ASYMPTOTIC APPROXIMATIONS FOR SYMMETRIC ELLIPTIC INTEGRALS

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Dedicated to Dick Askey and Frank Olver in gratitude for many years of friendship.

Abstract. Symmetric elliptic integrals, which have been used as replacements for Legendre's integrals in recent integral tables and computer codes, are homogeneous functions of three or four variables. When some of the variables are much larger than the others, asymptotic approximations with error bounds are presented. In most cases they are derived from a uniform approximation to the integrand. As an application the symmetric elliptic integrals of the first, second, and third kinds are proved to be linearly independent with respect to coefficients that are rational functions.

Key words. Elliptic integral, asymptotic approximation, inequalities, hypergeometric R-function

AMS(MOS) subject classifications. primary 33A25, 41A60 26D15; secondary 33A30, 26D20

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Abbreviated title

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1 Introduction

A recent table of elliptic integrals [9, 10, 11, 12, 13] uses symmetric standard integrals instead of Legendre's integrals because permutation symmetry makes it possible to unify many of the formulas in previous tables. Fortran codes for numerical computation of the symmetric integrals, which are homogeneous functions of three or four variables, can be found in several major software libraries as well as in the supplements to [9, 10]. For analytical purposes it is desirable to know how the homogeneous functions behave when some of the variables are much larger than the others. For all such cases we list in Section 2 asymptotic approximations (sometimes two or three approximations of different accuracy), always with error bounds. Proofs are discussed in Section 3. In most cases the approximations are obtained by replacing the integrand by a uniform approximation. Many of the results found by a different method in [16] have been improved by sharpening the error bounds or by finding bounds for incomplete elliptic integrals that are still useful for the complete integrals, which are then not listed separately. Cases not considered in [16] include two for a completely symmetric integral of the second kind and two for a symmetric integral of the third kind in which two variables are much larger than the other two.

We assume that x, y, z are nonnegative and at most one of them is 0. The symmetric integral of the first kind,

$$R_F(x,y,z) = \frac{1}{2}\int_0^\infty [(t+x)(t+y)(t+z)]^{-1/2} dt, \tag{1}$$

is homogeneous of degree $-1/2$ in x, y, z and satisfies $R_F(x,x,x) = x^{-1/2}$. The symmetric integral of the third kind,

$$R_J(x,y,z,p) = \frac{3}{2}\int_0^\infty [(t+x)(t+y)(t+z)]^{-1/2}(t+p)^{-1} dt, \quad p > 0, \tag{2}$$

is homogeneous of degree $-3/2$ in x, y, z, p and satisfies $R_J(x,x,x,x) = x^{-3/2}$. If $p = z$, R_J reduces to an integral of the second kind,

$$R_D(x,y,z) = R_J(x,y,z,z) = \frac{3}{2}\int_0^\infty [(t+x)(t+y)]^{-1/2}(t+z)^{-3/2} dt, \quad z > 0, \tag{3}$$

which is symmetric in x and y only. If two variables of R_F are equal, the integral becomes an elementary function,

$$R_C(x,y) = R_F(x,y,y) = \frac{1}{2}\int_0^\infty (t+x)^{-1/2}(t+y)^{-1}dt, \quad y > 0. \tag{4}$$

If $x < y$ it is an inverse trigonometric function,

$$R_C(x,y) = (y-x)^{-1/2}\arccos(x/y)^{1/2}, \tag{5}$$

and if $x > y$ it is an inverse hyperbolic function,

$$R_C(x,y) = (x-y)^{-1/2}\text{arccosh}(x/y)^{1/2} = (x-y)^{-1/2}\ln\frac{\sqrt{x}+\sqrt{x-y}}{\sqrt{y}}. \tag{6}$$

If the second argument of R_C is negative, the Cauchy principal value is [18, (4.8)]

$$R_C(x,-y) = \left(\frac{x}{x+y}\right)^{1/2} R_C(x+y,y), \quad y > 0. \tag{7}$$

If the fourth argument of R_J is negative, the Cauchy principal value is given by [18, (4.6)]

$$\begin{aligned}(y+p)R_J(x,y,z,-p) &= (q-y)R_J(x,y,z,q) - 3R_F(x,y,z) \\ &+ 3\left(\frac{xyz}{xz+pq}\right)^{1/2} R_C(xz+pq,pq), \quad p > 0,\end{aligned} \tag{8}$$

where $q - y = (z-y)(y-x)/(y+p)$. If we permute the values of x, y, z so that $(z-y)(y-x) \geq 0$, then $q \geq y > 0$.

