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Volterra Series Models for Nonlinear System Control

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Abstract

In this paper, a new non-linear Volterra series model is built based upon positive, negative and double step responses. The predictive control algorithm for non-linear system is then proposed, and the existence and uniqueness of the solution are proved mathematically. As a simulation study, the new predictive control algorithm is applied to a non-linear robot system.

1. Introduction

In the late 1970s', a new digital control algorithm, predictive control was applied in real control systems and its advantage over PID control was distinguished. Predictive control is a general scheme of a type of heuristic control algorithm developed independently in different fields, particularly in process control. It is characterized by: (1) using the impulse or step response as the model of the plant, (2) introducing the control law through a moving horizontal optimization, and (3) correcting the prediction values by a closed-loop algorithm [3], [4].

Up to the present, however, the application of predictive control has mainly been restricted to the systems that can be well described by a linear model because the impulse response model is only valid for the above-mentioned systems. Now, we will extend it to the non-linear systems especially to the robotics.

In robotics, there are many places where nonlinear processes exist. The non-linearity to be controlled includes motor dynamics, flexible link vibrations, harmonic drive stiffness, gear backlash, and full arm dynamics. For a single motor driven multi-degree-of-freedom manipulator, the related clutches and mechanics make a tremendous contribution to the non-linearity.

Traditional identification methods deliver a nominal model that only approximately describes the dynamics of a plant. In order to cope with this approximation, an increased realization so far is focusing on the quantification of the uncertainty in the robust control system and identification of a suitable nominal model for high performance control design. Recently, several researchers have extended the investigation for Volterra series to the control-relevant domain. [7], [9], [10], [11], [12].

Volterra series are utilized as the nominal model for a non-linear system. In this paper, this non-parametric model for non-linear system is built up and its representation by positive, negative and double step responses is presented at first. Unlike the impulse signal, since the step signal is easy to be generated, this method is really practicable. Based on this model, a predictive control algorithm for non-linear system is proposed. The existence and uniqueness of the solution are proved mathematically thereafter. Finally, a simulation study for a non-linear robot system is illustrated as a conclusion section.

2. Non-Parametric Predictive Model of Stable Non-Linear System

Predictive control is a general scheme of a type of heuristic control algorithm developed independently in different fields, particularly in process control. It is characterized by: (1) using the impulse or step response as the model of the plant, (2) introducing the control law through a moving horizontal optimization, and (3) correcting the prediction values by a closed-loop algorithm. Now, we will extend it to the non-linear systems.

2.1. Volterra Series Representation of Non-Linear System

Predictive control algorithm for linear system is based on the system I/O relationship. Considering that Volterra series is a generalization of impulse response representation, the I/O relationship of the non-linear system studied in this paper is represented with Volterra series [1] to derive a new predictive control algorithm:

$$\begin{aligned} y(k) &= \sum_{i=0}^{k-1} h_1(k-i)u(i) + \\ &\sum_{i_1=0}^{k-1}\sum_{i_2=0}^{k-1} h_2(k-i_1, k-i_2)u(i_1)u(i_2) + \cdots \\ &+ \sum_{i_1=0}^{k-1}\sum_{i_2=0}^{k-1}\cdots\sum_{i_n=0}^{k-1} h_n(k-i_1, k-i_2, \cdots, k-i_n)u(i_1)u(i_2)\cdots u(i_n) + \cdots, \\ &k \ge 1 \end{aligned} \tag{1}$$

For most of the plant in process industries, Volterra series with up to second-order kernel are enough to describe their main dynamic characteristics. So, (1) can be simplified as follows:

$$y(k)=\sum_{i=0}^{k-1}h_1(k-i)u(i) + \sum_{i=0}^{k-1}\sum_{j=0}^{k-1}h_2(k-i,k-j)u(i)u(j)+\cdots \quad (2)$$

To avoid static deviation, we used to choose its incremental form:

$$y(k)=\sum_{i=0}^{k-1}h_1(k-i)\Delta u(i) + \sum_{i=0}^{k-1}\sum_{j=0}^{k-1}h_2(k-i,k-j)\Delta u(i)\Delta u(j)+\cdots \quad (3)$$

(3) is the mathematical description to non-linear system's I/O throughout this paper. Now we derive the method to obtain the non-parametric predictive model.

