On an extension of distance hereditary graphs

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Received 1 January 2006; accepted 27 December 2007
Available online 25 February 2008

Abstract

Given a simple and finite connected graph $G$, the distance $d_G(u, v)$ is the length of the shortest induced $\{u, v\}$-path linking the vertices $u$ and $v$ in $G$. Bandelt and Mulder [H.J. Bandelt, H.M. Mulder, Distance hereditary graphs, J. Combin. Theory Ser. B 41 (1986) 182–208] have characterized the class of distance hereditary graphs where the distance is preserved in each connected induced subgraph. In this paper, we are interested in the class of $k$-distance hereditary graphs ($k \in \mathbb{N}$) which consists in a parametric extension of the distance heredity notion. We allow the distance in each connected induced subgraph to increase by at most $k$. We provide a characterization of $k$-distance hereditary graphs in terms of forbidden configurations for each $k \geq 2$.

Keywords: Distance heredity; Dilatation number; Forbidden configurations

1. Introduction

The major problem that we meet in several networks consists of the inefficiency of transmitting messages between processors. Even if the sender and the destination are still connected, the messages are delivered with a certain delay. The distance is the main function which expresses the connectivity between the sites and the faulty tolerance in these networks. It is computed by means of shortest paths in networks. The $k$-distance hereditary graphs can be used to represent communication networks. In fact, a network modelled as a $k$-distance hereditary graph ($k \in \mathbb{N}$) is characterized as follows: if two sites have failed in transmission, as they remain connected, the distance between them in the faulty network is bounded by the distance in the original network plus an integer $k$, $k \in \mathbb{N}$. This means that a graph $G$ belongs to the $k$-distance hereditary class if and only if the length of each induced $\{u, v\}$-path is at most the distance in the original graph plus $k$. Among these classes, we can find the well-known distance hereditary graphs ($k = 0$) which are introduced by Howorka [7] and characterized in several ways by Bandelt and Mulder [2] using both algorithmic and metric aspects. Moreover, Aïder [1] has introduced and studied the 1-distance hereditary graphs (which he called almost-distance hereditary graphs). Cicerone and Di Stefano [5] have used the notion of twin graph to characterize the $k$-distance hereditary graphs and have stated the coNP-completeness of the recognizing problem in this class.

In the present work, we characterize $k$-distance hereditary graphs in terms of forbidden subgraphs using the chord distance in cycles and the dilation number of a graph. We provide all forbidden configurations for each integer

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doi:10.1016/j.disc.2007.12.102
Fig. 1. Examples of chord distance of cycles (the chord vertices are filled).

$k \geq 1$. This paper is organized as follows. Notations and basic concepts used in this work are given in Section 2, while Section 3 introduces the $k$-distance hereditary graphs and provides some results which have been found in the framework of characterization of $k$-distance hereditary graphs. In Section 4, an improvement of this characterization is done in terms of forbidden subgraphs. Finally, Section 5 concludes the paper by listing some open problems.

2. Preliminaries definitions

We only consider finite, simple, loopless, connected and undirected graphs $G = (V, E)$ where $V$ is the vertex set and $E$ is the edge set. A subgraph of $G$ is a graph having all its vertices and edges in $G$. The neighbourhood of a vertex $u$, denoted by $N(u)$, consists in all the vertices $v$ which are adjacent to $u$. Given a subset $S$ of $V$, the induced subgraph $\langle S \rangle$ of $G$ is the maximal subgraph of $G$ with vertex set $S$. A sequence of pairwise distinct vertices $(u_1, u_2, \ldots, u_n)$ is a path (more precisely $(u_1, u_n)$-path) if $(u_i, u_{i+1}) \in E$ for $1 \leq i < n$ and is an induced path if $\langle \{u_1, u_2, \ldots, u_n\} \rangle$ has $(n - 1)$ edges. For convenience, the $(x, y)$ part of a path $P$ is denoted by $P(x, y)$. The length of a shortest induced path between two vertices $u$ and $v$ is called distance and is denoted by $d_G(u, v)$; moreover, the length of a longest induced path between the same vertices is denoted by $D_G(u, v)$.

