.3 Argues about High Frequency Trading

As a mathematical finance thesis, we could not weigh too much on the morality or regularity issues about the market trading business. Yet one has to care about the trillion-dollar liquidity in financial markets. What if the luxurious lucre enters a dirty pocket by exploiting tons of dollars from market participants? Or even if like the saying goes, market is a zero-sum arena, could there be regulatory concerns about HFT?

A well-known case in point is the flash-crash on May, 6th, 2010. Between 1:30pm and 2:00pm the Dow Jones Industrial Average (DJIA), S&P 500 Index, the June 2010 E-mini S&P 500 futures contract, and other derivatives like options, ETFs, etc. experienced a sudden nose-dive followed by a rapid rebound, with both more than five percent. Admittedly this tragedy (for some market players good news) happened as a consequence of global financial scenario with respect to the European debt crisis. Yet high-frequent market played an indispensable big role. Research analysis [2] concluded that high-frequency scenario brings about opportunity for financial equities aggressively traded *parallel to* the direction of price changes, steering further toward huge trading volume. To culminate, high-frequency traders were reluctant to the minus gain the dollars by the cumulative positions conducted by themselves.

Other issues remain, say, there is no real economic value created by HFT. Plus, certain trading strategies are a form of market manipulation to harm long-term investors, putting sand in the wheels of potential price (or true value) discovery.

CHAPTER 3

LIMIT ORDER BOOK RESEARCH—MATHEMATICAL METHODS AND MODELS

As a state-of-the-art topic in Mathematical Finance, LOB receives a bunch of mathematical attention herein, though they are more or less scattered as we will see. On the other hand, it is these distinguished viewpoints that have made LOB a heated research theme, blending well between market behavior and mathematical derivation.

Basically there are queueing problems and patience of investors, problems regarding optimal trading strategies in high frequency markets, models for characterizing risks and market volatility, models concerning equilibrium in multi-agent markets, among others. The classification is certainly flexible. Yet this kind would be more complete and reviewable in the study of limit order books.

As a thesis on this field, here we address various models about high frequency markets from existing literature. The review is, though, not merely done with classifying, citing and narrating other authors' work. Alternatively, here in this Chapter we put forward distinctively very simple but featured (model) set-ups for the four sections respectively. This will be done at the beginning of each sections so that a brief overlook of each modeling scenario is shown, which constitutes the author's own understanding of each type of models.

In this chapter we have before each section an 'overlook' of each type of models and methods. This contains common concern within them, for example, the concept of tick-data for queueing problems in limit order books. The overlook also contains a little appreciation of the author of his own.

3.1 Queueing Problems and Patience of the Investors

Overlook of queueing problems

►It is necessary to interpret first and foremost 'what is a queue'. As is mentioned, market orders are executed immediately against the best quote but pay a half spread while limit orders are stored in the order book and executed only when a market order crosses the spread. This forms the price dynamics together with the trading volumes at each price being executed. Assume that (q_t^a, q_t^b) is the pair of trading volume for ask and bid side at a given limit order book at time $t \geq 0$, with the corresponding bid and ask price being s_t^a and s_t^b. Thus the queues of $(s_t)_{t\geq 0}$ and $(q_t)_{t\geq 0}$ are formed with respect to both sides. This queue is tightly relevant with the order flow, thus the arrival rate is modeled, often as Poisson process with parameter λ, μ and θ respectively for limit, market and cancelation orders. The price could move up order down, according to order-arrival frequency and the type of them, also to the trading volumes.

Here what researchers concern is the tick-by-tick trading scenario. A/One tick is the minimum amount by which the price changes. 'Tick-to-tick' means the trading data has a time flow with respect to each pulse of price change, with the minimum possible change being the amount of a tick. The tick-data for trading is the empirical data to be calibrated and tested for the theoretical models regarding queueing problems. As a matter of fact these models are built on the foundation of the tick-data where the order flow boasts approximately the frequency of different sort of orders and the price moves according to the tick-data trading volumes. One important thing is the fundamental difference between tick-data and data with respect to discrete time points. The feature distinction lies in the uniformity of time sampled. By its construction one tick could occur at a time scale of between 10^{-3} to 10^{2} seconds (see [5]). There the plot of price-time graph has an non-uniform scale in x-axis, which could be a reason for functional central limit theorem to be used in some theoretical results in the queue models.

Mathematics for these models include stochastic processes, especially random walks for modeling the price behavior and Brownian Motioned for modeling the diffusion process of price dynamics. Also involved are the partial differential equations and central limit theorem, as used in the rationale for some main results' proofs.◀

Starting from a description of order arrivals and cancelations as point processes, the dynamics of a LOB is naturally described in the literature of queueing theory. In [22], the authors formulate a bivariate point process to jointly analyze trade and quote arrivals. Since the process has good properties and easy to be dealt with statistically, empirical literature like tick-by-tick data are easier to be plugged into the model, facilitating calibration and numerical simulation. In their paper, a new approach is introduced, to analyzing transaction price movements in financial markets. It relies on an approach that has been extended to include negative counting. The thrift form of the model constitutes two processes: the one for the price movement and another for its size. This approach is particularly suited for financial markets by simulations, with fairly low transaction intensities. It uses the method with integer counts that partitions the overall process of delta transaction prices into three separate but correlated parts.

We are given transaction prices $P(t_i), i = 1, ...n$, and the corresponding price change process $Y_i = P(t_i) - P(t_{i-1})$. The information flow $\{\mathcal{F}_i\}$ is available at the time when the transaction i takes place. Then the conditional probability could be got. In particular, $P(Y_i|\mathcal{F}_i)$ could be calculated. This leads to the form to be determined

$$\pi_{ij} = P(D_j = j|\mathcal{F}_{i-1}),$$

with D_j the direction of the transaction price change. In order to get π_{ij}, a multinomial logistic model is used. Meanwhile, the size of price changes can therefore be formulated. Let U be a distribution with the probability density function (called Negbin distribution)

$$f(u) = \frac{\Gamma(\kappa + u)}{\Gamma(\kappa)\Gamma(u+1)} \left(\frac{\kappa}{\kappa + \omega}\right)^{\kappa} \left(\frac{\omega}{\kappa + \omega}\right)^{u},$$

where the overdispersion of the distribution depends on the parameter $\kappa > 0$ and $\omega = \mathbb{E}[U]$. Such a distribution has an advantage of converging to a Poisson as $\kappa \to \infty$. Herein, the absolute price change $S_i = |Y_i|$ would have its first and second moment derived with respect to the above distribution.

In [10], the model is proposed in regards of the order-book dynamics as a complicated formed but theoretically doable queueing system. The computational instruments of Lapalace transform and its inverse is involved. There, the properties of hidden or 'iceberg' orders and their relationship with market order-book are studied, concerning the market effect on the best quotes. As I see it, there are two appreciations blending the mathematical modeling well with market scenario. First is the threshold between falling behind and exceeding the execution target size (underfill or over-fill the order book), i.e. the marginal penalty parameters. These are motivated by the correlation between (limit) order execution and price movements, essentially the market behavior of queueing systems. Second is the optimal dependent of two types of orders on the ratio that offsets marginal trading fees, and on the price distribution and the queue length. This kind of split quantitatively depicts the comparison between market and limit orders. The optimal strategy is analyzed, yet the paper shoot the star in regard of its derivation of order placement concerning queueing system.

In particular, the trader, in the background of K exchanges, is to acquire queue priority via his order placement decision to buy S shares of stock in the time horizon $[0, T]$, summarized by $X := (M, L_1, ...L_k) \in \mathbb{R}^{k+1}$ where $L_k(k = 1, ..., K)$ is the limit order size submitted, and M is the market order size accordingly. Market orders are supposed to be surely filled, while limit orders are lining in queues of $(Q_1, ...Q_K)$ of pre-existing limit orders, which is reasonable for empirical congruence. $C_k \in [0, Q_k]$ is the number of cancelation (pre-existing) orders and D_k counter-side marketable orders reaching the queue. Denote r_k to be the rebates of all exchanges. Let $\xi = (\xi_1, ...\xi_K)$ be the overall bid queue outflow. So far, the total amount traded by time T can be modeled as

$$A(X, \xi) = M + \sum_{k=1}^{K} (\xi_k - Q_k)_+ - (\xi_k - Q_k - L_k)_+.$$

Here λ_u and λ_o are modeled as the threshold between falling behind and exceeding the execution target size (under-fill or over-fill the order book), i.e. the marginal penalty parameters. The above λ_u and λ_o are mathematically expressed through comparison between $A(X, \xi)$ and S. For example, $\lambda_u > s + f$, where s is a half of the bid-ask spread at time 0 and f is the lowest available liquidity fee. Queueing system is also modeled as a background in which the (market/limit/cancelation) orders lie. Using the variables and $A(X, \xi)$ in the model, the paper builds up an optimal order placement problem the maximize an $E[v(X, \xi)]$, under the assumptions for parameters. The optimal allocations of market and limit orders,for single exchange (i.e. $K = 1$), is established, assuming the availability of the cumulative distribution function of ξ. When it comes to multiple trading venues, the main result of this paper shows that optimization of order allocation is equivalent to the attainment of threshold for pricing parameters. Specifically, an optimal allocation X^* satisfies $M^* > 0$ if

$$\lambda_u \geq \frac{2s + f + \max_k\{r_k\}}{\mathbb{P}\left(\cap_k(\xi_k \leq Q_k)\right)} - (s + \max_k\{r_k\});$$

satisfies $L_j^* > 0$ if

$$\mathbb{P}\left(\bigcap_{k \neq j}\{\xi_k \leq Q_k\} | \xi_j > Q_j\right) > \frac{\lambda_o - (s + r_j)}{\lambda_u + \lambda_o}.$$

Under the above two conditions, X^* solves a group of probability equations

$$\mathbb{P}\left(M^* + \sum_{k=1}^{K}[(\xi_k - Q_k)_+ - (\xi_k - Q_k - L_k^*)_+] < S\right) = \frac{s + f + \lambda_o}{\lambda_u + \lambda_o},$$

$$\mathbb{P}\left(M^* + \sum_{k=1}^{K}[(\xi_k - Q_k)_+ - (\xi_k - Q_k - L_k^*)_+] < S|\xi_j < Q_j + L_j^*\right) = \frac{\lambda_o - (s + r_j)}{\lambda_u + \lambda_o}.$$

The two equations show that an order allocation is optimal as long as it sets the probabilities of time-lagging the target quantity are identical to certain thresholds calculated via pricing parameters.

In [33] the authors study the high-frequency dynamics of the price in the limit order market, in which the arrivals of (market/limit/cancelation) orders are depicted as a Markovian queueing system. The paper presents a system of two copula queues standing at the buy and sell sides in the order book. Through strongly analytical tractability, it derives the distribution of price changes and their time durations, i.e. the dynamic of the Markovian queueing system. For example, the conditional distribution of the duration between price moves is given; the probability of a price up-moving is derived on the condition of the tick-state. Also studied is the relation between price volatility and order-flow by focusing on the diffusion limit of the price process. The Markoviantiy allows formula of transition probability for dynamics to be derived, with limit and stationary states, as well as results via functional central limit theory. Boasting news points of view from the mathematical finance, the paper treats the queue sizes as stationary random variables drawn from a probability measure on $\mathbb{N}^2$, which summarizes both sides (bid and ask) of the queue with the rest of the limit order book. The advantage of doing so is for the pair of order queue to be Markovian, good to be tractable, e.g. construction of central limit theory. The technical usage of mathematics is delicate, e.g. theorems or propositions quote results from Markov Chains and from standard proceeds in probability theory and partial differential equations, e.g. Central Limit Theorem, bivariate (symmetric) random walk, Dirichlet Problem in partial differential equation, et al. Moreover, the market model is intriguing, as mentioned. Because the paper deals mainly with the ideal literature of order flow and the queueing system dynamics, for empirical concern (likely) non-trivial case of, e.g. asymmetry condition on the joint distribution of queue sizes is concerned. Consequently under this condition and theories in auto-regression the up-moving price change is more probable to be followed by a down-moving one, which could be somehow interpreted as a market resilience as will be put forward in the next section of the literature review. Finally, the paper bears market data structure well into the theoretical concerns within it, especially for the conception therein. For instance, to fit in the relevant cases of many liquid stocks, it considers the case of a *balanced order book* where the intensity of market and cancelation orders is equal to that of limit orders. To estimate the price volatility, it could not be bothered with observing the price itself. Also, an intuitive interpretation is conveyed via the diffusion limit of the price volatility (tick-size$\rightarrow \infty$) that the higher queue size of the second-best queue is, the lower price volatility is.