2.2. Method to Obtain the Non-Parametric Predictive Model

In actual application, we are used to analyzing the system dynamic characteristic through positive, negative, and double step responses. It is reasonable to denote

$$x(k)=\begin{cases}1, k\geq 0\\ 0, k\prec 0\end{cases},$$

and choose the inputs as:

$u_p(k)=x(k)$; $u_n(k)=-x(k)$;
$u_{2p}(k,i,j)=x(k-i)+x(k-j)$;
$u_{2n}(k,i,j)=-x(k-i)-x(k-j)$.

We can get their zero state responses $y_p(k)$, $y_n(k)$, $y_{2p}(k,i,j)$, $y_{2n}(k,i,j)$, respectively.

As the system is time-invariant, we'd better notice the following two relationships:

$y_{2p}(k,i,j)=y_{2p}(k-i,0,j-i)$ and
$y_{2n}(k,i,j)=y_{2n}(k-i,0,j-i)$, for $0\leq i<j$.

Now it is easy to get

$$h_1(k)=(y_p(k)-y_n(k))/2, k\geq 1 \quad (4)$$

$$h_2(k,k)=(y_p(k)+y_n(k))/2, k\geq 1 \quad (5)$$

$$h_2(i,j)=(y_{2p}(i,0,i-j) + y_{2n}(i,0,i-j)-2h_2(i,i)-2h_2(j,j))/4, i>j\geq 1 \quad (6)$$

2.3. Non-Parametric Predictive Control Model For Non-Linear Systems

According to (4), (5) and (6), we can determine $h_1(k)$, $h_2(i,j)$, by $u_p(k)$, $u_n(k)$, $u_{2p}(i,0,i-j)$ and $u_{2n}(i,0,i-j)$, and their zero state responses $y_p(k)$, $y_n(k)$, $y_{2p}(i,0,i-j)$, $y_{2n}(i,0,i-j)$. If we specify $u(i)=0$, for $i<0$, the predictive model (3) can be rewritten as follows:

$$y(k+l|k)=\sum_{i=0}^{k+l-1}h_1(k+l-i)\Delta u(i|k) + \sum_{i=0}^{k+l-1}\sum_{j=0}^{k+l-1}h_2(k+l-i,k+l-j)\Delta u(i|k)\Delta u(j|k)+\cdots \quad (7)$$

where $y(k+l|k)$ is a predictive output of $y(k+l)$ at time $t=k\times T$; $\Delta u(i|k)=\Delta u(i)$, for $i<k$; $\Delta u(i|k)$ is an assumed value of $\Delta u(i)$ at time $t=k\times T$, for $i\geq k$.

In practical application, we should notice that there exists a truncating length N for $h_1(k)$ and $h_2(i,j)$. But due to length limitation, the problem is omitted here.

3. Existence and Uniqueness of the Solution of Predictive Control for Non-Linear Systems

3.1. Problem Description

In this paper, we choose the performance index for multi-step predictive control as follows:

$$J(k)=\sum_{i=1}^{M}[y_d(k+i)-y_m(k+i|k)]^2 r(i) + \sum_{i=1}^{M}\Delta u^2(k+i-1|k)w(i), k\geq 0 \quad (8)$$

Where

- M: length of control horizon;
- $r(i)$: weight coefficients of error with $r(i)>0$, for $i=1,2,\cdots,M$;
- $w(i)$: weight coefficients of control with $w(i)>0$, for $i=1,2,\cdots,M$;
- $y_d(k+i)$: desired value of $y(k+i)$.