A cycle $C_n$ of length $n$ in $G$ is a path $(u_1, u_2, \ldots, u_n)$ where also $(u_1, u_2) \in E$. A chord of a cycle $C_n$ is an edge that joins two nonconsecutive vertices of $C_n$. A hole, denoted $H_n$, is a cycle with $n$ vertices and without chords. For each cycle $C_n$, we denote by $K(C_n)$ the set of consecutive vertices $u_i, u_{i+1}, \ldots, u_{i+|K(C_n)|-1}$ in $C_n$ such that for each chord $e$ in $C_n$ there exists a vertex $x_e$ in $K(C_n)$ incident to $e$. We note that $K(H_n) = \emptyset$. The chord distance of a cycle $C_n$, denoted by $cd(C_n)$, is the minimum cardinal of $K(C_n)$. (See Fig. 1(a)). We note that $cd(H_n) = 0$. Let $\zeta_n$ be a family of cycles such that $\zeta_n = \{H_n; C_n$ with $cd(C_n) = 1\}$. A cycle on $n$ vertices will be referred to as a $1C_n$ if it is isomorphic to one element of $\zeta_n$. (See Fig. 1(b), where dashed lines represent possible chords in the cycle).

The twin graph of $G$ is the graph $G^* = (V^*, E^*)$ such that $V^* = V^1 \cup V^2$ and $E^* = E^1 \cup E^2 \cup E^3$, where $V^1 = \{v^1 \mid v \in V\}$, $V^2 = \{v^2 \mid v \in V\}$, $E^1 = \{(u^1, v^1) \mid (u, v) \in E\}$, $E^2 = \{(u^2, v^2) \mid (u, v) \in E\}$ and $E^3 = \{(u^1, v^2), (u^2, v^1) \mid (u, v) \in E\}$.

Definition 1. Let $k$ be a positive integer. A graph $G$ is $k$-distance hereditary if and only if for any connected induced subgraph $H$ of $G$ we have:

$$\forall u, v \in H, \quad d_H(u, v) \leq d_G(u, v) + k.$$ 

3. Related works

The class of $k$-distance hereditary graphs contains the well known distance hereditary graphs (when $k = 0$) introduced by Howorka [7] and characterized later by Bandelt and Mulder [2]. Further, A¨ıder [1] gave a characterization of $1$-distance hereditary graphs by providing all forbidden configurations:

Definition 2. Let $C$ and $C'$ be two cycles isomorphic to $1C_5$.

- A $2C_5$-configuration of type (a) is a graph obtained by combining $C$ and $C'$ by connecting by an induced path two vertices which are not endpoints of chords of a cycle. The induced path can be empty, and the related vertices are then identified. Fig. 2(a).
- A $2C_5$-configuration of type (b) is the cycle $C_8$ which is isomorphic to the graph of Fig. 2(b).
Notice that in Fig. 2 (and in all our figures), a dashed line represents an eventual edge and a dotted line represents a path, eventually empty (in this case, the end vertices coincide).

**Theorem 3** ([1]). A graph $G$ is 1-distance hereditary if and only if $G$ neither contains a $2C_5$-configuration, nor a cycle $1C_n$, for $n \geq 6$, as induced subgraph.

Bessedik [3] and Rautenbach [9] have characterized 2-distance hereditary graphs in terms of forbidden subgraphs:

**Theorem 4** ([3,9]). Let $G$ be a graph. $G$ is 2-distance hereditary if and only if $G$ does not contain any one of the following configurations as an induced subgraph:

1. $1C_n$, $n \geq 7$;
2. One of the graphs of Fig. 3;
3. One of the graphs of Fig. 4.