Especially for mathematical interest is the use of functional central limit theories, which can be summarized as a generalization of the classic central limit theorem. Assuming the exponential arrival rate of market order μ, limit order λ, cancelation order θ. The distribution under some special conditions could be given for V_i^a, V_i^b (the ask/bid queue size in change) and T_i^a, T_i^b (the random time pertaining to these changes). Considering price dynamics, let $q_t = (q_t^a, q_t^b)$ be the pair of order queue, and s_t be the price process, which is in a sense vogue in mathematical and financial interpretation. Now one gets a continuous-time process $X_t = (s_t^b, q_t^b, q_t^a)$ for analysis. On these setup, one can define the first time when bid/ask queue is depleted:

$$\sigma_a = \inf\{T_i^a, q^a_{T_i^a -} + V_i^a = 0\}$$

and σ_b similarly. Then the inter-moving duration is

$$\tau = \sigma_a \wedge \sigma_b.$$

Thus comes the distribution of τ

$$\mathbb{P}[\tau > t | q_0^a = a.q_0^b = b] = \sqrt{(\frac{\mu+\theta}{\lambda})^(a+b)}\psi_{a,\lambda,\theta+\mu}(t)\psi_{a,\lambda,\theta+\mu}(t)$$

where

$$\psi_{n,\lambda,\theta+\mu}(t) = \int_t^\infty \frac{n}{u} I_n(2\sqrt{\lambda(\theta+\mu)}u)e^{-u(\lambda+\theta+\mu)}du.$$

One of the highlights in this paper lies in the diffusion limit of price processes. Let $Z_n = X_1 + ... + X_n$, and $s_t = Z_{n_t}$. Here, when dealing with the diffusion behavior of s_n, the high frequency dynamics of the price is described by a piecewise constant stochastic process, we applied the specialized Central Limit Theorem (CLT) with respect to $\frac{s_{n \log nt}}{\sqrt{n}}$. The subscription is chosen with delicacy, driven by the choice of $t_n = t\zeta(n)$, i.e. the adjustment of time-scale over which the average number of order book events is of order n. It serves as a good application of generalized functional CLT into mathematical finance fields.

The patience of the investors in the limit-order market reflects popularity of market and the participants' expectation of the price dynamics. In the article [23] the author creates a model of price formulation in a LOB market, assuming finite time-horizon, continuity of time flow and no dividend paid. Bidders among askers purchase one asset before quitting the arena. Two types of traders are classified: patient and impatient, who arrive randomly to the market, choosing alternatively with respect to different order types. It models a random execution time and the corresponding expected utility function, and the Poisson arrival for these types of traders.

In particular, Let k be the maximum number of units that a market order can have with positive probability; let M be maximum number of limit orders in the order book. The explicit form of f_m, with m being the number of sellers, f being the utility function, is under the condition of the arrival rates and the impatience index r. Given the configuration of limit orders, price impact function are defined by

$$imp(i) = a_{i+1} - a_1,$$

where a_{i+1} is level of the i'th offer above the ask. The main result with respect to shape of order book is for $a_i(m)$the level of i'th limit order in a stationary Markovian state $m < M$. $a_i(m)$ has the expression as a weighed average of $\{f_{m-j}\}$, $j = i, ..., k$ i.e.

$$a_i(m) = \frac{\lambda_k f_{m-k} + \lambda_{k-1} f_{m-k+1} + ... + \lambda_i f_{m-i}}{\lambda_k + \lambda_{k-1} + ... + \lambda_i}$$

and the weights λ_i are the arrival rates of the i'th-unit impatient buyers. Further, important relation holds as an inequality about the λ

$$\lambda > \sum_{i=1}^{k} i\lambda_i.$$

3.2 Optimal Trading Strategies

Overlook of optimal trading problems

►In the theory of mathematical finance an investor maximizes her utility function by applying the (optimal) trading strategy on her portfolio. Likewise in the literature of limit order books, market makers are faced with market risks, inventory risks as well as execution risks while also having the objective to maximize the expected utility of the terminal profits. It is as a matter of fact a complex problem how to act with the limit order books, which is a dynamical-shaped business, and choose among the time and size to trade— between the steering of bid and ask.

It is a common sense that a big buy (market) order would infect the market by pushing upward or downward the market price. In the high frequency market this holds true. In such context, there exists a class of strategies that consists in simultaneously posting limit orders to buy and sell during the continuous trading session, and culminating to the best result of the relevant individual parties. Let p_t^b be the best bid price at time $t \geq 0$. For instance, a market participant, say, an investor would pose bid orders more expensive than the momentarily best bid, $p_t^b + \Delta$, for $\Delta \geq 0$, so that she would own goodness in executing the order but has the trade-off for humble lucre. She now would love to maximize her expected utility function— that is the setup of the optimal problems. In other (optimal) problems, investors has the choice of whether market or limit orders to pose; or how a market resilience, i.e. the opposite direction of a price change would happen when a big market order arrives. An important factor in optimal problems is the constraint for the expected utility function. The obvious one is the total amount of money at $t_0 = 0$ or the finite time-horizon $T \in (0, +\infty)$. ◄

As mentioned before, optimal strategy are analyzed against risks. Typically, risks in the high-frequency literatures include inventory risk as well as execution risk and adverse-selection risk ([11]). In [11] the authors modeled the trading strategies in limit order markets, mathematically speaking to solve a control problem characterized with quasi-variational system by dynamic programming methods. Posed in detail, the paper deals with the simultaneously posting of limit bid and ask orders both continuously. The paper deals with the microstructure of LOB, in particular, price/time priority. The authors, in my opinion, treats market stuffs with highly mathematical delicacy. For empirical concern, limit orders are modeled as continuous control yet the market orders discrete.

In a benchmark market-making model, a probability space $(\omega, \mathcal{F}, \mathbb{P})$ is fixed. The mid-price is a Markovian process P. δ is the tick size. A continuous, time-inhomogeneous Markov chain S_t is built up corresponding to P_t, with intensity matrix $(r_i j(t))_{1\leq i,j\leq m}$ where $\{1, 2, ..., m\}$ is its state space. $Q_t = (Q_t^a, Q_t^b)$ represent the possible choices of bid/ask quotes valued in a proper space. Denote Ba to be the best-ask quote; Ba_- to be the ask quote at best price minus the tick. Similarly are Bb and Ba_+. $\pi^b(Q_t^b, P_t, S_t)$ and $\pi^b(Q_t^b, P_t, S_t)$ are bid and ask of market makers. Consequently, we have

$$\pi^b(q^b, p, s) = (p - \frac{s}{2} + \delta 1_{q^b=Bb_+})(1-\rho), \pi^b(q^a, p, s) = (p + \frac{s}{2} - \delta 1_{q^a=Ba_-})(1+\rho),$$

with some $\rho \in (0, 1)$. Further let $L = (L^a, L^b)$ be the pair of limit ask-bid strategies, then for the market-maker strategies $\alpha = (Q^b, Q^a, L^b, L^a)$, the cash holdings X and the number of shares Y held by the agent follow the dynamics

$$dY_t = L_t^b dN_t^b - L_t^a dN_t^a,$$

$$dX_t = -\pi^b(Q_t^b, P_{t_-}, S_{t_-}) L_t^b dN_t^b + \pi^a(Q_t^a, P_{t_-}, S_{t_-}) L_t^a dN_t^a.$$

As a standard objective of the market maker, he wants to maximize the expectation of his profits at the terminal, i.e.

$$\max \mathbb{E}[U(X_T) - \gamma \int_0^T g(Y_t)dt],$$

where U is an increasing utility function, g is a non-negative convex function on the strategy and γ is a non-negative constant. Here, the minus sign in the equation indicates the penalizing character of the term $\int_0^T g(Y_t)dt$, since it is the variation of the invention.

In [27] the authors studied a small investor's optimal portfolio problem in the LOB background, with the optimal strategy derived via justifying the existence of the a post-derived price process (whose corresponding process concerns without transaction costs) of the risky asset, sitting by a imaginary market that can be represented in the original market with the same gain process of value process. The paper introduces a new model to analyze the trade-off between time-delay of investments and good prices for the corresponding investors. The elegance of the shadow one lies in the mathematical precision of treatment with the distinctive order types. The main result deals with the construction of the shadow price through the optimization of logarithmic utility, which is better-understood. Deriving the boundary problem of the risky asset price process, the paper has the main result of the relation between shadow price and optimal strategy in the limit-order market. Questions remain for example when the limit orders are huge enough to be let in the incoming orders from other participants: whether the boundary problem come across here still makes sense.

In the Merton problem for a limit order book, let $\underline{S}$ be the best bid-price and $\bar{S}$ be the best ask-price, where $\underline{S} \leq \bar{S}$. Limit buy and sell orders are executed at jump times of N^1 and N^2 respectively. Let M^B, M^S, L^B, L^S be predictable processes, with M^B, M^S non-decreasing the zero at starting time; L^B, L^S non-negative, and c an optional process. The quintuple (M^B, M^S, L^B, L^S, c) is called a *strategy*. To this end, we can define in a standardized way in math finance, the portfolio process (φ^0, φ^1)

to stand for the number of risk-free and risky assets respectively. Also standard definition is the admissible set for (φ^0, φ^1), $\mathcal{A}(\varphi^0, \varphi^1)$. Let δ be the time preference. The value function in this problem is modeled as

$$V(\varphi^0, \varphi^1) = \sup_{(\varphi^0,\varphi^1)\in\mathcal{A}(\varphi^0,\varphi^1)} E\left(\int_0^\infty e^{-\delta t}\log(c_t)dt\right).$$

As mentioned, one of the crucial concepts in the paper is a *shadow price process*, defined as a real-valued semimartingale $\tilde{S}_t$ satisfying $\forall t > 0$

$$\underline{S}_t \le \tilde{S}_t \le \bar{S}_t,$$

and that

$$\tilde{S}_t = \begin{cases} \underline{S}_t, & \text{if} \Delta N_t^1 = 1; \\ \bar{S}_t, & \text{if} \Delta N_t^2 = 1. \end{cases} \tag{3.1}$$

In the paper [31], the optimization submission strategies are discussed, especially of the strategies that maximizes the expected utility when dealing with bid and ask quotes in the LOB. The mid-price is set. A two-step strategy, called inventory strategy is presented, and compared with the best bid/best ask strategy and symmetric strategy (benchmark cases).

Stock mid-price modeled as a Brownian Motion, bringing about the measure for the risk of inventory. To maximize the terminal-time expected (exponential) utility of gain or loss profile, the specially-modeled bid (ask) price is introduced, making the agent neutral to determine the present portfolio and its marginal one, with the mean called the ‘indifference price’. Then the model extends it into infinite horizon. Further, the author derives the relevant bid and ask prices and computes the market impact with large-sized orders. The model of stochastic wealth and inventory come via the standard-modeled Poisson process for both orders. Aggregating these setups, the model attains the Poisson intensity for the order execution. When dealing with optimal bid and ask quotes, the authors use Hamilton-Jacobi-Bellman (HJB)

Equation to solve it, and further find the reservation bid and ask prices defined previously.

