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FULL LENGTH PAPER

# A note on Robinson–Ursescu and Lyusternik–Graves theorem

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**Abstract** The aim of this note is twofold. First, we prove an analogue of the well-known Robinson–Ursescu Theorem on the relative openness with a linear rate (restrictive metric regularity) of a multivalued mapping. Second, we prove a generalization of Graves Open Mapping Theorem for a class of mappings which can be approximated at a reference point by a bunch of linear mappings. The approximated non-linear mapping is restricted to a closed convex subset of a Banach space.

**Keywords** Open mapping · Restrictive metric regularity · Graves theorem · Robinson–Ursescu theorem · Convex constraints

**Mathematics Subject Classification (2000)** 49J52 · 49J53

A recent book by Dontchev and Rockafellar [3] shows that the study of (equivalent) concepts of non-linear analysis such as linear openness, metric regularity, and inverse Aubin property, is of a great importance. Since this book will possibly serve as a basic reference on the subject, we mostly follow the notation therein. For $X$ being a Banach

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space (always over $\mathbb{R}$), then $\|\cdot\|_X$ denotes its norm, $B_X(x,r)$ and $B_X^O(x,r)$ the closed and open ball with the center $x \in X$ and the radius $r \geq 0$, respectively. The closure and the interior of a subset $K$ of $X$ is denoted by $\overline{K}$, and $\operatorname{int} K$, respectively. By $T : X \rightrightarrows Y$ we denote a multivalued mapping from $X$ into another Banach space $Y$ with the domain $\operatorname{dom} T$, the graph $\operatorname{gph} T$ and the range $\operatorname{rge} T$. Such a mapping $T$ is called *relatively open at* $(x_0, y_0) \in \operatorname{gph} T$ if for each neighborhood $U$ of $x_0$ in $X$ the set $T(U)$ is a relative neighborhood of $y_0$ in $\operatorname{rge} T$, *relatively open at* $(x_0, y_0)$ *with a linear rate* if there are $c > 0$ and $\varepsilon_0 > 0$ such that $T(B_X(x_0, \varepsilon)) \supset \operatorname{rge} T \cap B_Y(y_0, c\varepsilon)$ for every $\varepsilon \in (0, \varepsilon_0]$, and finally, $T$ is called *relatively open around* $x_0$ *with a linear rate* if there are $c > 0$, $\varepsilon_0 > 0$, and a neighborhood $U$ of $x_0$ such that $T(B_X(x, \varepsilon)) \supset \operatorname{rge} T \cap B_Y(y, c\varepsilon)$ whenever $x \in U$, $y \in T(x)$, and $\varepsilon \in (0, \varepsilon_0]$. Replacing, the range of the mapping $T$ with the whole target space $Y$ we get the classical notions of openness at (around) the reference point.

The notion of relative openness for $T$ being single-valued and linear has been used to characterize locally uniformly rotund Banach spaces. Roughly spoken, local uniform rotundity (convexity) of the underlying Banach space is equivalent to equal relative openness of the restriction of quotient maps to its closed unit ball [13, Theorems 3.4 and 3.5]. It is rather surprising that a restriction of a linear map, defined on a three-dimensional space $X$, to the unit ball may not be relatively open. By [13, Lemma 4.4], the relative openness is guaranteed if $X$ is locally uniformly convex. If one considers more general sets than closed balls the situation is much more interesting. One of fundamental questions of convex analysis is the following: *When is the linear image of a closed convex set closed?* For example, this can be applied to find out, when the sum and the convolution of closed convex functions are closed; and to investigate uniform duality in conic linear systems. J. M. Borwein and H. Bauschke discussed the relative openness of the restriction $T|_C$ of a single-valued continuous linear mapping $T$ between Hilbert spaces to a closed convex cone $C$. It is easy to prove, that if $T|_C$ is relatively open at the origin then $T(C)$ is closed (see also [1, Theorem 2.2]). On the other hand, [1, Example 2.3] shows that $T(C)$ may be closed although $T|_C$ is not relatively open at 0. Moreover, [1, Theorem 3.11] provides a necessary and sufficient condition for both the closeness of $T(C)$ and the relative openness of $T|_C$. Recently, this question in finite dimensional setting was also addressed by G. Pataki (e.g., see [12, Theorem 1.1]).

Given a mapping $T : X \to Y$ from a Banach space $(X, \|\cdot\|_X)$ to a metric space $(Y, d)$ and a point $x_0 \in X$ in its domain, we say that $T$ is *metrically regular around* $x_0$ if there exist a number $\mu > 0$ and neighborhoods $U$ of $x_0$ and $V$ of $T(x_0)$ such that

$$\operatorname{dist}\left(x; T^{-1}(y)\right) \leq \mu d(T(x), y) \quad \text{whenever} \quad x \in U \quad \text{and} \quad y \in V,$$

where $\operatorname{dist}(x; T^{-1}(y)) := \inf\{\|x - v\|_X : T(v) = y\}$. Fix a mapping $T : X \to Y$ between Banach spaces for a while. It is well known (see e.g. [8, Theorem 1.52, p. 61]) that $T$ is metrically regular around $x_0 \in X$ if and only if it is open around $x_0$ with a linear rate. Undoubtedly, metric regularity plays an important role in variational analysis and optimization. Recently, Mordukhovich and Wang [9] defined that $T$ is *restrictively metrically regular* around $x_0$ if $T$ regarded as a mapping from $X$ into a metric space rge $T$, whose metric is induced by the norm on $Y$, is metrically regular around $x_0$. Following ideas of the proof of the above mentioned theorem from [8], we immediately get that *the relative openness around $x_0$ with a linear rate is equivalent to the restrictive metric regularity around the same point*.