In order to improve the robustness of the control system and make the control action as smooth as possible, we assume that the control variables considered in optimization horizon are equal, i.e.

$\Delta u(k|k) = \Delta u$, and

$\Delta u(k+1|k) = \cdots = \Delta u(k+M-1|k) \overset{\Delta}{=} 0$.

Based on (7), considering the control variables and closed-loop correction, we can build up a predictive model as follows:

$$
\begin{aligned}
& y_m(k+l\mid k) \\
&= \sum_{i=0}^{k+l-1} h_1(k+l-i)\Delta u(i\mid k) \\
&+ \sum_{i=0}^{k+l-1}\sum_{j=0}^{k+l-1} h_2(k+l-i,k+l-j)\Delta u(i\mid k)\Delta u(j\mid k) \\
&= y(k) + \sum_{i=1}^{k-1}[h_1(k+l-i) - h_1(k-i)]\Delta u(i) \\
&+ \sum_{i=0}^{k-1}\sum_{j=0}^{k-1}[h_2(k+l-i,k+l-j) - h_2(k-i,k-j)]\Delta u(i)\Delta u(j) \qquad (9) \\
&+ \sum_{i=1}^{l} h_1(i)\Delta u(k+l-i\mid k) \\
&+ 2\sum_{i=1}^{l}\left[\sum_{j=0}^{k-1} h_2(i,k+l-j)\Delta u(j)\right]\Delta u(k+l-i\mid k) \\
&+ \sum_{i=1}^{l}\sum_{j=1}^{l} h_2(i,j)\Delta u(k+l-i\mid k)\Delta u(k+l-j\mid k)
\end{aligned}
$$

$l = 1,2,\cdots,M, k \ge 0$, and $y(0) = 0$.

Now , it is easy to show that

$$
\begin{aligned}
& y_m(k+l\mid k) \\
&= y(k) + \sum_{i=0}^{k-1}[h_1(k+l-i) - h_1(k-i)]\Delta u(i) \\
&+ \sum_{i=0}^{k-1}\sum_{j=0}^{k-1}[h_2(k+l-i,k+l-j) - h_2(k-i,k-j)]\Delta u(i)\Delta u(j) \\
&+ \left\{h_1(l) + 2\sum_{j=0}^{k-1} h_2(l,k+l-j)\Delta u(j)\right\}\Delta u + \{h_2(l,l)\}\Delta u^2
\end{aligned}
\qquad (10)
$$

$l = 1,2,\cdots,M, k \ge 0$, and $y(0) = 0$. Denote

$$a(l) = h_2(l,l)$$

$$b(k,l) = h_1(l) + 2\sum_{j=0}^{k-1} h_2(l,k+l-j)\Delta u(j)$$

$$
\begin{aligned}
g(k,l) &= y(k) + \sum_{i=0}^{k-1}[h_1(k+l-i) - h_1(k-i)]\Delta u(i) \\
&+ \sum_{i=0}^{k-1}\sum_{j=0}^{k-1}[h_2(k+l-i,k+l-j) - h_2(k-i,k-j)]\Delta u(i)\Delta u(j) - y_d(k+l)
\end{aligned}
$$

$l = 1,2,\cdots,M, k \ge 0$, and $y(0) = 0$. Then, (8) can be rewritten as

$$
\begin{aligned}
J[k,\Delta u] &= \sum_{i=1}^{M}[a(i)\Delta u^2 + b(k,i)\Delta u + g(k,i)]^2\, r(i) + \Delta u^2 w(1) \\
&= a\Delta u^4 + b(k)\Delta u^3 + c(k)\Delta u^2 + d(k)\Delta u + e(k)
\end{aligned}
\qquad (11)
$$

Where

$$a = \sum_{i=1}^{M} a^2(i)\, r(i)\,;$$

$$b(k) = \sum_{i=1}^{M} 2a(i)\, b(k,j)\, r(i)\,;$$

$$c(k) = \sum_{i=1}^{M}[b^2(k,i) + 2a(i)g(k,i)]r(i) + w(1)\,;$$

$$d(k) = \sum_{i=1}^{M} 2b(k,i)g(k,i)\, r(i)\,;$$

$$e(k) = \sum_{i=1}^{M} g^2(k,i)\, r(i)\,.$$

Problem: The problem of predictive control for non-linear systems is equivalent to the finding of a control variable Δu^* which minimizes the performance index (11), i.e., $J[k,\Delta u^*] = \min_{\Delta u} J[k,\Delta u]$.