Assume the mid-price S_t solves

$$dS_t = \sigma dW_t, \quad S_0 = s,$$

where W_t is a standard Brownian Motion and σ is a constant related. The agents objective is to maximize her expected utility at the terminal T. In an ideal setting, an inactive trader not holding any limit orders holds q inventory stocks until T. The value function is

$$v(x, s, q, t) = \mathbb{E}_t[-\exp(-\gamma(x + qS_T))],$$

where x is her initial wealth and γ the risk-neutral rate. These lead to the reservation price

$$r(s, q, t) = s - q\gamma\sigma_2(T - t).$$

This price serves as an adjustment to the mid-price. For example, if the agent is short stock, i.e. $q < 0$, the price is above mid-price since he is to buy higher. For the agents trading through limit orders, let X_t be the wealth in cash

$$dX_t = p_t^a dN_t^a - p_t^b dN_t^b,$$

with $p_t^a, N_t^a, p_t^b, N_t^b$ defined exactly the same as in [11]. Let

$$\delta^b = s - p^b, \quad \delta^a = s - p^a.$$

The objective function will be

$$u(s, x, q, t) = \max_{\delta^a, \delta^b} \mathbb{E}_t[-\exp(-\gamma(x + qS_T))].$$

Further, let λ_a be the Poisson rate at which market orders hit the agent limit sell order. Similarly defined is λ_b. The solution of u follows the Hamilton-Jacobi-Bellman function

$$\begin{cases} u_t + \frac{1}{2}\sigma^2 u_s s + \max_{\delta^b} \lambda^b \delta^b [u(s, x - s + \delta^b, q + 1, t) - u(s, x, q, t)] \\ + \max_{\delta^a} \lambda^a \delta^a [u(s, x - s + \delta^a, q + 1, t) - u(s, x, q, t)] = 0, \\ u(x, s, q, T) = -\exp[-\gamma(x + qs)]. \end{cases} \tag{3.2}$$

Also concerning a large asset purchase, but in the face of a one-sided LOB, in [25] the authors put emphasis of the impact of market resilience where the order book will be shaped back to its original pattern before the tick happens. The investor aims to minimize his expected cost of purchasing (over a given horizon) a target amount of asset. The delicacy of the article is to employ stuffs from measure theory to derive the price after any lump purchase, and further, the total cost incurred by the investor using the corresponding strategy. The constructive method toward the solution of the optimization problem is to transform the unknown expression of the total cost into a convex minimization problem and to solve it in a doable (in fact, mathematically simple) way. The heuristics of the article lie in introducing a market resiliency function (modeled beforehand) with very good analytical properties (i.e. Lipschitz, monotone, etc.); moreover the ask price in the presence of big investment is modeled by solving the volume effect process determined by an Stochastic Differential Equation. Finally, the simplification is attained by turning the cost function into a convex one.

Let $T > 0$ be this given time-horizon. A_t is the ask-price of an asset ('one-sided' assumed to be ask-side, w.l.o.g.) without large investors, which is a continuous L_1 semimartingale. Let μ be a measure on $[0, M)$ with $M > 0$. It takes finite value on compact subset of $[0, M)$. Define a left-continuous cumulative distribution function

$$F(x) = \mu([0, x)).$$

For a measurable subset $B \subset [0, M)$, $\mu(B)$ is denoted as the number of limit orders with price

$$B + A_t = \{b + A_t, b \in B\}.$$

Let the positive constant $\bar{X}$ be the mandatory amount of shares the large investor purchases, and X_t be the cumulative amount purchased until time t. Naturally, $X_T = \bar{X}$. Denote

$$\Delta X_t = X_t - X_{t_-}.$$

Let the resilience function be a strictly increasing and local Lipschitz function on $[0, \infty)$

$$h(0) = 0, \quad h(\infty) = \lim_{x \to \infty} h(x) > \frac{\bar{X}}{T}.$$

The ask price with large investors is defined to be $A_t + D_t$, where D_t satisfies

$$D_t = \psi(E_t) := \sup x \geq 0 : F(x) < E_t.$$

Financially speaking, D_t is the price after any lump purchase by the investor at (or 'right after' because of right-continuity of D_t) time t. Decompose X_t into its continuous and pure-jump parts $X_t = X_t^c + \sum_{0 \leq s \leq t} \Delta X_s$, and let

$$\phi(x) = \int_{[0,x)} \xi d\xi, \quad x \geq 0,$$

$$\Phi(y) = \phi(\psi(y)) + [y - F(\psi(y))]\psi(y), \quad y \geq 0.$$

The cost function of the investor using strategy X over the time interval $[0, T]$ is formulated by

$$C(X) = \int_0^T (A_t + D_t) dX_t^c + \sum_{0 \leq t \leq T} [A_t \Delta X_t + \Phi(E_t) - \Phi(E_{t_-})].$$

Without loss of generality (w.l.o.g.) the cost function is technically simplified as

$$C(X) = \int_0^T D_t dX_t^c + \sum_{0 \leq t \leq T} [\Phi(E_t) - \Phi(E_{t_-})].$$

By real analysis, stochastic analysis and properties in convex functions, under the assumptions w.l.o.g., $C(X)$ has the form

$$C(X) = \Phi(E_T) + \int_0^T D_t h(E_t) dt.$$

Furthermore, under two distinct conditions with respect to empirical strategies, the paper gives the analytical form of the optimal solution by the delicate properties in convex analysis.

In a analogous literature to the above paper with respect to market resilience function, in [26] the authors studied the market microstructure on optimization with stochastic algorithm. Its main purpose as well is modeling a cost function and minimize it. Yet the procedure in this model is simpler and plainer than in the above paper. But the feature is to construct a execution process using non-homogeneous Poisson process with the intensity proportional to the relative (bid) price. This process form both the mean execution cost and penalization function with respect to the lagging behind target quantity (number of assets). The paper gives the analytical properties about the cost function, e.g. its (order-1 and order-2, the latter of which implies convexity) differentiability, chain rules. Tools for these stuffs involve Poisson calculus and so forth. The main result, i.e. solution of optimizing the cost function, lies in the application of Central Limit Theorem and using Euler Scheme in stochastic approximation. The argmin of the function is constructed as a limitation. One of the remarkable feature of this work is the feasibility of the solution, which is in fact implemented in some sense at the conclusion part of the paper. Since the construction of the model is mathematically less involving than the previous paper (though with the same concern of minimizing cost function and with similar construction of penalizing function), the derivation of argmin variable is less costly but more amiable in computation.

The dynamics of the fair price of a security is a given $(S_t)_{t\geq 0}$, non-negative continuous stochastic process. It could be for example the fair/intrinsic price. Talking about the design of algorithm, the paper proposed an execution process of buy orders to be a non-homogeneous Poisson process

$$\left(N_t^{(\delta)} = \tilde{N}_{\int_0^t \lambda(S_s-(S_0-\delta))ds}\right)_{0\le t\le T}$$

with varying intensity

$$\lambda(S_s - (S_0 - \delta)).$$

Here $0 \le \delta \le \delta_{\max}$ ($\delta_{\max} \in (0, S_0)$) denotes the depth of the LOB; the function λ is defined on $\mathbb{R}$, finite and non-increasing. Over $[t, t + \Delta t]$ the probability of a single buy order to be executed is

$$\lambda(S_s - (S_0 - \delta))\Delta t.$$

Let $Q_t \in \mathbb{N}$ be the size of portfolio invested in asset S. A market penalization function (i.e. the very similar *market resilience function* $h(\cdot)$ in the last paper) Φ is a non-decreasing and convex function ranged on real positive, with $\Phi(0) = 0$. The function of execution cost, given the above setup, could be built up as

$$C(\delta) = \mathbb{E}\left[(S_0 - \delta)\left(Q_T \wedge N_t^{(\delta)}\right) + \kappa S_T \Phi\left(\left(Q_T - N_t^{(\delta)}\right)_+\right)\right],$$

where κ is a (free) parameter representative of the non-execution aversion. $\kappa = 1$ implies the true cost function. The main goal is to solve

$$\min_{0\le\delta\le\delta_{\max}} C(\delta).$$

The paper lays more emphasis than the last paper, on the stochastic algorithm of its main problem. Particularly, it proposes the first framework of simulated date, i.e. assuming $(S_t)_{0\le t\le T}$ to be diffusion process

$$dS_t = b(S_t)dt + \sigma(S_t)dW_t, \quad S_0 = s_0 > 0.$$

Then S is replaced by its Euler Scheme. In the second framework of true market data, S is assumed as a stationary ergodic process and that a dataset is accessible at frequency T/m. After that, the paper puts forward stochastic approximation and convex optimization procedures to facilitate the plug-in of items on the main minimization problem. This follows the main result that depicts the attainment of the optimal δ^* in the two respective frameworks.

For the last parts of the paper, besides numerical experiments, the authors discuss the monotonicity, differentiability and other propositions of the function $C(\cdot)$. At the mean time, the monotonicity and convexity for $C(\cdot)$ are gotten, as long as some estimations hold for Q_T and κ.

3.3 Equilibrium in Multi-Agent Markets

Overlook of Equilibrium Problems in Limit Order Books

►In financial markets, market participants are playing the game to maximize their respective profit. Yet this is weird if the corresponding counterpart of a buyer and a seller, in the identical market with the same share of stock at the same trading period, get both the maximization. Put it more precisely, their respective utility functions are different so that an equilibrium would be attained. Questions emerge, for instance how to define this equilibrium and whether it exists; what would be the mathematical relation between both sides' utility functions and the equilibrium stroke; how to describe mathematically the ask side's competition toward the execution of limit order; does liquidity have something to do with equilibrium in a highly competitive limit-order market. Undoubtedly, order book dynamics in real market are built up by different individuals (kinds of investor or market makers). Thus, it is very important to catch up with the pace they make by modeling properly the evolution of liquidity provision, quantified risk-aversion of multi-investors (including but not limit to utility function thing) and equilibrium. Questions at hand for researchers to think over include the definition of different equilibrium (e.g. Nash Equilibrium), their existence and uniqueness, what different scenario for distinctive type of equilibrium, what application for an equilibrium, et al.

Generally speaking, the research of equilibrium in order-driven markets concerns two parts: the optimality for individual counterparts and the construction of the (Nash) equilibria. As it were, it reflects but some research pattern for the previous section, but more involved in game theory mathematically. ◄

The paper [6] deals with an n-player non-cooperative game in the sphere of LOB with sellers competing to fulfill the incoming order. Consider there increases an extra investor in a given scenario of a given number of market participants. Randomness comes as the amount X of asset the extra investor (say, buyer) asks for, which links with the best available price and the size of the offer. The rationale of herein is clear that right from the beginning, a standard setup of a single (seller) competitor's optimal problem (maximizing expected payoff) is built up. This is followed up by deriving the Nash Equilibrium of n-player, for a general class of X. The paper discusses the existence for Nash equilibrium of cross-class agent players. Finally it describes the asymptotic behavior where $n \to \infty$. The highlight of this paper is Nash equilibrium or even the equilibria itself, whose application is barely seen in the LOB literature. Another feature for the big picture is the involvement of order-book-shape into the existence of equilibrium, very interesting bonding. Nevertheless the paper lacks market data to justify their research, probably because the idealistic assumption of competitiveness. Pragmatically, information provided to each competitor is asymmetric, sometimes largely diversified in terms of transparency, with itself a difficult field.