Robinson–Ursescu theorem (e.g., see [3, Theorem 5B.4, p. 263]) has also found many applications. It can be stated as follows: *A set-valued mapping $T$ from a Banach space $(X, \|\cdot\|_X)$ into another Banach space $(Y, \|\cdot\|_Y)$ with closed convex graph is open at $(x_0, y_0) \in$ gph $T$ with a linear rate whenever $y_0$ is a core point of* rge $T$ *(e.g., an interior point)*. In the first part of this note, we prove an analogue of Robinson–Ursescu theorem on the relative openness of a multivalued mapping. We employ the following geometric properties of sets. Given a subset $K$ of a Banach space $X$ and $x_0 \in K$ we say that $K$ is *locally star-shaped at $x_0$* if there is $\varepsilon > 0$ such that

$$(1-\lambda)x_0 + \lambda x \in K \quad \text{whenever} \quad x \in K \quad \text{and} \quad \lambda \in [0, \varepsilon].$$

A set $K$ is said to be *locally conic at $x_0$* if there exist a neighborhood $W$ of $x_0$ in $X$ and a shifted cone $C$ with vertex $x_0$ such that $K \cap W = C \cap W$. We will prove the following statement.

**Theorem 1** *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces, let $T : X \rightrightarrows Y$, and let $(x_0, y_0) \in$ gph $T$. Suppose that dom $T$ is bounded, that gph $T$ is locally star-shaped at $(x_0, y_0)$, and that rge $T$ is locally conic at $y_0$. Then $T$ is relatively open at $(x_0, y_0)$ with a linear rate. In particular, if $y_0$ is an interior point of rge $T$, then $T$ is open at $(x_0, y_0)$ with a linear rate.*

The following examples show that no single assumption on the mapping in question of the theorem above can be dropped in general.

*Example 1* Let $T : \mathbb{R} \rightrightarrows \mathbb{R}$ be defined by

$$T(x) = \begin{cases} \{x, 1\} & \text{if } x \in [0, 1], \\ \emptyset & \text{otherwise.} \end{cases}$$

Clearly, dom $T = [0, 1]$ is bounded and rge $T = [0, 1]$ is locally conic at 1. But $T$ is not relatively open at (0, 1). Note that gph $T$ is not locally star-shaped at (0, 1).

*Example 2* Let $T : \mathbb{R} \rightrightarrows \mathbb{R}^2$ be defined by

$$T(r) = \begin{cases} \left\{(x, y) \in \mathbb{R}^2 : \ (x-r)^2 + y^2 \le r^2\right\} & \text{if } r \in [0, 1], \\ \emptyset & \text{otherwise.} \end{cases}$$

Then gph $T$ is convex, and hence locally star-shaped at (0, 0, 0), and dom $T = [0, 1]$ is bounded. But $T$ is not relatively open at (0, 0, 0). Note that rge $T = \left\{(x, y) \in \mathbb{R}^2 : \ (x-1)^2 + y^2 \le 1\right\}$ is not locally conic at (0, 0).

*Example 3* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$T(x, y, z) = \begin{cases} (y, z) & \text{if } (x-z)^2 + y^2 \le x^2, \ x \ge 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $\operatorname{dom} T = \{(x, y, z) \in \mathbb{R}^3 : (x - z)^2 + y^2 \le x^2, \ x \ge 0\}$ is a cone in $\mathbb{R}^3$, hence $\operatorname{rge} T$ is a cone in $\mathbb{R}^2$. Clearly, $T$ is not relatively open at $(0, 0, 0, 0, 0)$. Note that $\operatorname{dom} T$ is unbounded.

In the latter part of this note, we prove the following analogue of the well-known Graves' Theorem when the non-linear mapping in question is restricted to a closed convex subset of a Banach space and can be approximated at a reference point by a bunch of mappings from $\mathcal{L}(X, Y)$ (the space of all continuous linear operators from $X$ with values in $Y$ equipped with the operator norm).