3.2. Existence and Uniqueness of the Solution

Theorem 1 For Problem, there exists at least one solution Δu^*, which minimizes $J[k,\Delta u]$ and satisfies $\frac{\P J}{\P \Delta u}|_{\Delta u^*} = 0$.

Proof: Assume $a(1) = h_2(1,1) \ne 0$, and denote $w = w(1) > 0$. When $|\Delta u| \to +\infty$

$$\sum_{i=1}^{M}[a(i)\Delta u^2 + b(k,i)\Delta u + g(k,i)]^2\, r(i) \to +\infty$$

Thus there exists a large number $N > 0$, such that

$$\sum_{i=1}^{M}[a(i)\Delta u^2 + b(k,i)\Delta u + g(k,i)]^2\, r(i) > \sum_{i=1}^{M} g^2(k,i)\, r(i)\,,$$

$\left|\Delta u\right| \geq N$

Then, for $\left|\Delta u\right| \geq N$

$$J[k,\Delta u] > \sum_{i=1}^{M} g^2(k,i)\, r(i) + w\Delta u^2 > \sum_{i=1}^{M} g^2(k,i)\, r(i) = J[k,0] \tag{12}$$

If we denote

$$G = \max_{1\leq i\leq M}\left\{\left|a(i)\sqrt{r(i)}\right|, \left|b(k,i)\sqrt{r(i)}\right|, \left|g(k,i)\sqrt{r(i)}\right|, w\right\}$$

Then in the real interval $[-N, N]$, it holds that

$0 < J \leq MG^2(N^4 + 2N^3 + 3N^2 + 2N + 1) + GN^2 =$ constant.

It implies that J is continuous and bounded for $-N \leq \Delta u \leq N$. According to the Weierstrass Theorem, there is at least one Δu^* in the closed interval which minimizes $J[k,\Delta u]$. Based on (12), we have $J[k,-N] > J[k,0]$ and $J[k,N] > J[k,0]$. That means $\left|\Delta u^*\right| \neq N$ or Δu^* is an inner point of $[-N, N]$. Therefore the optimal Δu^* satisfies $\left.\frac{\P J}{\P \Delta u}\right|_{\Delta u^*} = 0$. Q.E.D.

Theorem 2 For $\Delta u_1 \neq \Delta u_2$, they are both the solutions of the Problem, only if $\Delta u_3 = -\frac{3b}{4a} - \frac{2bc - 12ad}{8ac - 3b^2}$ is a solution of the equation $\frac{\P J}{\P \Delta u} = 0$.

Proof : Based on $J = a\Delta u^4 + b\Delta u^3 + c\Delta u^2 + d\Delta u + e$, we have $\frac{\P J}{\P \Delta u} = 4a\Delta u^3 + 3b\Delta u^2 + 2c\Delta u + d$. By using polynomial division, J can be expressed as follows:

$$J = \left(\frac{\Delta u}{4} + \frac{b}{16a}\right)\frac{\P J}{\P \Delta u} + \left(\frac{c}{2} - \frac{3b^2}{16a}\right)\Delta u^2 + \left(\frac{3d}{4} - \frac{bc}{8a}\right)\Delta u + \left(e - \frac{bd}{16a}\right) \tag{13}$$