Consider a non-cooperative game with n players trading a given asset in a one-sided LOB. The i^{th} player owns an amount of κ_i shares. An external agent will buy amount X with the upper bound $\bar{P}$. Denote $\Phi_0(p)$ as the total amount of stock for sale at price $\leq p$. Let β be the variable standing for a particular share in possession of the new agent, $\phi(\beta)$ the price it puts on sale. Assume the incoming order has size X,

$$\beta(X) = \sup\{\beta \in [0, \kappa] : \beta + \Phi_0(\phi(\beta)) \leq X\}.$$

Further let $\phi_i : [0, \kappa_i] \mapsto [0, \bar{P}]$ be the pricing strategy for the i^th player. Define

$$\Phi_I(p) = \sum_{j \neq i} meas\left(\{\beta \in [0, \kappa_j] : \phi_j(\beta) \leq p\}\right).$$

Then the Nash Equilibrium is the optimal solution to the problem

$$\max: \quad J_i(\phi) = \int_0^{\kappa_i} (\phi(\beta) - p_0)\psi\left(\beta + \Phi_i(\phi(\beta))\right) d\beta.$$

The paper shows the sufficient conditions when this Equilibrium exists or not. The former uses backward induction while the latter uses counterproof to deduce the contradiction with optimization.

Another perspective to view the equilibrium is to pool it within the market of specific financial instrument, also various types of order formation, e.g. limit order market, hybrid market, etc. Yet this kind of model setup has special background. For instance, in [9] the authors mainly study the competition-proof of various market types and competition between different exchanges against liquidity and price improvement; at the mean time, derives multiple equilibria supported by distinctive preference rules with respect to these corresponding markets. The study is interesting. On one hand, the framework is macro, pertaining to competition between a pure limit-order market and a hybrid market with both a specialist and the LOB. The authors give a detailed market analysis of liquidity providing and competition in the hybrid market, described as time-line for sequence of events to submit and clear certain type of orders, constructing as a cornerstone for main results thereafter. Then for the counterpart of the exchanges, the active trader chooses the minimize her cost function, constructed as an (preliminary) optimization problem. After explaining the order execution mechanics and market-clearing process, the paper discusses the specialist's profit maximization problem as another part of the n-player setting. One important conception and illustrated in a figure here is the sequence of 'execution thresholds' for which the cleanup price is big enough when market order size triggers it ([9]). Collecting all the counterparties yields a Nash Equilibrium built-up, mainly with respect to value trader's marginal expected profits, financial experts's execution thresholds and market order arrival distribution. Then the equilibrium impact of inter-market competition on both limit order placement and the market order flow is investigated, discussing questions such as market centralization given heterogeneity of trading costs and pre and post occurrence of liquidity. These are of big empirical value. Yet little is seen in this article to extent these business.

Let $S_1^h, S_2^h, \ldots$ denote the total limit sells posted at price, $p_1, p_2, \ldots$ in the hybrid market; similarly are $S_1^p, S_2^p, \ldots$. Denote $Q_j^h = \sum_{i=1}^{j} S_i^h$, $Q_j^p = \sum_{i=1}^{j} S_i^p$. Let B^p be the market buy size at price p. For the Nash Equilibrium, the paper proposes the threshold for order size under which the execution is triggers, by

$$\theta_j^h = \max\{B^h | p_h(B^h) < p_j\}.$$

Moreover, given that an active trader buys x shares, the total liquidity premium $\tau(x)$ is minimized:

$$\tau(x) = \min_{B_h, B_p : B_h + B_p = x} T_h(B_h) + T_p(B_p),$$

where T denotes the according cost schedules. In this equilibrium, also defined is e_j, the expected marginal profit at price p_j. It satisfies

$$S_j^p = 0 \to_j^p \leq 0$$

and

$$S_j^p > 0 \to_j^p = 0.$$

The last two conditions are for value traders who arrive randomly at the initial time, submit limit orders if profitable, and then leave.

The main result is to compute the equilibria of various types relevant. To obtain this goal, let F_h and F_p be the distribution from which the probabilities $P(B^h > \theta_j^h)$ and $P(B^p > \theta_p^h)$ are computed. Let H be the inverse of the distribution F. Define

$$H_j = \begin{cases} H\left(\frac{c_j/\alpha}{p_j - v}\right), & \frac{c_j/\alpha}{p_j - v} \leq 1; \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

where c_j the cost when submitting limit order at price p_j, v a constant connected to the price aggregation $(p_j)_{j\in\mathbb{Z}}$ and α is a constant. Given some assumptions, in the equilibrium the pure LOB satisfies

$$H_j \geq H_{j-1}.$$

For the hybrid market it satisfies

$$\beta_j \leq \theta^h_{j+1}$$

and

$$p^h(\beta_j) = p^j.$$

Based on similar literature according to competition among liquidity providers and equilibrium of order books, there are plenty of stuffs worthy to be sought mathematically and financially, e.g. equilibrium acting with different sizes of market orders; equilibrium types in term of different types of orders.

Consider a limit-order market and a uniform price market. A random information flow occurs at time τ, distributed exponentially with parameter r. Let $\tilde{v} = 1$ or 0 be the value of a risky asset. Consider, n being fixed, orders of size $i = 1, 2, ..., n$. The market order arrives randomly with Poisson process, with intensity β for both buy and sell orders. Denote m_t to be the expected asset value at time t, given information to market order arrivals. Let $a_i(m)$ denote the conditional expectation of the asset given a buy order of size i at time t when $m_{t_-} = m$. Similarly for $b_i(m)$. In a limit order market, for each i, a limit sell order has a price $a^L_{i+}(m_{t_-})$. In a uniform price market, the cost of buy orders of size i will be $ia^U_i(m_{t_-})$. The counting processes for liquidity trades are denoted by Z, for informed trades by X, for total trades by $Y = X + Z$, all of which come with subscripts $+$ and $-$ denoting buy and sell, respectively. Define $X_i = X^+_i - X^-_i$. In the game modeled, equilibrium occurs where the functions $\theta_i^+(m, v)$ and $\theta_i^-(m, v)$ satisfy

$$X_{it}^+ - \int_0^t \theta_i^+(m_{s_-}, \tilde{v})ds$$

and

$$X_{it}^- - \int_0^t \theta_i^-(m_{s_-}, \tilde{v})ds$$

are martingales with respect to the informed trader's information. Meanwhile, the evolution (i.e. jump) of m_t reads

$$dm_t = f(m_{t_-})dt + \sum_{i=1}^{n}[a_i(m_{t_-}) - m_{t_-}]dY_{it}^+ + \sum_{i=1}^{n}[b_i(m_{t_-}) - m_{t_-}]dY_{it}^-$$

due to the proposition of dynamical elements, e.g. f, a_i, b_i...

In the limit order market, let $J(m, v)$ be the value function for the informed trader. In equilibrium it yields

$$J(m, 1) = i - \sum_{j=1}^{i} a_{j+}(m) + J(a_i(m), 1); \quad J(m, 0) = \sum_{j=1}^{i} b_{j+}(m) + J(b_i(m), 0).$$

Likewise for $\theta_i(m, \cdot)$ being positive, $J(m, \cdot)$ satisfies relevant inequalities. Also concerning the informed trader's optimization yields

$$rJ(m, v) = \frac{\partial J(m, v)}{\partial m} + \sum_{i=1}^{n} \beta_i[J(a_i(m), v) - J(m, v)],$$

and naturally the boundary conditions accordingly. Similarly proposed in the equilibrium for the uniform price market are $J(m, \cdot)$ and the price dynamics.

When it comes to working orders, in the equilibrium it yields

$$ia_i(m) > ja_j(m) + (i-j)a_{i-j}(a_j(m)), \quad ib_i(m) > jb_j(m) + (i-j)b_{i-j}(b_j(m)),$$

where $m \in (0, 1)$ arbitrarily and $1 < j < i$.

Finally, the interesting thing concerning work orders and block orders is that the Nash equilibrium with work order in uniform price market is equivalent to that of block orders in limit order market. The proof employs the model itself plus arbitrage theories.

As is seen from the review of above two papers, liquidity and its role in equilibrium is multivariate as regards market study. In particular, how to link the differently accepted measures of liquidity? In a more general literature, the paper [21] deals with a model to imply a market view from client trades: it builds a profitability function and responding effects of the market maker's decision is given. A tricky part of the paper is that while trading volumes are hard to model, limit-orders are not. But both are, at least constitute a majority of, the measure for liquidity. The paper links these two stuffs. The choice of alpha as a main elements of research here is interesting. By Pontryagin's Principal for stochastic maximization, i.e. expectation function, using properties of backward stochastic differential equation, the alpha for the optimizor is solved. Worthy to mention is that the paper uses numerous results from and even sets its background in stochastic analysis, stochastic processes and stochastic partial differential equations.

Consider the market with n clients of heterogeneous beliefs on price, and one market maker. A filtered probability space given $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. W_t^k is a $k-$dimensional $\mathbb{P}-$Wiener process. Given the drift a_t and diffusion σ_t, the price process p_t reads

$$dp_t = a_t dt + \sigma_t dW_t.$$

Consider the cost process c_t that is sufficiently smooth in t, financially acting as the liquidity fee for the market maker. Let the i^{th} client has the position L^i. Denote l_t^i for the trading volume. α is the standard notation for market α. From the client's perspective, her optimization problem reads

$$S = \sup_l \mathbb{E}_{\tilde{\mathbb{P}}} \left[\int_0^\infty e^{-\beta t} (L_t dp_t - c_t(l_t) dt) \right].$$

Moreover, let

$$\gamma_t(\alpha) = \sup_{l \in \mathrm{supp}(c_t)} (\alpha l - c_t(l))$$

be the Legendre transform of c_t. The paper shows that the implied α for the optimization to hold satisfies

$$\alpha_t = \mathbb{E}_{\tilde{\mathbb{P}}}\left[\int_t^\infty e^{-\beta(s-t)} dp_s | \mathcal{F}_t\right]$$

where β is an enough large constant. More explicitly could α_t be written as the Ito process

$$d\alpha_t = \beta\alpha_t dt - dp_t + \theta_t d\tilde{W}_t.$$

Here $\tilde{\mathbb{P}}$ and $\tilde{W}_t$ are the clients' probability measure and filtration. θ_t is a measure of intelligence of a client over the price process p_t.

In solving the market maker's control problem

$$\int_0^\infty e^{-\beta t}\mathbb{E}[L_t\langle id, \beta t\rangle] + \langle -L_t\beta id + (id - \bar{\alpha}_t)\gamma_t' - \gamma_t - \epsilon f \cdot g_t, \mu_t\rangle$$

let g_t be the admissible control and μ_t be the measure derived from the approximation of the above optimization problem, $\epsilon = \beta\sigma^{-2}$, the main result yields

$$g_t(\alpha) = \frac{e^{m_t(\alpha)/\epsilon}}{\int e^{m_t(\alpha)\epsilon} d\mu_t(\alpha)}.$$

When talking about liquidity and equilibrium, one important factor is no doubt the determinants of equilibrium, its dynamics and the critical 'point' for its existence or uniqueness. In [30], there is a model of price formation in a limit order market, deriving the equilibrium order placement strategies. The highlight of trader's patience and market resilience is precious for the paper. Patience are described as a waiting cost for per unit time, which has the empirical implication such as portfolio managers (patient) or market speculators (impatient). An equilibrium of trading is defined as pairs of strategies for order placement with respect to patient and impatient traders, so as to maximize the expected profit determined by a trader's valuation and the best quotes. When working with the equilibrium order placement strategies it is natural to turn to spreads evolution in between transactions, by the definition of equilibrium. Thus comes market resiliency, whose determinants and relations with market heterogeneity (traders' behavioral differences) are discussed mathematically and pragmatically. Numerical examples are carried out in terms of heterogeneity, patience, resilience mentioned earlier. In equilibrium, a concise formula is derived concerning relation between market resiliency and duration between trades. Furthermore, simple numerical examples are displayed regarding different market resiliency, different heterogeneity and different patience of traders. Yet question remains as what themselves interact, for example, what brings heterogeneity bigger, a more or less resilient market? If questions are harder, then is there a correlation? What to do with real markets?