Other characterizations dealing with distance properties or chord distance ones were established for each $k$-distance hereditary graphs ($k \in \mathbb{N}$):
Definition 5. Let $G$ be a graph and $u$ and $v$ be any two vertices of $G$:

- The dilation number of the pair $\{u, v\}$ is given by:
  \[ \delta_G(u, v) = D_G(u, v) - d_G(u, v). \]
- The dilation number of the graph $G$, denoted by $\delta(G)$, is the maximum dilation over all possible pairs of vertices in $G$, that is,
  \[ \delta(G) = \max_{\{u, v\} \in G} \delta_G(u, v). \]

Lemma 6 ([5]). A graph $G$ is $k$-distance hereditary if and only if $\delta(G) \leq k$.

Lemma 7 ([5]). Let $G$ be a graph and $k \geq 0$ be an integer. Let $G^*$ be the twin graph of $G$. Then, $G$ is $k$-distance hereditary if and only if $G^*$ is $k$-distance hereditary.

Theorem 8 ([5]). Let $G$ be a graph and $k \geq 0$ be an integer. $G$ is a $k$-distance hereditary graph if and only if $cd(C_n) \geq (n - k)/2 - 1$ for each cycle $C_n$, $n > k + 4$, of the twin graph $G^*$.

The implicit form of this result incites us to check for the forbidden subgraphs of $k$-distance hereditary graphs. That is the main aim of the next section.

4. Characterization of $k$-distance hereditary graphs

It is easy to check that the class of $k$-distance hereditary graphs is closed undertaking connected induced subgraphs. Therefore, each cycle $C_{n+k'}$ is a forbidden induced subgraph in a $k$-distance hereditary graph for $k' > k$, since it can be seen as a union of a shortest $\{u, v\}$-path of length 2 and a longest one of length $2 + k'$. This configuration remains nonallowed if all the chords have a common end vertex. However, we can build forbidden configurations using the following operations:

Definition 9. Let $C_{n_1}$ and $C_{n_2}$ be two disjoint cycles having vertex set $V_1 \cup \{v_1\}$ and $V_2 \cup \{v_2\}$ respectively and edge set $E_1$ and $E_2$ respectively such that if $C_{n_1}$ (resp. $C_{n_2}$) has chords, the vertex $v_1$ (resp. $v_2$) is adjacent to a chord vertex in $C_{n_1}$ (resp. in $C_{n_2}$) and nonincident to any chord of $C_{n_1}$ (resp. $C_{n_2}$):

- For each integer $r$, the $r$-composition of $C_{n_1}$ and $C_{n_2}$ is the graph $C_{n_1} \varphi_r C_{n_2}$ obtained by connecting $C_{n_1}$ and $C_{n_2}$ by a path of end vertices $v_1$ and $v_2$ and of length $r$.
- $1C_{n_1} \varphi_{-1} 1C_{n_2}$ is the graph $G$ having a vertex set $V = V_1 \cup V_2$ and edge set $E = E_1' \cup E_2' \cup \{(x, y) \mid x \in N(v_1); y \in N(v_2)\}$ where $E_i' = \{(x, y) \in E_i \mid x, y \in V_i\}$ for $i = 1, 2$. 
The operation $\varphi_{-1}$ is the split composition applied to $C_{n_1}$ and $C_{n_2}$ with respect to $v_1$ and $v_2$ (see Fig. 5). This operation is the inverse of the decomposition operation introduced in [6]. In [4], split composition has been used, for example, to build distance hereditary graphs, complete graphs, complete bipartite graphs and trees. We can easily observe that the restriction of this operation to cycles such as $C_{n_1}$ and $C_{n_2}$, is associative.

Definition 10. A $k$-configuration is any forbidden induced subgraph of a $k$-distance hereditary graph. This configuration is strict if its dilation number is equal to $k + 1$. Otherwise (when the dilation number overtakes this value), the configuration is said to be large.