**Theorem 2** *Let $K$ be a non-empty closed bounded convex subset of a Banach space $(X, \|\cdot\|_X)$, let $F$ be a continuous mapping from $K$ into a Banach space $(Y, \|\cdot\|_Y)$, let $x_0 \in K$, and let $R > 0$ be such that $K \subset B_X(x_0, R)$. Suppose that $\mathcal{T}$ is a compact convex subset of $\mathcal{L}(X, Y)$, and that $\delta > 0$ is such that for each two distinct elements $k_1$ and $k_2$ of $K$ there is $T \in \mathcal{T}$ such that*

$$\|F(k_1) - F(k_2) - T(k_1 - k_2)\|_Y < \delta \|k_1 - k_2\|_X. \tag{1}$$

*Suppose that there is $r > \delta R$ such that*

$$T(K - x_0) \supset B_Y(0, r) \quad \text{for each} \quad T \in \mathcal{T}. \tag{2}$$

*Then $F$ is open around $x_0$ with a linear rate; more precisely, for every constant $c \in (0, r/R - \delta)$ there exists a neighborhood $U$ of $x_0$ in $K$ such that*

$$F\big(K \cap B_X(x, \varepsilon)\big) \supset B_Y\big(F(x), c\,\varepsilon\big) \quad \text{whenever} \quad x \in U \quad \text{and} \quad \varepsilon \in (0, R].$$

The theorem above provides the same conclusion as [2, Theorem 2 (A1)]. However, the assumptions of these two differ. In the latter case, the non-linear mapping is approximated by a multivalued mapping $T : X \rightrightarrows Y$, with closed convex graph, such that $K - K \subset \operatorname{dom} T$. In addition, it is supposed that for a sufficiently small $\delta > 0$ there is $\varepsilon > 0$ such that

$$\|F(k_1) - F(k_2) - z\|_Y \le \delta \|k_1 - k_2\|_X \tag{3}$$

for each $k_1, k_2 \in K \cap B_X(x_0, \varepsilon)$ and for each $z \in T(k_1 - k_2)$. Further, it is assumed that for some $r > 0$ we have

$$T(K - x_0) \supset B_Y(0, r). \tag{4}$$

Clearly, the condition (1) is much weaker than (3). This is balanced by (2) which is obviously stronger than (4).

*Proof of Theorem 1* We may assume without a loss of generality that $x_0 = 0$ and $y_0 = 0$. Put $M = \operatorname{rge} T$ and $B = \operatorname{dom} T$.

Let $U$ be a neighborhood of 0 in $B$. We `claim` that $T(U)$ is a neighborhood of 0 in $M$. Note that, there is $a \in (0, 1)$ such that

$$t\,B \subset B \quad \text{and that} \quad t\,T(B) \subset T(tB) \quad \text{whenever} \quad t \in [0, a]. \tag{5}$$

To prove (5), as the graph of $T$ is locally star-shaped at (0, 0), there is a constant $a \in (0, 1)$ such that

$$t \operatorname{gph} T \subset \operatorname{gph} T \quad \text{for each} \quad t \in [0, a]. \tag{6}$$

Hence, the first inclusion in (5) is obvious. To prove the second one, let $t \in [0, a]$ be arbitrary. Fix any $z \in tT(B)$. Find $y \in T(B)$ and $x \in B$ such that $y \in T(x)$ and $z = ty$. Then, by (6), $t(x, y) \in \operatorname{gph} T$, thus $ty \in T(tx)$. As $tx \in tB$, we infer that $z = ty \in T(tx) \subset T(tB)$. Since $z$ was arbitrary, (5) is proved.

Now, find a neighborhood $V$ of 0 in $X$ such that $V \cap B \subset U$. Fix an arbitrary $t \in [0, a]$. Since $B$ is bounded, taking $a$ smaller if necessary, we may assume that

$$tB \subset V. \tag{7}$$

Consequently, (5) and (7) yield that

$$tM = t\,T(B) \subset T(tB) = T(tB \cap B) \subset T(V \cap B) \subset T(U).$$

The set $tM$ is a neighborhood of 0 in $M$, hence so is $T(U)$. Indeed, if $W$ is a convex neighborhood of 0 and $C$ is a cone such that $M \cap W = C \cap W$, then $tW$ is also a neighborhood of 0 and

$$\begin{aligned} tM \cap tW &= t(M \cap W) = t(C \cap W) = tC \cap tW = C \cap tW \\ &= (C \cap W) \cap tW = (M \cap W) \cap tW = M \cap tW, \end{aligned}$$

where the latter set is a neighborhood of 0 in $M$. As $T(B) = M$, (5) implies that $tM \subset M$. Therefore $tM$ is a neighborhood of 0 in $M$. Our claim is proved.

Find constants $\alpha > 0$ and $\beta > 0$ such that

$$T\left(B \cap B_X(0, \alpha)\right) \supset M \cap B_Y(0, \beta). \tag{8}$$

We shall find $c > 0$ and $\varepsilon_0 > 0$ such that

$$T(B \cap B_X(0, \varepsilon)) \supset M \cap B_Y(0, c\varepsilon) \quad \text{whenever} \quad \varepsilon \in (0, \varepsilon_0]. \tag{9}$$

Since the graph of $T$ is locally star-shaped at (0, 0), similarly as in the proof of (5), find a constant $a \in (0, 1)$ such that

$$tB \subset B \quad \text{and} \quad t\,T\big(B \cap B_X(0, \alpha)\big) \subset T\big(t(B \cap B_X(0, \alpha))\big) \quad \text{whenever} \quad t \in [0, a]. \tag{10}$$

Since $M$ is locally conic at 0, there exists $\gamma \in (0, \beta)$ and a cone $C$ in $Y$ such that

$$M \cap B_Y(0,r) = C \cap B_Y(0,r) \quad \text{whenever} \quad r \in (0,\gamma]. \tag{11}$$