For this single-security market, the determinant range of admissible price is $[B, A]$. The best bid and ask quotes are a, b, with the spread $s = a - b$. The market orders arrive through a Poisson process with parameter λ and with inter-arrival time τ. Let V_b and V_s be the buyer and seller valuation. Δ is the tick size. Meanwhile, patient traders afford the per time waiting cost of δ_P; same for δ_I with impatient traders, with their respective proportion being θ_P and θ_I. p_b and p_s are respective execution prices. Denote $T(j)$ as the expected waiting time for $j-$tick limit orders execution. We have

$$\pi_i(j) = j\Delta - \delta_i T(j)$$

being the payoff of a trader submitting j-tick limit order. Let $j_i^* = CF\left(\frac{\delta_i}{\lambda\Delta}\right)$ where $CF(x)$ is the ceiling function, i.e. the smallest integer $\geq x$. In equilibrium, various properties concerning are obtained, e.g. the expected inter-trade time is (given the heterogeneity for traders)

$$T(n_1) = \frac{1}{\lambda}; T(n_h) = \frac{1}{\lambda}\left[1 + 2\sum_{k=1}^{h-1} \rho^k\right] \forall h = 2, 3, ..., q-1$$

and

$$T(j) = T(n_h) \quad \forall j \in \langle n_{h-1}+1, n_h \rangle \quad \forall h = 1, 2, ..., q-1.$$

Here $\rho = \frac{\theta_P}{\theta_I}$; $n_1, ...n_q$ are q (pieces of) randomly formulated spreads; and $\langle j_1, j_2 \rangle$ stands for the set $\{j_1, j_1+1, ..., j_2\}$ with $j_1 < j_2$.

Furthermore, let R be the measure of market resiliency, i.e. the probability that the spread reverts to its competitive level before the next transaction occurs. It satisfies $R = 1$ for homogeneous markets and $R = (\theta_p)^{q-1} < 1$ for heterogeneous markets.

Next the paper considers two types of markets, the fast market and the slow one, with their λ being λ_F and λ_S respectively. In the Markovian literature for $\{n_h\}_{h=1}^{q}$, in the equilibrium yields

$$n_h(\lambda_F) \leq n_h(\lambda_S), \text{for} \quad h \leq q_S;$$

$$n_h(\lambda_F) \leq n_{q_S}(\lambda_S), \text{for} \quad q_S \leq h \leq q_F.$$

CHAPTER 4

DIFFICULTIES OF MODELING FOR HIGH FREQUENCY MARKETS

4.1 Price Dynamics of Coupling Between Bid and Ask Price

As described in the first chapter, the time-t bid price $b(t)$ determines the boundary condition for sell limit order placement because any sell order placed at or below $b(t)$ will at least partially match immediately. A similar role is played by the ask price $a(t)$ for buy orders (c.f. [35]). As a matter of fact, almost all the models reviewed in this thesis allow limit orders at only two fixed prices, i.e. the best bid and the best ask. This assumption restricts those models to use standard results from queuing theory and to compute the mathematical properties such as the expected number of stored limit orders or the expected time to execute the orders.

One approach is to apply economics literature. In the paper [5] it is observed how this nonlinear coupling makes modeling the LOB such a difficulty. The paper [14] extended this into stages allowing any given order types and price patterns, assuming that these processes focused $t > 0$, with non-rebounded price behavior.

Another approach is to assume that limit orders are placed at a fixed distance from the mid-price, and that the limit prices of these orders are then randomly shifted and shuffled until they culminate to transactions. It is this random shuffling that causes price diffusion. This assumption takes advantage of the analogy to a standard diffusion model in the physics literature. See for reference e.g. the papers [13], [22], [33] and [26].

4.2 Price Dynamics in Determination of the 'True' Market Price

The price dynamics is the result of the interaction between the incoming order flow and the order book. As we have seen the most popular model of the price is that of mid-price. As the arithmetic average the best bid $b(t)$ and best ask $a(t)$, mid-price

$$m(t) = \frac{1}{2}a(t) + b(t)$$

is easy to be applied to deriving the optimal execution and optimal market making problems to facilitate an bird-over of mathematical perspective of market microstructure. It is also clearly to be calibrated easily.

On the other hand, one should not trust mid-price as a 'skeleton key' while dealing with so various kinds of market microstructure. Take, as a very simple but typical example, in a short time-scale (let's say 0.1 seconds or less) a volatile market where oscillation is big. If at one moment there is a huge market sell order into the order book, then the change of the one-sided price, say $b(t)$ would go tremendously larger than the other side $a(t)$, culminating to the deviation of the mid-price $m(t)$. On other words, it is possible that there is price jump as a sell (buy) market order arrival which is executed at a price smaller (larger) than the best bid (best ask) price at the moment consecutively after the market order arrival. Moreover, no evidence that mid-price returns had a significant impact on order arrival or cancelation rates (Poisson) was found. The problem here is universal for the limit order market models. From both mathematics and financial aspects, it is necessary to handle this bid-ask liquidity imbalance issue. One way to do this, is to draw from regression method.

To put the question more mathematically feasible, it turns out that how could we researches deal with the discontinuous price paths in the high-frequency trading circumstances? This is reasonable for resolving the original question since mid-price loses rationality only when the price dynamics goes through a discontinuous formality.

Here a few more delicate concepts with respect to limit order market is introduced, concerning trade-through [1] and trade-sign [2]. For the imbalance between two sides, a (bid-ask) volume ratio is introduced, corresponding to the i-th depth right before the k-th trade inside the order book by $W_{t_k-1}(i)$. Then the conditional probability of the negative trade-sign given $W_{t_k-1}(i) \geq x$ is calculated, for some $x \in \mathbb{R}_+$. Theoretically speaking, this < 0 trade-sign implies that a foreseeing ask market order reaches making the next trade entering a threshold by a ask market order. A side effect is achieved that this conditional probability is observed computationally copulated to this ratio in regards of the unit depth. This is crucial to measure whether the shape of an existing order-book is balanced or not.

Using Logistic regression (in statistics) for the relationship between liquidity and trade-sign is derived and an analysis on the prediction of the tick price jump occurrence by logistic regression is attained. Thus the original question is partially resolved.

(See for reference e.g. the papers [7], [32].)

[1] Trade-through can be regarded as the momentum convert of market price revoked by a market order.

[2] An indicator function of order, usually market ones indicating that pattern of LOB and empirical trading volumes are constructive for the trade foresight.

4.3 Iceberg Orders, Trade-through and Arrival Rates

As I see it, one of the thrilling characters of LOB lies in its highly strategic manner. There are tricky parts in market making. As long as these tricks are legal, nothing is to blame. From a 'maximization of utility function' viewpoint, it is worth digging into these plays.

One such tricks is an iceberg order. An iceberg order is a trading strategy. It refers to a large kind of transaction in the equity markets involving separating a relatively large order into smaller pieces of orders. A hidden signal with amount in iceberg orders, they are treated by market makers as a feasible way of kind of promoting the follow-up participants to trade identical to their direction(s). The latter may well ignore the possible manipulation (by the former) they are faced with.

Specifically, one is interested in for example the impact of iceberg orders on the price and order flow dynamics in limit order books. Commensurate is reached that such orders bring about hidden liquidity ('latent liquidity' as put in some articles), a result from the market dust-followers.

But now financial questions arise, such as the indirect effect that they are likely to have on the strategic behavior of other market participants. Naturally one would ask the question on the detection: to what degree would market makers figure out the existence of such tricky parts and predict the trading volume that lies behind the curves? If they can be detected by and large, what is the rationale of dealing with them, i.e. for one of the counterpart to submit and for the other to avoid? What is the motivation therein? Meanwhile, the concerning of the impact of iceberg orders on liquidity suppliers using limit orders is also of common concern.

Mathematically speaking, for one thing the model of iceberg order itself is a hard and miscellaneous problem. Actually the exact time point of these orders (or rather, the 'average' time duration of successive partition of iceberg orders) is totally unknown and have little hint for their behavioral regularity. Even the most common stochastic process, Poisson Process for modeling the incoming orders (whether it be market, limit or cancelation) has little room for them. In reality, researchers often find it a bit easier— sometimes by predicting price movement with respect to these behind-the-curve liquidity via tick-data. This affords great work of both computational and financial efforts and wisdom.

In contrast, trade-through is much more mathematically feasible. it usually happens when the sitting best quotes are hardly viewed in their volumes and hence insufficient for the incoming order to fill into. Then comes the 2nd-best quote in the order book to continuing matching the incoming order. This is just a vivid example of the queueing system previously described in detail in this thesis. As it were, the essential mechanism within trade-through is largely distinguished with iceberg orders, though sharing the apparent appearance as partitioning order sizes. For the former, Poisson Process could be frequently plugged into.

As tick-by-tick data shows, the string of trade-through is often modeled as Poisson Process, as far as mean waiting time until the next trade-through is concerned. And this in turn facilitates the calibration as well as the mathematical analysis of trade-through ([25],[32]). One such Process, a point process with time-varying intensity parameter

$$\lambda(t) = \lambda_0(t) + \sum_{t_i<t} \sum_{j} C_j \exp[-D_j(t - t_i)]$$

where λ and λ_0 denote the arrival rate in the relevant literature; where the t_i denote the times of previous arrivals and the C_j and D_j are parameters controlling the intensity of arrivals. This process is very useful in characterizing the reaching pace of orders as a functional of arrival rates momentum and of the amount of arrivals momentum. In studying empirical data from several different asset classes, this processes displays its strengths (see for reference [19]). For exogenously non-stochastic rate concerned, it is a good instrument for modeling. But arrival rate itself has some complexity, which will be studied in a further way later.

Difficulty appears when market is volatile. In reality, Markovian is literarily interpreted as the independence between future and past positions, given the current position; or mathematically

$$\mathbb{P}(X_{n+1} = y | X_n = x, X_{n-1} = x_{n-1}, ..., X_1 = x_1, X_0 = x_0) = \mathbb{P}(X_{n+1} = y | X_n = x).$$

If the price of the next moment (or more precisely, the next given time-scale) is chaotic or does not follow the probabilistic properties with respect to the past or even the current, there is a big problem. In some literature, this is concerned with the zero-intelligence or econ-physics[3].

For the price processes, another issue discussed beforehand in this thesis is the true price versus mid-price (for bid and ask sides) which interrupts the model procedure. Still another interesting question is a gap between theoretical and empirical literature that a wide variety of time series related to LOB have been reported to exhibit long memory. That is sometimes regarded as the long time-scale auto-correlation problem.

In the existing literature, order arrival are modeled as Poisson Processes. For simplicity, the authors denote λ, μ, and θ to be the Poisson parameter for the limit, market and cancelation orders respectively. Unfortunately, the interesting and important question of the corpula relation for these parameters have not been treated

[3]This is a controversial conception. To put it concisely, statistical mechanics theory is plugged into financial study, e.g. symmetry, equilibrium etc. For example, the Hurst Index depicting the long-term memory of stock prices as discussed later for auto-correlation. Generally, it takes into account the ideality that every market-participant is of zero-intellectual with barely any information accessability. But it neglects the manned factors within the market.

in the majority of the literature. In fact, λ, μ, and θ are thought of as different and independent things. To be discussed, too, in the next Chapter. (See for further reference Papers [5], [30], [11], [24] and [17].)