It is not difficult to see that if we link a set of $p1C_{4+k_i}$, $1 \leq k_i < k + 1$, $i = 1 \ldots p$, with $\sum_{i=1}^{p} k_i = k + 1$, using the $r$-composition, $r \geq 0$, we obtain a configuration which is not allowed as a subgraph in any $k$-distance hereditary graph (See Fig. 3(a) for the case $k = 2$):

Definition 11. Let $G$ be a graph, $H$ be a subgraph of $G$ and $k$ be an integer ($k \geq 1$). $H$ is a a strict $k$-configuration of type (a) if $H$ is isomorphic to the following graph:

$$1C_{4+k_1}\varphi_{r_1}1C_{4+k_2}\varphi_{r_2} \ldots 1C_{4+k_p-1}\varphi_{r_{p-1}}1C_{4+k_p}$$

where $2 \leq p \leq k + 1$; $r_j \in \mathbb{N}$, $1 \leq j \leq (p - 1)$; $\sum_{i=1}^{p} k_i = k + 1$.

We note that a $2C_5$-configuration of type (b) is the split composition of two $1C_5$ (Fig. 2(b)). Hence, we define the following forbidden configuration in a $k$-distance hereditary graph:

Definition 12. Let $G$ be a graph and $u$, $v$ be two nonadjacent vertices of $G$. The pairwise $\{u, v\}$ is a cycle-pair if there exists a shortest induced $\{u, v\}$-path $p_G(u, v)$ and a longest induced $\{u, v\}$-path $P_G(u, v)$ such that $V(p_G(u, v)) \cap V(P_G(u, v)) = \{u, v\}$.

The cycle induced by $V(p_G(u, v)) \cup V(P_G(u, v))$ is denoted by $C_n[u, v]$ where $n = d_G(u, v) + D_G(u, v)$.

Definition 13. Let $G$ be a graph having dilatation number more than $k$ ($k$ a positive integer) and $H$ be a subgraph of $G$.

$H$ is a configuration of type (b) if there exists a cycle-pair $\{u, v\}$ such that $H$ is isomorphic to $C_n[u, v]$ where: $D_G(u, v) = d_G(u, v) + k'$ with $k' > k$ and $d_G(u, v) > 2$.

Lemma 14. Let $G$ be a $k$-distance hereditary graph, $k \geq 1$, and $H$ be an induced subgraph of $G$.

If $H$ is a strict $k$-configuration of type (b), then there exists a cycle pair $\{u, v\}$ such that $H$ is isomorphic to a cycle $C_{2p+k+3}[u, v]$ with

$$cd(C_{2p+k+3}) = p \quad \text{where} \quad 2 \leq p \leq k + 1.$$
\[\text{Case 1. } k_1 + k_2 = k + 1 C_n \text{ contains a } k\text{-configuration of type (a). Contradiction.}\]

\[\text{Case 2. } k_1 + k_2 < k + 1 \text{ Let } C_n' \text{ be the cycle induced by the vertices of } C_n \text{ except } u, y_1, \ldots, y_{r-1}.\]

\[\text{Subcase 2.1. Assume that the vertex } x_2 \text{ is incident to any chord in } C_n'. \text{ Let } (x_1, y_1) \cup Q(y_1, v) \text{ is an induced } \{x_1, v\}\text{-path of length } 2k - k_1 + 3. \text{ Thus, the dilatation number of the pair } \{x_1, v\} \text{ in the cycle } C_n' \text{ is at least } k - k_1 + 1. C_n' \text{ is a } (k - k_1)\text{-configuration of type (b) having a chord distance } k + 1. \text{ By virtue of recurrence hypothesis, } C_n' \text{ strictly contains a } (k - k_1)\text{-configuration.}\]

If this graph is a cycle, it forms with \(C_n\) a \(k\)-configuration strictly included in \(C_n\). Contradiction with the minimality condition of \(C_n\).