If there exists $\Delta u_1 \neq \Delta u_2$,
s.t. $J[k,\Delta u_1] = J[k,\Delta u_2] = \min_{\Delta u} J[k,\Delta u]$, then

$$\left.\frac{\P J}{\P \Delta u}\right|_{\Delta u_1} = \left.\frac{\P J}{\P \Delta u}\right|_{\Delta u_2} = 0$$

According to (13), we can obtain:

$$\begin{aligned} &J[k,\Delta u_1] - J[k,\Delta u_2] = \\ &\left(\frac{c}{2} - \frac{3b^2}{16a}\right)(\Delta u_1^2 - \Delta u_2^2) + \left(\frac{3d}{4} - \frac{bc}{8a}\right)(\Delta u_1 - \Delta u_2) \\ &= (\Delta u_1 - \Delta u_2)\left[\left(\frac{c}{2} - \frac{3b^2}{16a}\right)(\Delta u_1 + \Delta u_2) + \left(\frac{3d}{4} - \frac{bc}{8a}\right)\right] \\ &= 0 \end{aligned}$$

Since $\Delta u_1 \neq \Delta u_2$, we have

$\left(\frac{c}{2} - \frac{3b^2}{16a}\right)(\Delta u_1 + \Delta u_2) + \left(\frac{3d}{4} - \frac{bc}{8a}\right) = 0$, i.e.

$$\Delta u_1 + \Delta u_2 = \frac{2bc - 12ad}{8ac - 3b^2} \tag{14}$$

From the relation of the roots and coefficients of the equation $\frac{\P J}{\P \Delta u} = 0$,

$$\Delta u_1 + \Delta u_2 + \Delta u_3 = -\frac{3b}{4a} \Rightarrow \Delta u_3 = -\frac{3b}{4a} - \frac{2bc - 12ad}{8ac - 3b^2}$$

Q.E.D

Since the calculations of a, b, c, d, and e are based on the non-parametric model and the previous control values, and it is almost impossible to satisfy the necessary condition. Therefore, we can claim that the optimal Δu^* is unique under the probabilistic meaning.

Theorem 3 For real numbers Δu_1 and Δu_2, with $\Delta u_1 < \Delta u_2$, and they are both the solutions of the Problem, if and only if

$\Delta u_3 = -\frac{3b}{4a} - \frac{2bc - 12ad}{8ac - 3b^2}$ is the solution of the equation $\frac{\P J}{\P \Delta u} = 0$;

The criterion function Δ of the equation $\frac{\P J}{\P \Delta u} = 0$ is less than zero;

$\frac{\P^2 J}{\P \Delta u^2}\Big|_{\Delta u_3} = 0$, with $\Delta u_1 < \Delta u_3 < \Delta u_2$, and $J[k,\Delta u_3]$ is a local maximum value.

Proof: Omitted.

4. Algorithm of Predictive Control for Non-Linear Systems

4.1. Solution of Predictive Control

The equation $\frac{¶J}{¶\Delta u} = 4a\Delta u^3 + 3b\Delta u^2 + 2c\Delta u + d = 0$ can be rewritten as

$$y^3 + py + q = 0 \qquad (15)$$

Where $y = \Delta u + \frac{b}{4a}$;

$$p = \frac{2c}{4a} - \frac{3b^2}{16a^2} = \frac{8ac - 3b^2}{16a^2};$$

$$q = \frac{d}{4a} - \frac{b^3}{64a^3} - \frac{2bc}{16a^2} + \frac{2b^3}{64a^3} = \frac{8a^2d + b^3 - 4abc}{32a^3}.$$

Let $\Delta = (\frac{q}{2})^2 + (\frac{p}{3})^3$ be the criterion function of equation (15).