A completely symmetric integral of the second kind is not as convenient as R_D for use in tables because its representation by a single integral is more complicated [7, (9.1-9)]:

$$R_G(x,y,z) = \frac{1}{4}\int_0^\infty [(t+x)(t+y)(t+z)]^{-1/2}\left(\frac{x}{t+x}+\frac{y}{t+y}+\frac{z}{t+z}\right)tdt. \tag{9}$$

It is symmetric and homogeneous of degree 1/2 in x, y, z, and it satisfies $R_G(x,x,x) = x^{1/2}$. It has a nice representation by a double integral that expresses the surface area of an ellipsoid [7, (9.4-6)]. It is related to R_D and R_F by (58) and by

$$6R_G(x,y,z) = x(y+z)R_D(y,z,x) + y(z+x)R_D(z,x,y) + z(x+y)R_D(x,y,z), \tag{10}$$

$$6R_G(x,y,0) = xy[R_D(0,x,y) + R_D(0,y,x)]. \tag{11}$$

Legendre's complete elliptic integrals K and E are given by

$$
\begin{aligned}
K(k) &= R_F(0, 1-k^2, 1), && (12)\\
E(k) &= 2R_G(0, 1-k^2, 1)\\
&= \frac{1-k^2}{3}[R_D(0, 1-k^2, 1) + R_D(0, 1, 1-k^2)], && (13)\\
K(k) - E(k) &= \frac{k^2}{3} R_D(0, 1-k^2, 1), && (14)\\
E(k) - (1-k^2)K(k) &= \frac{k^2(1-k^2)}{3} R_D(0, 1, 1-k^2). && (15)
\end{aligned}
$$

Approximations and inequalities for K, E, and some combinations thereof are given in [1, 2, 3]. If the error terms in (30), (31), and (53) are omitted, the approximations reduce to the leading terms of well-known series expansions of K and E for k near 1 [15, p. 54] [4, 900.06, 900.10]. If the series for K is truncated after any number of terms, simple bounds for the *relative* error are given in [14, (1.17)]. A generalization of this series to $R_F(x, y, z)$ with $x, y << z$ is given in [14, (1.14)-(1.16)], again with simple bounds for the relative error of truncation.

The various functions designated by R with a letter subscript are special cases of the multivariate hypergeometric R-function,

$$R_{-a}(b_1, \ldots, b_n;\ z_1, \ldots, z_n),$$

which is symmetric in the indices $1, \ldots, n$ and homogeneous of degree $-a$ in the variables $z_1, \ldots, z_n$. Best regarded as the Dirichlet average of x^{-a} [7, § 5.9], it is a symmetric variant of the function known as Lauricella's F_D. By the method of Mellin transforms, series expansions are obtained in [8, (4.16)-(4.19)] that converge rapidly if some of the z's are much larger than the others and if the parameters satisfy $\sum_{i=1}^{n} b_i > a > 0$. Thus the leading terms of these series provide asymptotic approximations for all except R_G among the functions

$$
\begin{aligned}
R_F(x, y, z) &= R_{-\frac{1}{2}}(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2};\ x, y, z), \quad R_C(x, y) = R_{-\frac{1}{2}}(\tfrac{1}{2}, 1;\ x, y), && (16)\\
R_J(x, y, z, p) &= R_{-\frac{3}{2}}(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 1;\ x, y, z, p), \quad R_D(x, y, z) = R_{-\frac{3}{2}}(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2};\ x, y, z), && (17)\\
R_G(x, y, z) &= R_{\frac{1}{2}}(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2};\ x, y, z). && (18)
\end{aligned}
$$

However, error bounds for the approximations are more easily derived by the methods of the present paper. Another function that is used repeatedly in obtaining error bounds is [7, Ex. 9.8-5]

$$R_{-1}(\tfrac{1}{2},\tfrac{1}{2},1;\,x,y,z) = \int_0^\infty [(t+x)(t+y)]^{-1/2}(t+z)^{-1}dt \tag{19}$$

$$= 2R_C((\sqrt{xy}+z)^2,(\sqrt{x}+\sqrt{y})^2 z). \tag{20}$$

In Section 4 the asymptotic approximations are applied to show that $R_F(x,y,z)$, $R_D(x,y,z), R_J(x,y,z,p)$, and $(xyz)^{-1/2}$ are linearly independent with respect to coefficients that are rational functions of x, y, z, and p. An Appendix contains some elementary inequalities that are used in obtaining error bounds.