If the \((k - k_1)\)-configuration is a union of cycles \(C\) and \(C'\) joined by an induced path, this one forms with \(C_n\) a \(k\)-configuration in \(C_n\). Contradiction.

\[\text{Subcase 2.2. Now, let us assume that there exists no chord incident to } x_2 \text{ in } C_n' \text{ and let } p' \text{ denotes the chord distance of } C_n'. \text{ If } p' > k - k_1 + 1, \text{ then using the same arguments used in the previous subcase, we find a } k\text{-configuration strictly included in } C_n, \text{ contradiction.}\]

Otherwise, let us denote by \(x_a\) the nearest vertex to \(x_2\) which is incident to a chord in the cycle \(C_n'\). The index \(a\) is at least equals to \(k_1 + 2\) since \(p' \leq k - k_1 + 1\). Therefore, \(P(x_{a-1}, x_1) \cup (x_1, y_1) \cup Q(y_1, v)\) is an induced \(\{x_{a-1}, v\}\)-path of length at least \((2k + 3)\). The dilatation number of the cycle \(C_n'\) satisfies:

\[\begin{align*}
\delta(C_n') &\geq \delta_{C_n}(x_{a-1}, v) = D_{C_n}(x_{a-1}, v) - d_{C_n'}(x_{a-1}, v) \\
\delta_{C_n}(x_{a-1}, v) &\geq D_{C_n}(x_{a-1}, v) - d_{C_n'}(x_{a-1}, v) |_{a=k_1+2} \\
&\geq 2k + 3 - (k - k_1 + 2) \\
&\geq k + k_1 + 1 > k + 1.
\end{align*}\]

Thus, the \(k\)-configuration \(C_n\) is neither strict nor minimal. Contradiction to the hypothesis. Consequently, each strict \(k\)-configuration of type (b) having chord distance more than \(k + 1\) is not minimal. \(H\) is a cycle \(C_n[u, v]\) where \(2 \leq cd(C_n[u, v]) = p \leq k + 1.\)

**Definition 15.** Let \(G\) be a graph and \(H\) be an induced subgraph of \(G\). \(H\) is a \(k\)-configuration of type \((a,b)\) if \(H\) is obtained:

- either by joining one (or more) \(k_i\)-configuration(s) of type (a) and one (or more) \(k_j\)-configuration(s) of type (b) with respect to the \(r\)-composition \(r \in \mathbb{N}\) such that \(\sum_i k_i + \sum_j k_j \geq k + 1;\)
- or by joining two (or more) \(k_i\)-configurations of type (b) with respect to the \(r\)-composition \(r \in \mathbb{N}\) such that \(\sum_i k_i \geq k + 1.\)

We note that, among the forbidden subgraphs of 2-distance hereditary, introduced in Section 3, the \(1C_5\varphi_1\varphi_1C_5, r \in \mathbb{N}\) is a 2-configuration of type \((a, b)\). See the last graph of Fig. 3(b). However, the graph \(1C_5\varphi_11C_6, r \in \mathbb{N} \cup \{-1\}\) is a large 2-configuration. (See Fig. 4.)

**Lemma 16.** Let \(G\) be a \(k\)-distance hereditary graph, \(k \geq 1\), and \(H\) be an induced subgraph of \(G\).

If \(H\) is a strict \(k\)-configuration, then \(H\) is isomorphic to one of the following graphs:

1. \(1C_4+k;\)
2. For each integer \(p\) such that \(2 \leq p \leq k + 1\), the graph

\[1C_4+k_1\varphi_11C_4+k_2\varphi_2\ldots1C_4+k_{p-1}\varphi_{p-1}1C_4+k_p\]

where \(1 \leq k_i \leq k\) for \(i = 1, \ldots, p; r_j \in \mathbb{N} \cup \{-1\}\) for \(j = 1, \ldots, p - 1;\) and \(\sum_{i=1}^{p} k_i = k + 1;\)
(3) All graph obtained from (ii) by replacing each cycle $C'$ having a chord distance $cd'$, $cd' \geq 2$, and a dilation number $k'$, $2 \leq k' \leq k + 1$ by a strict $(k' - 1)$-configuration of type (b).

**Proof.** If $H$ is a strict $k$-configuration in the graph $G$, then there exists a pair of vertices $\{u, v\}$ in $H$ such that $D_G(u, v) = d_G(u, v) + k + 1$. We denote by $P$ (respectively $Q$) the shortest (respectively the longest) induced $\{u, v\}$-path in $G$. Thus, either the vertices $u$ and $v$ are linked by a cycle $C_n[u, v]$ or there exists at least a section of the path $Q$ (eventually reduced to one point) belonging to $P$. We suppose that $H$ is minimal else there exists a pair $\{x, y\}$ in $H$ having dilatation number $k$ and constituting a $k$-configuration strictly included in $H$.