Put $\varepsilon_0 = \min\{\alpha, \alpha a\}$. Fix an arbitrary $\varepsilon \in (0, \varepsilon_0]$. Put $\lambda = \varepsilon/\alpha$. Note that $\lambda \le \varepsilon_0/\alpha \le a \ (\le 1)$. Hence using the first inclusion in (10), we arrive at

$$B \cap B_X(0,\varepsilon) \supset \lambda B \cap \lambda B_X(0, \varepsilon/\lambda) = \lambda\,[B \cap B_X(0,\alpha)]. \tag{12}$$

Further, put $c = \gamma/\alpha$, hence $c\varepsilon = \lambda\gamma \le \gamma$. A combination of (12), (10), (8), and (11) yields

$$\begin{aligned} T(B \cap B_X(0,\varepsilon)) &\supset T\,(\lambda\,[B \cap B_X(0,\alpha)]) \supset \lambda T\,(B \cap B_X(0,\alpha)) \\ &\supset \lambda\left[M \cap B_Y(0,\gamma)\right] = \lambda\left[C \cap B_Y(0,\gamma)\right] \\ &= C \cap B_Y(0,\lambda\gamma) = M \cap B_Y(0,\lambda\gamma) = M \cap B_Y(0,c\varepsilon). \end{aligned}$$

The inclusion (9) is proved, and so is our theorem. □

Before proving our second main result, let us mention the following approximation lemma. The proof of it combines ideas from [11] and [2].

**Lemma 1** *In addition to the assumptions of Theorem 2, put* $\rho = \frac{\delta R}{r}$, *let* $t \in (0, 1-\varrho]$, *put* $s = t/(1-\varrho)$, *and pick* $v \in Y$ *with* $\|v\|_Y < tr$. *Then there are* $x_1 \in (1-t)x_0 + tK$, $T_0 \in \mathcal{T}$, *and* $x \in (1-s)x_0 + sK$ *such that*

$$T_0(x_1 - x_0) = v, \quad F(x) = F(x_0) + v, \tag{13}$$

$$\|x - x_1\|_X < \varrho s R, \quad \text{and} \quad x - x_1 \in \varrho s(K - x_0). \tag{14}$$

*Proof* Without a loss of generality, we may assume that $x_0 = 0$ and $F(0) = 0$. If $v = 0$ then we put $x = 0$, $x_1 = 0$, and the assertion is satisfied with any $T_0 \in \mathcal{T}$. From now on, suppose that $v \neq 0$. Put $s_0 = 0$ and

$$s_n = \sum_{i=0}^{n-1} t\varrho^i \quad \text{for each} \quad n \in \mathbb{N}.$$

Note that $s_n < s \le 1$ for each $n \in \mathbb{N}$. We will define sequences $(x_n)_{n\in\mathbb{N}}$ in $K$, $(y_n)_{n\in\mathbb{N}}$ in $Y$ and $(u_n)_{n\in\mathbb{N}}$ in $K$, such that these satisfy for each $n = 0, 1, 2, \ldots$ the following conditions:

(i) $x_n \in s_n K$,
(ii) $y_n = v - F(x_n)$,
(iii) $\|y_n\|_Y < t\varrho^n r$,
(iv) $u_n \in t\varrho^n K$,
(v) $\|u_n\|_X < t\varrho^n R$,
(vi) $x_{n+1} = x_n + u_n$.

Putting $y_0 = v$, we infer that the conditions (i)–(iii) are fulfilled for $n = 0$. Further, we proceed inductively. Suppose that for an index $N \geq 0$ the elements $x_N$, $y_N$ satisfying (i)–(iii) with $n = N$ are determined already, and for each non-negative integer $n < N$, the elements $x_n$, $y_n$ and $u_n$ are defined, and all the conditions (i)–(vi) hold true. If $y_N = 0$, we put $u_N = 0$, $x_{N+1} = x_N$ and $y_{N+1} = 0$. Further, let $y_N \neq 0$. By (iii), there is $\alpha \in (0, t\varrho^N)$ such that

$$\|y_N\|_Y < \alpha r. \tag{15}$$

Let us define a set-valued mapping $\Phi$ from $\mathcal{T}$ into $\alpha K$ by

$$\Phi(T) = \{k \in \alpha K : T(k) = y_N\} \quad \text{for} \quad T \in \mathcal{T}.$$

Clearly, for each $T \in \mathcal{T}$, the set $\Phi(T)$ is a closed convex subset of $\alpha K$ and, by (15) and (2), it is also non-empty. We will prove that $\Phi$ is lower semi-continuous on $\mathcal{T}$. Fix an arbitrary $T \in \mathcal{T}$ and let $H$ be an open set in $X$ such that $\Phi(T)$ intersects $H$. We will show that for $\widehat{T} \in \mathcal{T}$ close enough to $T$ we have $\Phi(\widehat{T}) \cap H \neq \emptyset$. To see this, pick an arbitrary $h$ in $\Phi(T) \cap H$. Since $H$ is open, there is $\lambda \in (0, 1)$ such that