4.4 High Frequency Trading for Different Types of Equities

In this subsection, we put forward in a pastiche an interesting type of problem in high frequency trading. Up till now limit order market is studied mainly for stock market. Occasionally foreign exchange market is concerned, for it is one of the most liquidate market in the financial field, not to mention its importance in cross-national economy.

This is understandable. But when it comes to derivative market, high frequency trading could probably play a more subtle role. Here is the deal: high frequency trading in limit order book impacts on both the derivative markets and the underlying equity (say, stock) markets. For a given equity, say, Citigroup stock and its call option, both the underlying stock and the derivative is traded with high frequency. Here we put forward one mathematical question and one financial question to highlight the academic and empirical importance of high frequency trading in the derivative limit order book.

Mathematically, the option price is correlated with underlying equity. Say, for a American call option, assume K to be the strike of stock, $S(t)$ the time-t ($t \leq T$ where T is the maturity) stock price, then the corresponding time-t call option price is

$$C(t) = \max(0, S(t) - K).$$

For the stock, orders come with, say, arrival rate parameter. This impacts stock price (market price) but also the option price derived by the above equation. There is copula relationship between the volatility σ of stock and option. Is the correlation positive or negative? Hard to say. Option has its own limit order book, with *its* trading liquidity changes according to its underlying. Needless to say it is affected by the price of its underlying. These are two separate but interconnected issue: 'LOB on its own+stock affecting option'. In the Geometric Brownian Motion setup,

$$S(t) = S(0)e^{[\mu(t)-\frac{1}{2}\sigma(t)^2]t+\sigma(t)B(t)}$$

where $B(t)$ is a standard Brownian Motion. Here μ and σ are functions of t standing for drift and volatility respectively. Thus the order book dynamics affect μ as the market price expectation and also the σ which is the stock oscillation. These factors both impacts on option price, which is already hard to model and study. At the meantime, stock price has its effects in return to its underlying— affecting the stock order book. This tangling question is complicated and affords further study. For the author of this thesis, the question is of big interest for him.

Financially speaking, market participant would be keen on maximizing the expected utility. This is certainly true for options investors. For like European options, the closer to execution has effect on underlying stock. Now comes a plausible hypothesis: an investor wants his stock to go further up yet unfortunately both the economic surrounding and the technical analysis (e.g. bar-line or MACD) of the stock may not support the price tendency on his will. As a bright investor he would turn to options market for assistance. Trading on the corresponding options accordingly the exertion would be on the underlying stock. Yet how exactly would he trade? It is meaningful, arguably, in that the derivative market has comparatively usual for manipulation, and hence the visibility for other participants (like, the counterpart) to penetrate into.

CHAPTER 5
A BRIEF STUDY OF A TYPICAL MODEL

In this closing chapter, a special model [33] of LOB is studied. The reason for choosing this model is straightforward. First, as a Markovian model (with respect to bid and ask market depth) it uses 'good' mathematics involved in. Second, it reflects some categories of big interest in HFT research, e.g. how to model incoming queues of order flow, market resilience, dynamics of price change and the impact of queues on it, diffusion limit of price change, and so forth. Third, this model is standard in the sense of its convenience in tick-data calibration. Finally, the model is proper for discussion of generalization since it involves stock market only— note that the dynamics for different equity market are subtle and afford fecundity in mathematical research and financial interpretation.

Here five questions are put forward, allowing for data simulation to testify the model itself and would in turn help to better the model.

For convenience of simulation and calibration, we have a data set based on one-day Foreign Exchange market tick data.

5.1 Market Interpretation of Incoming Order Flow Parameters

Question: Market interpretation of λ, μ and θ. Why

$$\lambda \leq \mu + \theta.$$

Discussion: In the stochastic process, that a Poisson process has a parameter λ infers the average number of events per unit of time is $\frac{1}{\lambda}$. To put it another way, λ is the average *frequency* of the occurring events where in our case is the incoming (limit) orders. The same interpretation for μ and θ.

By definition, a Poisson arrival with intensity λ implies that the number of arrivals in any interval of length T has a Poisson distribution with parameter λT. In the order book, the limit order with arrival rate λ makes the queue increase by one unit on the according side. The same holds true for either market order or cancelation. For λ, it is a decreasing function of the distance to the bid/ask side since most orders are placed very close the current (market) price.

The assumption, as discussed previously in the last Chapter, makes things much ideal and easily computed that the arrival events are mutually independent for both the across-type orders and within one type of order. By the properties of Poisson Processes, limit orders arrive at a distance from the opposite best quote at independently exponential point $t > 0$ with rate λ, similarly for market and cancelation.

There are significant connection among λ, μ and θ. Take the bid side and the ask order for example. If a ask limit order pops put, the bid side would increase; for a market or cancelation order, it is depleted. Thus the effect of market and cancelation orders are the same, different with limit orders. Hence the inequality

$$\lambda < \mu + \theta$$

means the shape of order book is changing more frequently and the market price experiences more oscillation. The equality

$$\lambda = \mu + \theta$$

implies a 'balanced' order book in the sense that the flow of limit orders is offset by that of market and cancelation orders. On the other hand if

$$\lambda > \mu + \theta$$

with the probability

$$\frac{\mu + \theta}{\lambda + \mu + \theta} > 1/2$$

of price up-moving per unit and with the probability

$$\frac{\lambda}{\lambda+\mu+\theta} < 1/2,$$

of price down-moving per unit, then the pair (q_a, q_b) of order sizes would blow up to ∞ with positively probability. Note that for discussion simplicity the above formulas are under the condition that only unit of change occurs in queue size convert. In effect even if this assumption is weakened there still exists the probability$> \frac{1}{2}$ of the queue size getting ever bigger.

Due to the limitation of the dataset we get, it is hard to tell which order type a given order is based on the price change information in the order book. Yet it is worth to notice that for foreign exchange markets which is far more liquid than stock markets, it is reasonable to assume that all the orders we analyze and simulate are limit orders. In fact from what it calibrated, this assumption does make sense.

5.2 Measure of Market Liquidity

Question: How to measure the market liquidity in the model by these parameters?

Discussion: Liquidity is an inevitable issue when one talks about limit order market. To measure the liquidity in the limit order book is comprehensive. In the paper although liquidity is not frequently risen up, it is actually analyzed and formulated mathematically.

Liquidity is first and foremost involved with trading volume. Clearly, when the queue pair size for ask and bit at time t : $q_t = (q_t^a, q_t^b)$ has bigger oscillation, the market is locally (on a small interval with the center point of t) more involving, meaning more liquidate. But this is cerntainly not merely a piece of cake. As proposed earlier, market makers could pose dark pool orders or trade-through orders on their own interest in order to make profit with delicate and revealed manner. If this happens then the it is unwise to measure liquidity just considering trading volumes.

In the model,

$$|\lambda - (\mu + \theta)| = (\mu + \theta) - \lambda$$

is a good way to measure liquidity for the reason that this expression takes into account the dynamics of market price, for μ is the market arrival rate that leads the true price to change. Combined q_t with $(\mu + \theta) - \lambda$ is to model the short-term (high-frequency-scaled) liquidity of the market. Yet another question emerges: what if the market makers manipulate the price so that it goes up shortly and goes down thereafter? This involves with the long-term market behavior and is less reckoned by the model.

5.3 Durations Between Consecutive Price Changes

Question: How to simulate the duration between consecutive price changes?

Discussion: The Poisson arrival rates λ, θ, μ are the measures of frequency of the coming orders. One question of both the research and empirical interest is the duration between consecutive price changes. The term 'tick-by-tick data' refers to this very conception. To make it specific, this tick data boasts the time-duration of not the uniform period, say one second, but random times. The essence here is the consecutive price dynamics and the probability of the increase (decrease) of the converting of price.

In the model the authors derived continuously a formula with respect to the dynamics and afford easily the simulation by the market tick data. With tick data we can compute the distribution function of the bid and ask queue trading volume and test the model with empirical results. Worthy of being mentioned is the non-symmetric behavior between n and p for the bid and ask volume respectively. (Theoretical analysis and sketch of proof c.f. the paper.)

To simulate an indefinite integral like Equation (4) in the paper, i.e. 'the probability of the values of duration giving bid and ask volumes' ([33]), we could use several methods like Taylor expansion of the integrand, by using the relation between series and indefinite integral, and so forth. But the better way to evaluate the Bessel function of the first kind, a 2-order ODE, is via the indefinite integral-series theory.

5.4 Auto-correlation of Price Changes

Question: What determines the negative auto-correlation of price changes? Likely, merely

$$\mathrm{Cov}(X_k, X_{k+1})$$

doesn't make sense. So, do $\mathrm{Cov}(X_k, X_{k+1})$, $\mathrm{Cov}(X_{k+1}, X_{k+2})$, ... i.e. AR-process of order-n. Take $n = 3, 5$, etc.

Discussion: In stochastic process theory ([36]), a discrete-time process $X_t, \quad t = 0, 1, 2, ...$ is said to be an Autoregressive Process of Order p (AR(p)) if $\exists a_1, ..., a_p \in \mathbf{R}$ and a zero-mean white noise $Z(t)$ such that for $s > 0$ and

$$\mathbf{E}(Z(t)Z(t+s)) = 0$$

we have

$$X(t) = \sum_{s=1}^{p} a_s X(t-s) + Z(t).$$

For this model the discretization of X_t is an autoregressive process. It has something to do with the price auto-correlation, but not so simple as the model originally points out especially when approaching the change of time-scale of price dynamics. On the other hand, long-term and short-term concerns of price changes are different as to the order-n.

One apparent question, however, would be about the probability of two successive price moves in the same direction. In the model, this probability p_{cont} is taken for granted to be constant, which makes easy the n-order covariance between moves in price. Yet the model fails to justify the assumption of the 'constant' proposition. Might it be that the successive price changes does NOT even has a probability distribution? In reality we observe from empirical data that different trading volume and price direction renders different auto-correlation of price changes, or put it more specifically, they interacts with each other. Plus the condition of $p_{cont} = \frac{1}{2}$ is worth study in the model, where it lacks discussion. In that case it is implied from the model that the price changes $(X_1, X_2, ..., X_k)$ is uncorrelated. Then the case the degenerated.

5.5 Estimation of Price Volatility

Question: Estimation of volatility with respect to order flow. Test whether the result of the model makes sense. Plus, how far could this model go if the underlying equity convert from stock into other types, like foreign exchange or options?

Discussion:

From the dataset at hand, it fits more into the balanced order book, since market and cancelation orders approximately coincides with limit orders, which is indicated by the good balance between two sides of the order book.

1. By the cumulative two-sided trading volumes gotten from HFT tick-data, compute the geometric average of the size of the bid queue and the size of the ask queue after a price change, $\sqrt{D(f)}$.

2. Choose the comparatively larger time scale τ to $\tau_0 = (\lambda)^{-1}$. Solve the equation of n :

$$n \ln(n) = \frac{\tau}{\tau_0}.$$

3. Compute the price volatility (diffusion) σ from

$$\sigma = \delta \sqrt{\frac{n\pi\lambda}{D(f)}}.$$

4. Plot from the tick database the price dynamics, and evaluate the true market volatility with the corresponding time scale chosen above.

5. Do comparison.

5.6 Main Results from Data Simulations

5.6.1 On the distribution of order arrivals. It is sufficient to test that the time duration, i.e. the difference between the time(s) when a price jump happens, conforms an exponential distribution. Here We use the clock-time of the price moves and calculate the time-duration via excel, before plotting the graph of the dynamics of these durations and testing the exponential proposition.

Far from ideality, Kolmogorov-Smirnov Test (KS-Test) shows that neither the full tick-data nor its partial data has the exponential distribution, which is attained by randomly generating an exponential distribution and do the two-sample test with the empirical, where the significance value is a defaulted $\alpha = 0.05$. This is understandable because the Poisson distribution assumption (for the time spots of price change) is a nice mathematical tool for analysis of financial propositions of HFT yet lacks empirical justification for market data to be so.