Case 1. The pair $\{u, v\}$ is a cycle pair and $H$ is isomorphic to a cycle $C_n[u, v]$ induced by the paths $P$ and $Q$. If the chord distance of $C_n[u, v]$ is at most 1, $C_n[u, v]$ is equivalent to the configuration of case (i). Otherwise, the cycle $C_n[u, v]$ is a $k$-configuration of type (b). Its chord distance is at most $k + 1$ according to the previous lemma. Therefore, $H$ is either isomorphic to the configuration of case (ii) for $r_j = -1$, $j = 1, \ldots, p - 1$ or isomorphic to another strict $k$-configuration of type (b).

Case 2. If there exists at least one subpath of $P$ (eventually reduced to one vertex different than $u$ and $v$) included in $Q$, then $H$ is isomorphic to a union of cycles joined by these subpaths. Since $H$ is strict and minimal, there exists $p$ cycles $2 \leq p \leq k + 1$. Let us denote these cycles by $C_{n_1}, C_{n_2}, \ldots, C_{n_p}$. For each index $i$, $1 \leq i \leq p$, $cd_i$ and $k_i$ are the chord distance and the dilatation number of $C_{n_i}$ respectively. Since $H$ is strict and minimal, the dilatation numbers $k_i$ satisfy $1 \leq k_i \leq k$ and $\sum_{i=1}^{p} k_i = k + 1$.

If $cd_i \leq 1$ for each index $i$, $1 \leq i \leq p$, then the cycle $C_{n_i}$ is equivalent to $1C_{4+k_i}$. Therefore, $H$ is isomorphic to the graph of case (ii) where $r_j \in \mathbb{N}$, $1 \leq j \leq p - 1$. If there exists several indices (not all) $i_0$ such that $C_{n_{i_0}}$ has a chord distance at least 2 and a dilatation number $k_{i_0}$, then this cycle is a strict $(k_{i_0} - 1)$-configuration of type (b).

$H$ is isomorphic to $1C_{4+k_1}C_{4+k_2} \ldots C_{4+k_p}$ ($r_j \in \mathbb{N}$, $1 \leq j < p - 1$) which is a $k$-configuration of type (a,b).

We have to note that the composition of $1C_{4+k_i}$, $\sum_{j=1}^{p-1} k_j = k_{i_0}$, with respect to $\varphi_{-1}$ is a strict $(k_{i_0} - 1)$-configuration of type (b).

Finally, if all cycles have at least a chord distance equals to 2, then $H$ is isomorphic to $C_{1}C_{-1}C_{2} \ldots C_{-1}C_{p}$ where $C_i$ is of type (b) $i = 1, \ldots, p$ and $\sum_j k_j = k + 1$. Therefore, $H$ is isomorphic to a strict $k$-configuration of type (a,b). □

**Theorem 17.** Let $G$ be a graph and $k$ an integer ($k \geq 2$). $G$ is a $k$-distance hereditary graph if and only if $G$ does not contain any of the following subgraphs:

1. $1C_{4+k_i}(k' \geq k + 1)$;
2. A strict $k$-configuration;
3. A large $k$-configuration having dilatation number $k'$ and $k + 1 < k' \leq 2k$.

**Proof.** **Necessary condition.** It is clear that if $G$ contains one of these configurations its dilatation number is greater than $k$ and therefore $G$ is not $k$-distance hereditary graph.

**Sufficient condition.** Let $G$ be a connected graph and assume that $G$ is not $k$-distance hereditary graph. Thus, $G$ contains a pair of vertices $\{u, v\}$ such that $D_G(u, v)$ is at least $d_G(u, v) + k + 1$. Among all the pairs of vertices satisfying this property denoted $\Delta$, we choose the pair $\{u, v\}$ such that $d_G(u, v)$ is as small as possible. We denote by $P$ (respectively $Q$) the smallest (respectively the longest) induced $\{u, v\}$-path in $G$. We are referring to this choice of vertices as the minimality condition. $H$ is the subgraph of $G$ induced by the vertices of the paths $P$ and $Q$.