$$\lambda h + (1 - \lambda) B_X(0, R) \subset H. \tag{16}$$

Putting $\Delta = (1 - \lambda)\,(\alpha r - \|y_N\|_Y)$, we have $\Delta > 0$ by (15). Let $\widehat{T} \in \mathcal{T}$ be such that $\|\widehat{T} - T\|_{\mathcal{L}(X,Y)} \leq \Delta$. We `claim` that $\Phi(\widehat{T}) \cap H$ is non-empty. Indeed, as $h \in \Phi(T)$, we have $T(h) = y_N$ so that

$$\begin{aligned} \|y_N - \widehat{T}(\lambda h)\|_Y &= \|T(h) - \widehat{T}(\lambda h)\|_Y \\ &\leq \|T(h) - T(\lambda h)\|_Y + \|T(\lambda h) - \widehat{T}(\lambda h)\|_Y \\ &\leq (1 - \lambda)\|y_N\|_Y + \Delta = (1 - \lambda)\alpha r. \end{aligned}$$

Using this and (2), we find $\tilde{h} \in K$ such that

$$y_N - \widehat{T}(\lambda h) = (1 - \lambda)\alpha \widehat{T}(\tilde{h}). \tag{17}$$

Put $\hat{h} = \lambda h + (1 - \lambda)\alpha\tilde{h}$. The convexity of $\alpha K$ implies that $\hat{h} \in \lambda\alpha K + (1 - \lambda)\alpha K \subset \alpha K$. Hence (17) yields that $\widehat{T}(\hat{h}) = y_N$, thus $\hat{h} \in \Phi(\widehat{T})$. Finally, as $\alpha < 1$ and $K \subset B_X(0, R)$, using (16) we obtain that $\hat{h} \in H$ which finishes the proof of the claim. As $T$ was an arbitrary element of $\mathcal{T}$, the lower semi-continuity of $\Phi$ on $\mathcal{T}$ is proved.

By Michael's selection theorem, [4, Theorem 7.53, p. 362], there exists a continuous selection $\Phi_N$ for $\Phi$, hence the definition of $\Phi$ implies that

$$T(\Phi_N(T)) = y_N \quad \text{for each } T \in \mathcal{T}. \tag{18}$$

Since $y_N \neq 0$, it follows that $\Phi_N(T) \neq 0$ for any $T \in \mathcal{T}$, thus $\Phi_N$ is a mapping into $K_\alpha$ where

$$K_\alpha := \alpha K \backslash \{0\}.$$

The inequality $\alpha < t\varrho^N$, the inclusion (i), with $n := N$, and the convexity of $K$ yield that

$$x_N + K_\alpha \subset s_N K + t\varrho^N K \subset K.$$

Therefore we may define a function $f$ from $K_\alpha$ into $Y$ by

$$f(h) := F(x_N + h) - F(x_N) \quad \text{for} \quad h \in K_\alpha.$$

Further, let us define a set-valued mapping $\eta$ from $K_\alpha$ into $\mathcal{T}$ by

$$\eta(h) = \{T \in \mathcal{T} : \|T(h) - f(h)\|_Y \leq \delta \|h\|_X\} \quad \text{for} \quad h \in K_\alpha.$$

For each $h \in K_\alpha$, the set $\eta(h)$ is closed, convex and, by assumption (1), non-empty as well. We `claim` that $\eta$ is lower semi-continuous on $K_\alpha$. To prove this, fix any $h \in K_\alpha$, and let $\Omega$ be an open set in $\mathcal{T}$ such that $\eta(h)$ intersects $\Omega$. Pick any $T$ from $\eta(h) \cap \Omega$. According to (1), there is $\tilde{T} \in \mathcal{T}$ such that

$$\|\tilde{T}(h) - f(h)\|_Y < \delta \|h\|_X.$$

As $\Omega$ is open and $T \in \Omega$, there is $\lambda \in (0, 1)$ such that $T_\lambda := (1 - \lambda)T + \lambda\tilde{T} \in \Omega$. Put

$$V = \{k \in K_\alpha : \|T_\lambda(k) - f(k)\|_Y < \delta \|k\|_X\}.$$

Since

$$\begin{aligned} \|T_\lambda(h) - f(h)\|_Y &\leq (1 - \lambda)\|T(h) - f(h)\|_Y + \lambda \|\tilde{T}(h) - f(h)\|_Y \\ &< (1 - \lambda)\delta\|h\|_X + \lambda\delta\|h\|_X = \delta\|h\|_X, \end{aligned}$$

the choice of $T$ and $\tilde{T}$ yields that $h \in V$, hence employing the continuity of both $T_\lambda$ and $f$, we infer that $V$ is a neighborhood of $h$ in $K_\alpha$. According to definitions of $\eta$ and $V$, $T_\lambda \in \eta(\hat{h})$ for $\hat{h} \in V$ so that $\eta(\hat{h})$ intersects $\Omega$ for each $\hat{h} \in V$, whence the claim follows.