Nonetheless, there are two things to be treated with care. One is that although the arrival rate does not conform to a Poisson process, the model has a good fitness with market data, with respect to price volatility and up-moving probability, as will be seen in the next few sections.

The other and more valuable character from tick-data, is about the similarity of time-duration distribution within the data itself. Specifically, the whole size of tick-data, i.e. the number of price changes is 53217. We separate the whole sample into six sub-intervals: (0,10000], (10000,20000], (20000,30000], (30000,40000], (40000,50000] and (50000,53217]. First we do the 2-sample KS-test for two pairs within the six sub-intervals and find only one pair has the $H = 0$ in $\alpha = 0.05$ and p-value=0.0670, in fact the two pairs are successive. Now we sub-divide the pairs respectively to test what happens. Follow the above step, we find whichever the smaller groups of tick-data that causes the similarity of distribution, or $H = 0$, and find to corresponding α's and p-values.

The most important result is that

$$H = 0, \quad \alpha < p - \text{value}$$

happens with the two samples both corresponding to the local peak trading volume of bid or ask. More clearly put, two groups of time-duration data has the same distribution (via KS-Test) if both of them belong to the time period of locally biggest trading volumes. See Figure 5.1 and compare it with what is stated in the following paragraph.

For example, let x_k $(k = 0, 1, ...4)$ be the $[10000 * k, 10000 * (k + 1)]$-th time-duration in the tick-data; x_{ab} be the $[10000 * a + 1000 * b, 10000 * a + 1000 * (b + 1))$-th time-duration in the tick-data, and we have some pairs with the acceptance of hypothesis testing: $H = 0$,

$$\text{KS-Test}(x_{33}, x_{35}) \implies H = 0, \text{p-value} = 0.3638$$

etc. These sub-intervals of time-duration show clearly in the graph that the corresponding bid and ask volumes both attain their local 'peak'. These p-values implies that one can accept the hypothesis of the same distribution in Kolmogorov-Smirnov Test for the sub-intervals of time-duration. See Table 5.1 for reference.

On the other hand, the majority of the time-duration groups shows no common distribution with one another, as long as they do not have both the peak trading volumes.

The above discovery indicates an important and interesting phenomenon in high-frequency trading: market participants would have the 'following the crowd' effect where the trading volume could go higher and higher at a relatively very short

Table 5.1. KS-Test Results with $H = 1$, for samples of 10000-sized ticks

Pairs of Sampled Data	p-value
(x_0, x_1)	5.9216e-023
(x_0, x_2)	4.5982e-093
(x_0, x_3)	2.5145e-167
(x_0, x_4)	1.8509e-144
(x_1, x_2)	2.4731e-035
(x_1, x_3)	1.9239e-078
(x_1, x_4)	1.3314e-071
(x_2, x_3)	3.6524e-012
(x_2, x_4)	2.2664e-008

Table 5.2. KS-Test Results with $H = 0$, for samples of 10000-sized ticks

Pairs of Sampled Data	p-value
(x_3, x_4)	0.0669

time slot. Such general behavior, as (mini time-scale) time goes by, leads to the short term market oscillation. Suppose that we had an quote-driven market rather than an order-driven one, and the trading frequency was far less than we are now, the picture would not be the same. Would it become that the size of the order determined the market trend. This could bring about another interesting question that we will not touch right now.

Table 5.3. KS-Test Results with $H = 0$, for samples of 1000-sized ticks

Pairs of Sampled Data	p-value
(x_{11}, x_{12})	0.2145
(x_{12}, x_{13})	0.0609
(x_{15}, x_{16})	0.4592
(x_{31}, x_{32})	0.1171
(x_{32}, x_{33})	0.6034
(x_{33}, x_{35})	0.3638
(x_{33}, x_{37})	0.1445

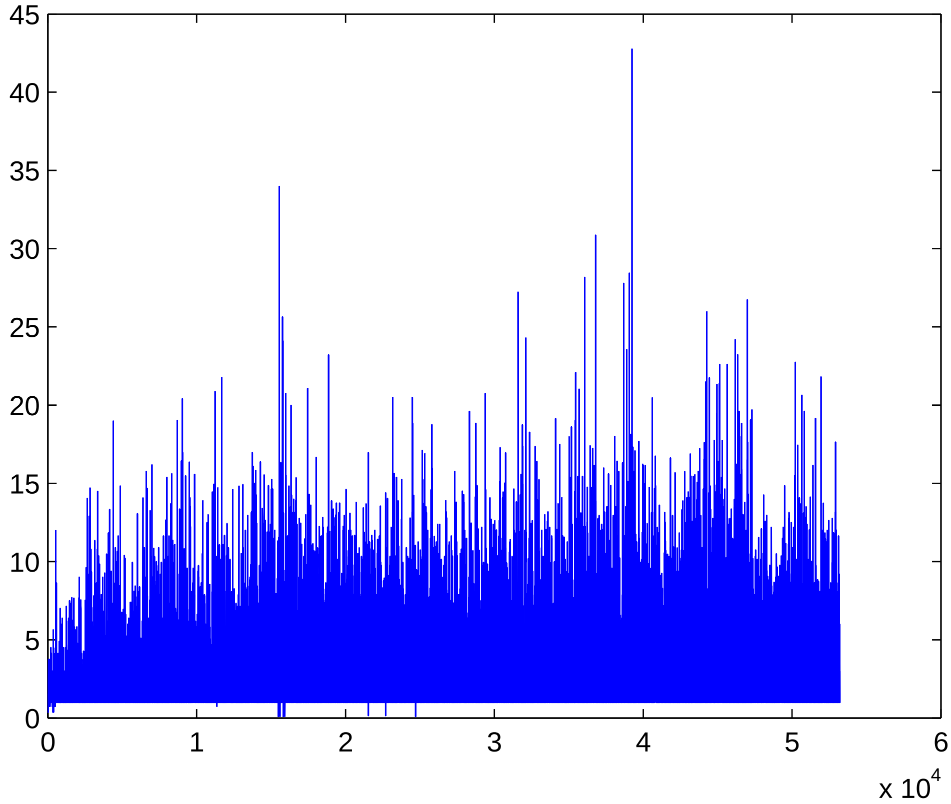

Figure 5.1. Bid volume of the whole tick-data

5.6.2 On the up-move probability in the price dynamics. From price paths (bid and ask, e.g. in Figure 5.1) in the tick data, the price movements is acquired by the logic sentences by numbers '1','-1' and '0' for up, down and remaining respectively. Pick up the (bid and ask, with both prices and volumes) price pairs with respect to '1', i.e. the up-moving ones. Calculate the (joint) conditional probabilities of these up-moving pairs according to bid and ask queues, via MATLAB codes. Plot the probabilities as functions of pairs of queues. Compare with the theoretical joint distribution $\phi(n,p)$ in the model.

The conditional probability of up-moving price changes in the empirical scenario is calculated by the definition of conditional probability:

$$P(n,p|\delta=1)=\frac{P(n,p;\delta=1)}{P(n,p)}=\frac{\#(n,p;\delta=1)}{\#(n,p)}$$

where n,p stand for bid and ask depth respectively; δ here is the price moving direction, with logical numbers stated above; $\#(\cdot)$ is the frequency a random result '$\cdot$' happens. Owing to MATLAB, a nice computing software with strong user-packages to process the computation, the probability with respect to every pairs of (n,p) turns out.

The procedures of computation are described as follows:

a. Let B_0 be the 900×3 matrix characterizing a basic sample of 900 price changes in tick-data, with only the bid and ask volumes and the indicators '1', '0' and '-1'; let P_0 be the up-moving subset of matrix B_0, but only with two columns since the third one should be all '1' and thus omitted for the reasons of saving storage.

b. Use the 'unique' command in MATLAB to get the B and P : the same values as in B_0 and P_0 but with no repetitions.

c. Use the 'hist' command in MATLAB to calculate the numbers of each repeating rows, for both B and P in B_0 and P_0, respectively.

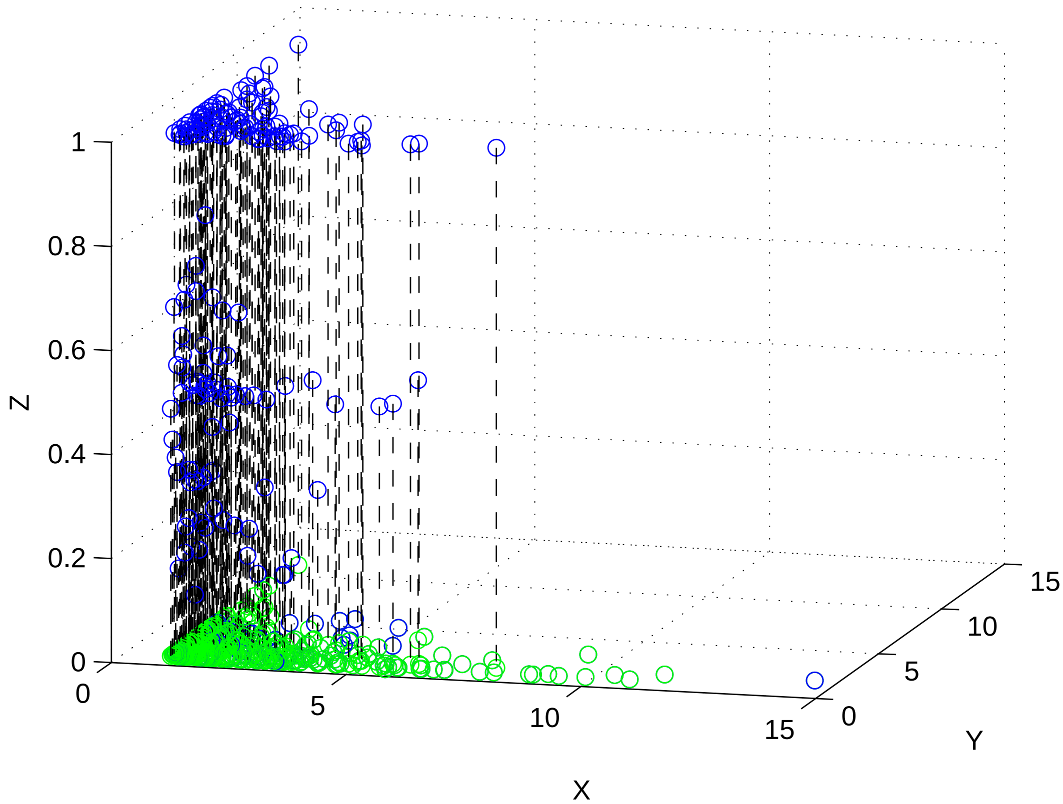

Figure 5.2. x:bid size y:ask size z:conditional probabilities of price up-moving

d. Use 'for' command with dual loops in programming, as regards the lengths (i.e. number of rows in a matrix) of both B and P while using 'if' sentences in order to determine whether any given row in P occurs in B, and if so how many times this occur. Notice that we use the 'break' command in the 'for-loop' so that the order of each unique rows in P could exactly correspond to those in B, hence making the conditional probability computation correct (with neither repetition nor neglect while being counted).

e. Plot the 3-Dimensional graph with the 'x', 'y', 'z' labels being the bid, ask volumes and the (according) conditional probabilities with '1' occurring. Plot the dots for each point so as to be clearer looking, as in Figure 5.2.

Experiments with several samples of B_0 show that the theoretical model yields good empirical fitness. The reason might be the good assumption for the price changes to be bivariate random walk, as the authors of the model did. Therefore it is mathematically doable for the $\phi(n, p)$ to have the analytical form, while from our FX-tick-data it fits well. Moreover, since foreign exchange market is very liquid and the price changes quite frequently (indeed 'high-frequency' trading), it is reasonable to embed the model into the data-set we choose because random walk has the better asymptotic propositions when the sample size is big enough and the time-duration is tiny enough.