Case 1. $P \cap Q = \{u, v\}$. The paths $P$ and $Q$ constitute a cycle $C_n$ where $n = d_G(u, v) + D_G(u, v)$. We observe that $d_G(u, v) \geq 2$ and the length of the cycle $C_n$ is at least $5 + k$.

If $d_G(u, v) = d_G(u, v) + k + 1$, this cycle is either $1C_{5+k}(d_G(u, v) = 2)$ or a strict $k$-configuration of type (b) $(d_G(u, v) > 2)$. Otherwise, if $d_G(u, v) = d_G(u, v) + k'(k' > k + 1)$ and $d_G(u, v) = 2$, the cycle-pair $\{u, v\}$ induces a cycle $1C_{4+k'}$.

Now, let us consider the case where $\delta_G(u, v) = k' > k + 1$ and $d_G(u, v) > 2$. The subgraph $H$ is a large $k$-configuration. Let us show that $k' \leq 2k$.

We assume that $x$ is the nearest vertex to $u$ in $P$ having a neighbour in $Q$. Let $x'$ be its neighbour in $Q$ such that $x'$ is the nearest vertex to $v$ (see Fig. 7).
On one hand, the distance \(d_Q(u, x')\) is at most \(3 - d_P(u, x) + k\) otherwise, the cycle induced by \(Q\{u, x'\} \cup (x', x) \cup P(x, u)\) is isomorphic to \(1C_{4+p}\) (\(p \geq k + 1\)) and there exists a vertex in \(P(u, x)\) which forms with \(x'\) a pairwise satisfying the condition (\(\Delta\)) with a distance less than \(d_G(u, v)\). Contradiction to the minimality condition.

On the other hand, if \(d_Q(u, x') \leq d_P(u, x) + 1\), then we have

\[
\begin{align*}
  d_Q(x', v) + 1 &= d_Q(u, v) - d_Q(u, x') + 1 \\
  &\geq d_Q(u, v) - (d_P(u, x) + 1) + 1 \\
  &\geq d_P(u, v) + k' - d_P(u, x) \\
  &\geq d_P(u, v) + k'.
\end{align*}
\]

Then, \((x, x') \cup Q(x', v)\) is an induced \([x, v]\)-path having a length at least \(d_G(x, v) + k'\). Thus, the pair \([x, v]\) verifies the propriety (\(\Delta\)) with \(d_G(x, v) < d_G(u, v)\). Contradiction.

Consequently, \(d_P(u, x) + 2 \leq d_Q(u, x') \leq 3 - d_P(u, x) + k\).

**Subcase 1.1.** If \(d_P(u, x) = 1\), then \(3 \leq d_Q(u, x') \leq 2 + k\). More precisely, \(k' < 2k + 1\). Otherwise the pair \([x, v]\) satisfies the condition (\(\Delta\)) with \(d_G(x, v) < d_G(u, v)\), since

\[
\begin{align*}
  \delta_G(x, v) &= D_G(x, v) - d_G(x, v) \\
  &\geq d_Q(x', v) + 1 - d_P(x, v) \\
  &\geq (d_P(u, v) + 2k + 1 - d_Q(u, x')) + 1 - d_P(x, v) \\
  &\geq d_P(u, v) + 2k + 1 - (2 + k) + 1 - d_P(x, v) \geq k + 1.
\end{align*}
\]

Thus, it contradicts the minimal condition (\(\Delta\)).