By Michael's selection theorem, [4, Theorem 7.53, p. 362], there is a continuous selection $\eta_N$ for $\eta$, that is a continuous function from $K_\alpha$ into $\mathcal{T}$ such that if $u \in K_\alpha$ and $T := \eta_N(u)$, then

$$\|F(x_N + u) - F(x_N) - T(u)\|_Y \leq \delta \|u\|_X. \tag{19}$$

As $\mathcal{T}$ is a compact convex subset of $\mathcal{L}(X, Y)$, the composite mapping $\eta_N \circ \Phi_N$ from $\mathcal{T}$ into $\mathcal{T}$ has a fixed point, $T_N$ say, by Schauder's fixed point theorem [4, Corollary 12.40, p. 542]. Put

$$u_N = \Phi_N(T_N). \tag{20}$$

Hence $u_N \in \alpha K$. Since $\alpha < t\varrho^N$ and $K \subset B_X(0, R)$, (iv) and (v) with $n := N$ are proved. Define

$$x_{N+1} = x_N + u_N. \tag{21}$$

As $K$ is convex, (i) and (iv) reveal that

$$x_{N+1} \in s_N K + t\varrho^N K \subset s_{N+1} K.$$

Thus (i) holds true with $n := N + 1$. Combining (20) and (18) we infer that

$$T_N(u_N) = T_N(\Phi_N(T_N)) = y_N. \tag{22}$$

Since $T_N$ is a fixed point of $\eta_N \circ \Phi_N$, (20) reveals that $T_N = \eta_N(u_N)$. Therefore, by the very definition of the mapping $\eta_N$, (19) holds true with $u := u_N$ and $T := T_N$. This together with (21) and (22) yields that

$$\|F(x_{N+1}) - F(x_N) - y_N\|_Y \leq \delta \|u_N\|_X. \tag{23}$$

Define $y_{N+1}$ by (ii) with $n := N + 1$. Then using (ii) with $n := N$ we arrive at

$$y_{N+1} = y_N - (F(x_{N+1}) - F(x_N)).$$

This, (23), and (v) reveal that

$$\|y_{N+1}\|_Y < \delta t \varrho^N R = t\varrho^{N+1} r.$$

The induction step is finished. Note that, as $y_0 = v \neq 0$, an element $T_0 \in \mathcal{T}$ was found in such a way that, in consequence of (21) and (22), $T_0(x_1) = T_0(u_1) = y_0 = v$. By virtue of (vi) and (v), there exits

$$\lim_{n\to\infty} x_n = \sum_{n=0}^{\infty} u_n =: x.$$

Using (i), we infer that $x_1 \in tK$ and $x \in sK$. By (ii) and (iii), employing the continuity of $F$ we get $F(x) = v$. Further,

$$x - x_1 = x - u_0 = \sum_{n=1}^{\infty} u_n. \tag{24}$$

Hence, the inequality in (14) follows from (v). A convexity argument, together with (24) and (iv), yields the inclusion in (14). Our lemma is proved. □

*Proof of Theorem 2* First, putting $\rho = \frac{\delta R}{r}$, we `claim` that

$$F\big(K \cap B_X(x_0, \varepsilon)\big) \supset B_Y^O(F(x_0), (1-\varrho)r\varepsilon/R) \quad \text{whenever} \quad \varepsilon \in (0, R];$$

Indeed, assume that $x_0 = 0$ and $F(0) = 0$. Let $\varepsilon \in (0, R]$ be arbitrary. Put $t = (1-\varrho)\varepsilon/R$ and $s = \varepsilon/R$. In consequence Lemma 1, for each $v \in Y$ with $\|v\|_Y < tr$ there is $x \in sK \subset K$ such that $F(x) = v$. As $K \subset B_X(0, R)$, we have $sK \subset B_X(0, \varepsilon)$, whence the claim follows.

Now, let $c \in \big(0, (1-\varrho)r/R\big)$ be arbitrary. As $\varrho = \delta R/r < 1$, there is $\tilde{r} \in (0, r)$ such that

$$\tilde{\varrho} := \delta R/\tilde{r} < 1 \quad \text{and} \quad c < (1-\tilde{\varrho})\tilde{r}/R.$$

Since $\mathcal{T}$ is compact, $\mathcal{T} \subset B_{\mathcal{L}(X,Y)}(0, \gamma)$ for some $\gamma > 0$. Put

$$U = \left\{x \in K : \|x - x_0\|_X \le \frac{1}{2\gamma}(r - \tilde{r})\right\}.$$

Fix an arbitrary $T \in \mathcal{T}$ and $x \in U$. Then

$$\|T(x_0 - x)\|_Y \le \gamma \|x - x_0\|_X \le \frac{1}{2}(r - \tilde{r}).$$

Using (2), we infer that

$$T(K - x) = T(K - x_0) + T(x_0 - x) \supset B_Y\big(T(x_0 - x), r\big).$$

Therefore, $T(K - x) \supset B_Y(0, \tilde{r})$. Applying the claim with $x$, $\tilde{r}$, and $\tilde{\varrho}$ instead of $x_0$, $r$, and $\varrho$, respectively, we get the conclusion. The theorem is proved. □