(i)If $\Delta > 0$, there is only one real solution of the equation:

$$\Delta u^* = (-\frac{q}{2} + \sqrt{\Delta})^{\frac{1}{3}} + (-\frac{q}{2} - \sqrt{\Delta})^{\frac{1}{3}} - \frac{b}{4a}$$

(ii) If $\Delta \le 0$, we define $S = (-\frac{p}{3})^{\frac{3}{2}}$ and $q = \frac{1}{3}\arccos(-\frac{q}{2S})$. Get

$$\Delta u_1 = 2\sqrt[3]{S}\cos q - \frac{b}{4a}$$

$$\Delta u_2 = \sqrt[3]{S}\cos(q + \frac{2p}{3}) - \frac{b}{4a}$$

$$\Delta u_3 = \sqrt[3]{S}\cos(q + \frac{4p}{3}) - \frac{b}{4a}$$

Choose Δu^* among Δu_1, Δu_2 and Δu_3 by comparing $J[k,\Delta u_1]$, $J[k,\Delta u_2]$, $J[k,\Delta u_3]$ so that $J[k,\Delta u^*] = \min_{\Delta u_1,\Delta u_2,\Delta u_3} J[k,\Delta u]$.

4.2. Algorithm

Step 1: Input the kernel of Volterra Series $h_1(i), h_2(i,j), (i,j = 1,2,\cdots,N)$, let $k = 0$. Assume $\Delta u(-1) = \Delta u(-2) = \cdots = \Delta u(-N) = u(-1) = y(0) = 0$. Calculate $a(i), (i = 1,\cdots,M)$, and a.

Step 2: Calculate $y_d(k+i), b(k,i)(i = 1,\cdots,M)$, and $b(k)$.

Step 3: Input $y(k)$. Calculate

$g(k,i), (i = 1,\cdots,M), c(k), d(k), e(k), p, q, \Delta$.

Step 4: Calculate Δ.
If $\Delta > 0$, then go to step 5; else go to step 6.

Step 5: Calculate

$$\Delta u^* = (-\frac{q}{2} + \sqrt{\Delta})^{\frac{1}{3}} + (-\frac{q}{2} - \sqrt{\Delta})^{\frac{1}{3}} - \frac{b}{4a}.$$

Go to step 7.

Step 6: Calculate Δu_1, Δu_2 and Δu_3, and then choose Δu^* among Δu_1, Δu_2 and Δu_3 so that $J[k,\Delta u^*] = \min_{\Delta u_1,\Delta u_2,\Delta u_3} J[k,\Delta u]$.

Step 7: Output $u(k) = u(k-1) + \Delta u^*$, $k \Leftarrow k+1$. If $k*T \ge t_f$, then exit; or else go to step 2.

5. Simulation Studies in Control

The new predictive control algorithm has been simulated for a nonlinear with the model:

$$\dot{q} = -0.1755q(t) - 0.05088q(t)u(t) - 1.745u(t) + 0.003671u^2(t) - 0.4768u^3(t)$$

By choosing the tuning parameters as $T = 0.5$, $N = 60$, $M = 10$, $r(i) = 0.15^i$, $w(1) = 0.4$ and $q_d = 3.0$, We get the following satisfactory simulation results (Figure. 1.: [1]: $u(t)$; [2]: $q_d = 3.0$; [3]: $q(t)$).

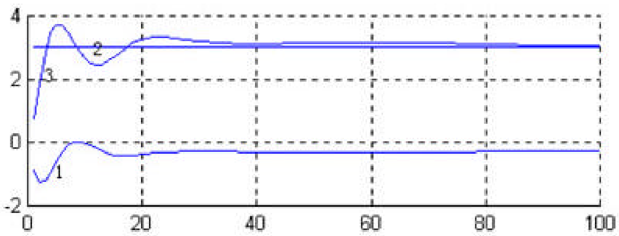

Fig. 1. Results of simulation of proposed control method

6. Conclusion

In this paper, we have proposed a new method for modeling non-linear system with Volterra series. The related predictive control algorithm for non-linear system is also presented. The proposed design is based upon positive, negative and double step responses. Since the existence and uniqueness of the solution are proved mathematically, we can give out the practical algorithm for robot control systematically.

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