**Subcase 1.2.** If \(d_P(u, x) > 1\), then \(Q(u, v)\) has a length \(d_G(u, v) + k'\) with \(k + 1 < k' \leq 2k - 2\). Otherwise, the longest induced \([x, v]\)-path verifies

\[
\begin{align*}
  D_G(x, v) &\geq d_Q(x', v) - 1 = d_Q(u, v) - d_Q(u, x') + 1 \\
  &\geq (d_P(u, v) + 2k - 1) - (3 - d_P(u, x) + k) + 1 \\
  &\geq (d_P(x, v) + 2d_P(u, x) - 4) + (k + 1) \\
  &\geq d_P(x, v) + k + 1.
\end{align*}
\]

Then, the pair \([x, v]\) satisfies the condition (\(\Delta\)) with \(d_G(x, v) < d_G(u, v)\). Contradiction.

Consequently, if \(\delta_G(u, v) > k + 1\), then \(C_n[u, v]\) is a large \(k\)-configuration of type (b) having a dilatation number \(k'\) such that \(k' \leq 2k\).

**Case 2.** \(P \cap Q \neq [u, v]\). There exists at least a subpath of \(P\) (eventually reduced to one point) belonging to the path \(P \cap Q\). Let us denote these subpaths respectively by \(P(y_1, x_2), P(y_2, x_3), \ldots, P(y_{s-1}, x_s)(s \geq 2)\). It is obviously clear that \(d_G(u, y_1) \geq 2; d_G(x_i, y_i) \geq 2 \) for each \(i = 2, \ldots, (s - 1)\) and \(d_G(x_s, v) \geq 2\).

The subgraph \(H\) is isomorphic to cycles \(C_{n_1}, C_{n_2}, \ldots, C_{n_s}\) joined by the paths \(P(y_1, x_2), P(y_2, x_3), \ldots, P(y_{s-1}, x_s)\). See Fig. 8. For each index \(i, 1 \leq i \leq s\), we denote by \(\delta_i\) the dilatation number of \(C_{n_i}\). It is easy to verify that the sum of all the dilatations \(\delta_i\) is equal to \(\delta_G(u, v)\). Furthermore, for each index \(i\), \((1 \leq i \leq s)\), the dilatation number \(\delta_i\) is at most \(k\) otherwise there exists one of these pairs of vertices satisfying the minimal condition (\(\Delta\)). Contradiction.

Let us suppose that \(D_G(u, v) = d_G(u, v) + k + 1\). If \(d_G(u, y_1) = d_G(x_i, y_i) = d_G(x_s, v) = 2, (i = 2, \ldots, (s - 1))\), then the subgraph \(H\) is a strict \(k\)-configuration of type (a). In the other case, if there exists a cycle \(C_{n_i}\) induced by a pair
of vertices at a distance at least 2, $C_{n_i}$ is a strict $(\delta_i - 1)$-configuration of type (b). Then $H$ is a strict $k$-configuration of type (a,b).

Finally, let us suppose that $D_G(u, v) = d_G(u, v) + k'(k' > k + 1)$.

The integer $k'$ satisfies $k' \leq 2k$, since $1 \leq \delta_i \leq k$ and $\sum_{i=1}^{r} \delta_i = \delta_G(u, v)$.

Using the same arguments as above, $H$ is isomorphic either to a large $k$-configuration of type (a) or to a large $k$-configuration of type (a,b) with both of them having a dilatation number $k'$. □

5. Open questions

In this paper, we have characterized $k$-distance hereditary graphs which represent a parametric generalization of distance hereditary class. We have introduced all kinds of $k$-configuration for each $k \geq 1$. Due to the additive concept of these configurations, most of them can be obtained by a recurrent relationship with respect to certain operations in graphs. Despite this, the results provided other interesting problems that remain open.

In the literature, several papers deal with the distance hereditary notion in algorithmic, metric and combinatorial perspectives. For instance, in [2,7], characterizations using these aspects were given. Can we extend some of these results in the $k$-distance hereditary graphs (nevertheless for the case $k = 1$) by providing algorithmic or metric characterizations or by setting some interesting properties used to resolve certain optimizing problems in these graphs?

In [8], Oum characterizes distance hereditary graphs. In fact, he proved that a graph $G$ is distance hereditary if and only if it has a rank width of at most one. Thus, the interesting question which comes to mind is: “Is a $k$-distance hereditary exactly a graph with a bounded rank width for $k \geq 1$?”

References