Ioffe [7] initiated the use of the strict pre-derivatives to approximate a non-smooth mapping $F$ from a Banach space $X$ into a Banach space $Y$ around the reference point $x_0 \in X$. Let $K$ be a subset of $X$ which contains $x_0$. A homogenous set-valued mapping $A : X \rightrightarrows Y$ is called the *strict pre-derivative of $F$ at $x_0$ relative to $K$* if for each $c > 0$ there exists $\Delta > 0$ such that

$$F(x_1) \in F(x_2) + A(x_1 - x_2) + c\|x_1 - x_2\|_X B_Y(0, 1), \tag{25}$$

whenever $x_1, x_2 \in K \cap B_X(x_0, \Delta)$. We focus on the case when the strict pre-derivative is generated by a family of continuous linear operators in such a way that there is a subset $\mathcal{T}$ of $\mathcal{L}(X, Y)$ such that $A(x) = \{T(x) : T \in \mathcal{T}\}$ for each $x \in X$. It is well known that the generalized Jacobian in the sense of Clarke is a strict convex-valued pre-derivative for a locally Lipschitz-continuous function between finite dimensional spaces $X$ and $Y$ (see [7, Corollary 9.11]). Páles [10] provides conditions ensuring that a homogenous set-valued mapping with closed convex values can be represented by a convex set of continuous linear operators. Clearly, the strict pre-derivative of $F$ at $x_0$ (relative to $X$) reduces to a singleton if and only if $F$ is strictly differentiable at $x_0$. This notion was used implicitly by Graves [5] in the proof of his well-known open mapping theorem.

Páles [11] defined two quantities associated to any subset $\mathcal{T}$ of $\mathcal{L}(X, Y)$. The first, *measure of non-compactness of* $\mathcal{T}$, denoted by $\chi(\mathcal{T})$, is defined by

$$\chi(\mathcal{T}) = \inf \left\{ r > 0 : \ \mathcal{T} \subset \bigcup \left\{ B_{\mathcal{L}(X,Y)}(L, r) : \ L \in \mathcal{F} \right\}, \ \mathcal{F} \subset \mathcal{T} \text{ finite} \right\}.$$

And the latter, *modulus of (linear) openness of* $\mathcal{T}$ *on* $K$ *at* $x_0$, denoted by $\sigma(\mathcal{T}, K, x_0)$, is defined by

$$\sigma(\mathcal{T}, K, x_0) = \inf \{\sigma(T, K, x_0) : \ T \in \mathcal{T}\},$$

where $\sigma(T, K, x_0) := \sup \left\{ r > 0 : \ B_Y(T(x_0), r) \subset T(K \cap B_X(x_0, 1)) \right\}$ is the *modulus of (linear) openness of* $T$ *on* $K$ *at* $x_0$.

**Corollary 1** *Let* $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ *be Banach spaces, let* $K$ *be a closed convex subset of* $X$ *which contains* $x_0 \in X$*, and let* $F : K \to Y$ *be a continuous mapping. Suppose that* $F$ *possesses the strict pre-derivative at* $x_0$ *relative to* $K$ *generated by a subset* $\mathcal{T}_0$ *of* $\mathcal{L}(X, Y)$ *such that* $\chi(\mathcal{T}_0) < \sigma(\mathcal{T}_0, K, x_0)$*. Then* $F$ *is open around* $x_0$ *with a linear rate.*

*Proof* We may assume without a loss of generality that $x_0 = 0$ and $F(0) = 0$. Put $\widetilde{K} = K \cap B_X(0, 1)$. Take $\alpha, \beta > 0$ such that $\chi(\mathcal{T}_0) < \alpha < \beta < \sigma(\mathcal{T}_0, K, x_0)$. Let $\varepsilon > 0$ be such that $\chi(\mathcal{T}_0) + 2\varepsilon < \alpha$. By the assumption, there is a finite set $\mathcal{T}_1 := \{T_1, \ldots, T_n\} \subset \mathcal{T}_0$ such that $\mathcal{T}_0 \subset \mathcal{T}_1 + B_{\mathcal{L}(X,Y)}(0, \chi(\mathcal{T}_0) + \varepsilon)$. Denote by $\mathcal{T}$ the closed convex hull of $\mathcal{T}_1$. Clearly, $\mathcal{T}$ is convex and compact. Since $\mathcal{T}_0$ generates the strict pre-derivative of $F$ at 0 relative to $K$, find $\Delta > 0$ such that (25) is satisfied (for $A(x) := \{T(x) : \ T \in \mathcal{T}_0\}$, $x \in X$, and $c := \alpha - \chi(\mathcal{T}_0) - 2\varepsilon$). Fix any two distinct elements $x_1$ and $x_2$ of $K \cap B_X(0, \Delta)$. Use (25) to find $T_0 \in \mathcal{T}_0$ such that

$$\|F(x_1) - F(x_2) - T_0(x_1 - x_2)\|_Y \le (\alpha - \chi(\mathcal{T}_0) - 2\varepsilon)\|x_1 - x_2\|_X.$$

Pick $T \in \mathcal{T}_1 \subset \mathcal{T}$ such that

$$\|(T - T_0)h\|_Y \le (\chi(\mathcal{T}_0) + \varepsilon)\|h\|_X \quad \text{for each} \quad h \in X.$$