5.6.3 On the diffusion behavior of price dynamics and price volatility.

Test the price dynamics (approximately when tick size $\delta \to 0$ or time-scale $n \to \infty$) and whether it is approximately a Brownian Motion.

We choose two data-sets with both having ten minutes in duration, thus $\tau =$ 10(min) in the discussion of the diffusion theorem of the model. One is from 2:00am to 2:10am when the trading volumes are scarce with regards to the whole trading day. The other is from 8:00pm to 8:10pm (Figure 5.3) with higher frequency of tick-data and price changes. We have the following steps for computation and testing:

a. Unify the units of each variable and constants.

b. Solve the equation for n, the average number of orders during the 10-min interval.

c. By the 'unique' and 'hist' commands in MATLAB (as in the previous section of figuring out conditional probability) to $\sqrt{D(f)}$, the geometric average of the size of the bid queue and the size of the ask queue after a price change. Here $f(i,j)$ is the distribution of the cases when bid and ask price pairs become (i,j), and

$$D(f) = \sum_{i,j} ijf(i,j).$$

d. Compute the scaling factor of the diffusion price-change process, using the theoretical formula.

e. Test whether the price-process divided by the scaling factor is indeed a standard Brownian Motion. Here we use 'normplot' (Figure 5.4) command and 'qqplot' (Figure 5.5) in MATLAB to have a graph of a normal probability plot for the re-scaled price process. The more linearized the figure is, the closer to normal distribution for the difference of the re-scaled price process, and hence the closer to Brownian Motion for the process itself (by Levy's Characterization of Brownian Motion. Note that we

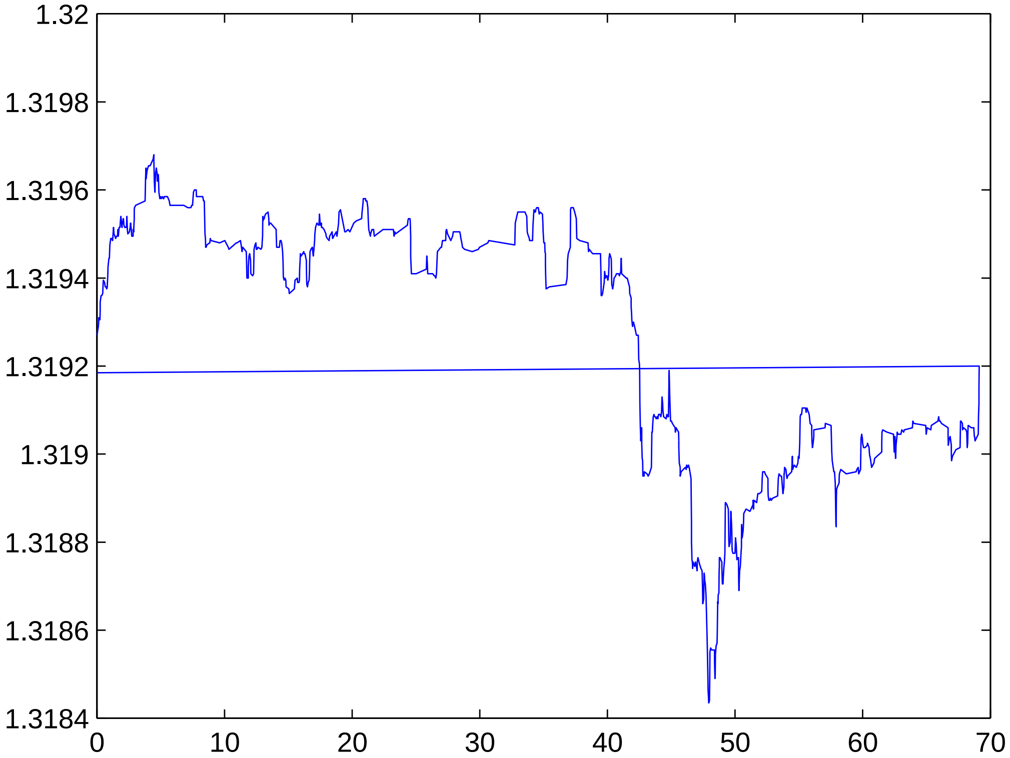

Figure 5.3. Tick-data from 8:00-8:10pm; horizontal: $\times 10$ seconds, vertical: price

might well assume the process to be continuous, both by graph and by definition of continuity.)

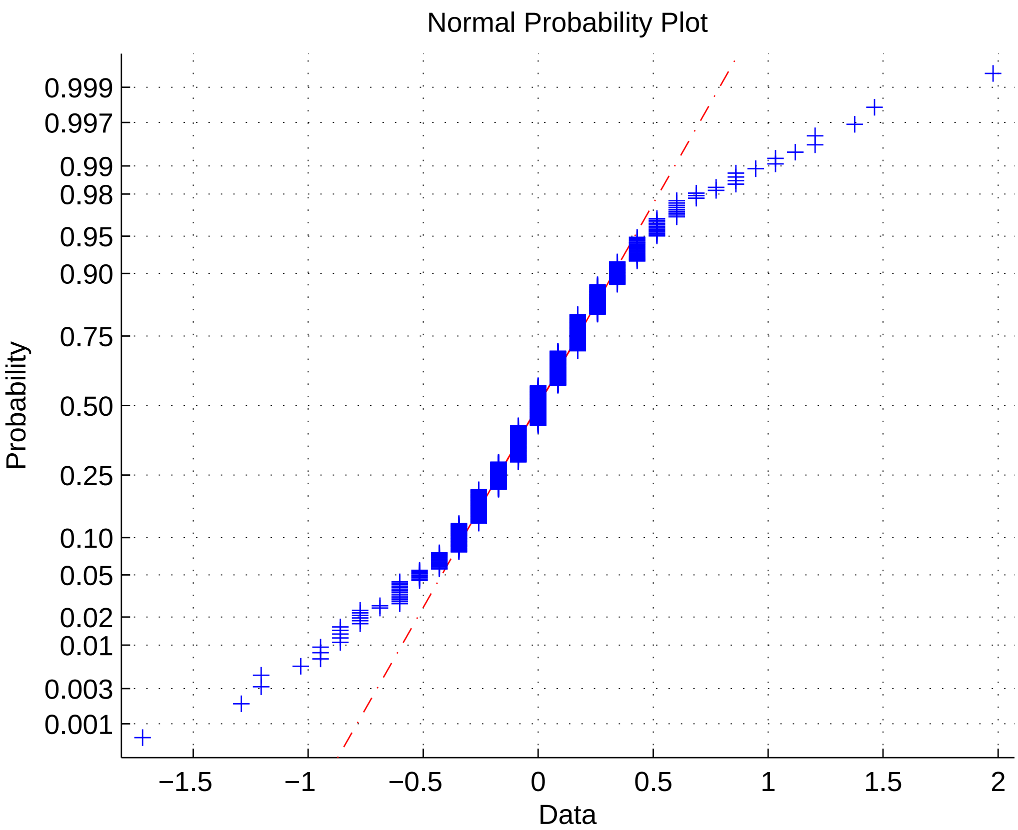

Figure 5.4. Normal probability plot of the scaled price change, 8:00-8:10pm tick-data

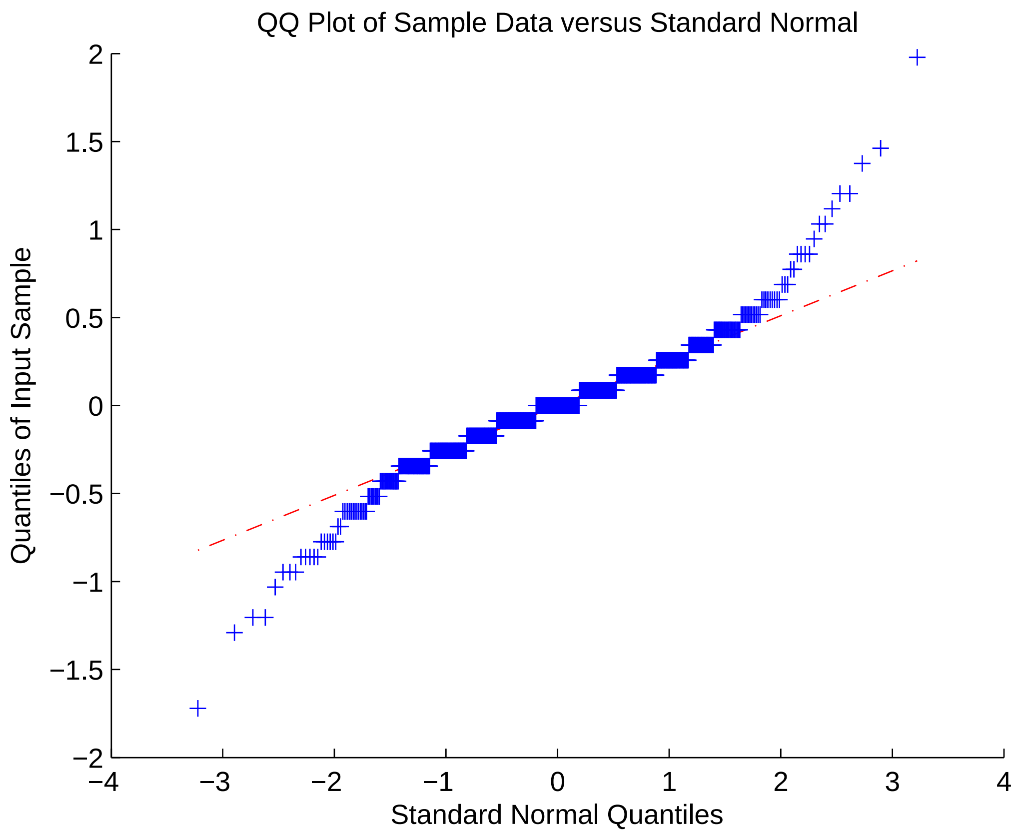

Figure 5.5. Normal probability plot of the scaled price change, 8:00-8:10pm tick-data

5.7 Conclusions

This model is described as a Markovian queueing system of limit order book. Here several core features of the model is tested and evaluated by computer simulation.

Tests of time-duration distribution show that there is generally no common distribution for the whole tick-data, nor for the data of ten thousand ticks. It is thus reasonable to suspect, at least for foreign exchange markets, whether the arrival rate could be modeled as a Poisson Process. Many existing models, as have been reviewed and studied in the previous chapters, assume the Poisson arrival of limit orders. A guess might be put forward that under 'some' condition the Poisson arrival holds true for these orders. That could be a further research topic.

Study of the diffusion behavior of empirical data show that they simulates (scaled) Brownian Motion in the model pretty well for a big probability, or with exception at the tail probabilities. That is, the simulation shows a heavy-tail behavior for normal distribution for the empirical data as regards their theoretical behavior in the model. Further work would be to make the model modified and more delicate in order to fit more data into it. Or if that is improper under the model's mechanism, the model set-up would be either generalized or specified as of the diffusion price behavior.

When it comes to the up-moving probability of price changes, our simulation shows good fitness with the theoretical formulae. This further justifies the fundamental assumption for the price to be the stochastic process of a random walk. As a matter of fact, when we test the data, the rationale of the price moving also follows the '1' or '-1' (up or down) direction. This would be a relatively simple but important part in the research of price dynamics.

The model is user-friendly at processing the empirical data, since it has good analytical tractability. Suppose it becomes more delicate to fit the market reality, this process may get harder. Such a trade-off seems inevitable, which researchers has their own balance of concern.

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