Hence, the triangle inequality and the fact that $x_1 \neq x_2$ reveal that

$$\|F(x_1) - F(x_2) - T(x_1 - x_2)\|_Y \le (\alpha - \varepsilon)\|x_1 - x_2\|_X < \alpha\|x_1 - x_2\|_X.$$

We `claim` that

$$T(\widetilde{K}) \supset B_Y(0, \beta) \quad \text{whenever} \quad T \in \mathcal{T}. \tag{26}$$

To prove this, fix an arbitrary $T \in \mathcal{T}$. Let $\varepsilon > 0$ be such that $\beta + 2\varepsilon < \sigma(\mathcal{T}_0, K, x_0)$. Hence, there is $\widetilde{T} := \sum_{i=1}^{n} \lambda_i T_i$ for some $\lambda_i \geq 0$ with $\sum_{i=1}^{n} \lambda_i = 1$ such that $\|T - \widetilde{T}\|_{\mathcal{L}(X,Y)} < \varepsilon$. As

$$T_i(\widetilde{K}) \supset B_Y(0, \beta + 2\varepsilon) \quad \text{for each} \quad i = 1, \dots, n,$$

we infer that

$$\widetilde{T}(\widetilde{K}) \supset B_Y(0, \beta + 2\varepsilon).$$

Now, Theorem 2 with $\widetilde{K}$, 1, $T|_K$, $\{\widetilde{T}\}$, $\varepsilon$, and $\beta + 2\varepsilon$ instead of $K$, $R$, $F$, $\mathcal{T}$, $\delta$, and $r$, respectively, implies the claim.

Put $R = \min\{\Delta, 1\}$ and $\widehat{K} = \widetilde{K} \cap B_X(0, R) = K \cap B_X(0, R)$. As $RK \subset K$, (26) says that

$$T(\widehat{K}) \supset T(R[K \cap B_X(0, 1)]) \supset B_Y(0, R\beta) \quad \text{whenever} \quad T \in \mathcal{T}.$$

Since $\alpha < \beta$, putting $r = R\beta$ and $\delta = \alpha$, we infer that all the assumptions of Theorem 2, with $K$ replaced by $\widehat{K}$, are satisfied. The proof is finished. □

Actually, Theorem 2 is a particular case of the Corollary 1 corresponding to the zero measure of non-compactness. This result extends an earlier work by Páles [11] in such a way that an additional convex constraint is present. One can find many other possible extensions of the classical Lyusternik–Graves theorem, e.g. one can consider metric space setting only (see [3, Theorem 5E.1]). A comprehensive treatment in this setting can be found in [6]. We learned from a referee that Theorem 2 can be proved also more directly using modern regularity theory instead of Michael's selection theorem. Namely, [6, Theorem 2b, p. 516] would yield the statement if one calculates the strong slope [6, Definition 4, p. 515] of appropriately chosen distance functions. The convexity of the constraint set would play a role in this calculation. In fact, Theorem 2 fails if the set in question is not convex. Although it is not the aim of this note, we would like to point out that one can deduce many other statements from Lemma 1, e.g. concerning the properties of $F^{-1}$; the existence, uniqueness, and continuous dependence on the right-hand side of the solution to the non-linear operator equation $F(x) = y$ (for $y$ in a vicinity of $F(x_0)$) as can be seen from Corollary 7, Corollary 8, and Theorem 9 in [2].

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## References

1. Bauschke, H., Borwein, J.M.: Conical open mapping theorems and regularity. Proc. Cent. Math. Appl. **36**, 1–10 (1999)
2. Cibulka, R.: Constrained open mapping theorem with applications. J. Math. Anal. Appl. **379**, 205–215 (2011)

3. Dontchev, A.L., Rockafellar, R.T.: Implicit Functions and Solution Mappings. A View from Variational Analysis. Springer, Dordrecht (2009)
4. Fabian, M., Hájek, P., Habala, P., Montesinos, V., Zizler, V.: Banach Space Theory: The Basis for Linear and Nonlinear Analysis. CMS Books in Mathematics. Springer, New York (2011)
5. Graves, L.M.: Some mapping theorems. Duke Math. J. **17**, 111–114 (1950)
6. Ioffe, A.D.: Metric regularity and subdifferential calculus. Uspehi Mat. Nauk **55**, 103–162 (2000)
7. Ioffe, A.D.: Nonsmooth analysis: differential calculus of nondifferentiable mappings. Trans. Am. Math. Soc. **266**, 1–56 (1981)
8. Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation I: Basic Theory. Springer, Berlin (2006)
9. Mordukhovich, B.S., Wang, B.: Restrictive metric regularity and generalized differential calculus in Banach spaces. Int. J. Math. Math. Sci. **50**, 2653–2680 (2004)
10. Páles, Z.: Linear selections for set-valued functions and extensions of bilinear forms. Arch. Math. **62**, 427–432 (1994)
11. Páles, Z.: Inverse and implicit function theorems. J. Math. Anal. Appl. **209**, 202–220 (1997)
12. Pataki, G.: On the closedness of the linear image of a closed convex cone. Math. Oper. Res. **32**, 395–412 (2007)
13. Reif, J.: A characterization of (locally) uniformly convex spaces in terms of relative openness of quotient maps on the unit ball. J. Funct. Anal. **177**, 1–15 (2000)