The results in this paper provide upper and lower approximations that approach the elliptic integrals as selected ratios of the variables approach zero. Approximations that approach the integrals as all variables approach a common value have been found by other methods. For example, the theory of hypergeometric mean values yields upper and lower algebraic approximations for all the integrals in this paper [5, Thm. 2], while truncation of Taylor series about the arithmetic mean of the variables gives approximations with errors that may be positive or negative. Successive applications of the duplication theorem for R_F, making its three variables approach equality, provide ascending and descending sequences of successively sharper (and successively more complicated) algebraic approximations to R_F and R_C [6]. Transcendental approximations that approach R_F when only two of its variables approach equality are furnished by

$$R_C\left(x,\frac{y+z}{2}\right) \le R_F(x,y,z) \le R_C(x,\sqrt{yz}), \qquad yz \neq 0, \tag{21}$$

which follows from (71). The inequalities can be sharpened by first using Landen or Gauss transformations of R_F [7, § 9.5] to make y and z approach equality. If $x = 0$ the Gauss transformation reduces to replacing $\sqrt{y}$ and $\sqrt{z}$ by their arithmetic and geometric means, and each R_C-function becomes $\pi/2$ divided by the square root of its second argument. Therefore, in the complete case the procedure reduces to the algorithm of the arithmetic-geometric mean [7, (6.10-6)(9.2-3)] and provides ascending and descending sequences of algebraic approximations, of which leading members are shown in (33).

2 Results

We assume throughout that x, y, and z are nonnegative and at most one of them is 0. The last argument of R_C, R_D, and R_J is assumed to be positive (see (7) and (8)).

C1. $\boldsymbol{R_C(x,y)}$ **with** $\boldsymbol{x << y}$**.**

$$R_C(x,y) = \frac{\pi}{2\sqrt{y}} - \frac{\sqrt{x}}{y} + \frac{\pi x \theta}{4y^{3/2}}\,, \tag{22}$$

where $1/(1+\sqrt{x/y}) \leq \theta \leq 1$ with equalities iff $x = 0$.

C2. $\boldsymbol{R_C(x,y)}$ **with** $\boldsymbol{y << x}$**.** Two approximations of different accuracy are

$$R_C(x,y) = \frac{1}{2\sqrt{x}}\left(\ln\frac{4x}{y} + \frac{y}{2x-y}\ln\frac{\theta_1 x}{y}\right) \tag{23}$$

$$= \frac{1}{2\sqrt{x}}\left[\left(1+\frac{y}{2x}\right)\ln\frac{4x}{y} - \frac{y}{2x} + \frac{3y^2}{4x(2x-y)}\ln\frac{\theta_2 x}{y}\right], \tag{24}$$

where $1 < \theta_i < 4$ for $i = 1, 2$. The first approximation implies

$$R_C(x,y) < \frac{1}{2\sqrt{x}(1-y/2x)}\ln\frac{4x}{y}. \tag{25}$$

F1. $\boldsymbol{R_F(x,y,z)}$ **with** $\boldsymbol{x, y << z}$**.** Let $a = (x+y)/2$, $g = \sqrt{xy}$, and $\rho = max\{x,y\}/z$. Then

$$R_F(x,y,z) = \frac{1}{2\sqrt{z}}\left(\ln\frac{8z}{a+g} + \frac{r}{2z}\right), \tag{26}$$

where

$$\frac{g}{1-g/z}\ln\frac{2z}{a+g} < r < \frac{a}{1-a/2z}\ln\frac{8z}{a+g}.$$

The upper bound implies

$$R_F(x,y,z) < \frac{1}{2\sqrt{z}(1-a/2z)}\ln\frac{8z}{a+g}. \tag{27}$$

A sharper lower bound and a higher-order approximation are given by

$$R_F(x,y,z) = \frac{1}{2\sqrt{z}}\left(\ln\frac{8z}{a+g} + \frac{ar_1}{2z}\right) \tag{28}$$

$$= \frac{1}{2\sqrt{z}}\left[\left(1+\frac{a}{2z}\right)\ln\frac{8z}{a+g} - \frac{2a-g}{2z} + \frac{3(3a^2-g^2)r_2}{16z^2}\right], \tag{29}$$

where

$$\ln\frac{z}{2a} < \frac{\ln(1/\rho)}{1-\rho} < r_i < \frac{1}{1-a/2z}\ln\frac{8z}{a+g}\,, \quad i=1,2.$$

By (12) this implies (since $4k^2 < 4-k'^2$ if $k^2<1$)

$$K(k) = \ln\frac{4}{k'} + \frac{k'^2}{4-k'^2}\ln\frac{\theta_1}{k'} \tag{30}$$

$$= \left(1+\frac{k'^2}{4}\right)\ln\frac{4}{k'} - \frac{k'^2}{4} + \frac{9k'^4}{16(4-k'^2)}\ln\frac{\theta_2}{k'}, \tag{31}$$

where $0 < k' = \sqrt{1-k^2}$ and $1<\theta_i<4$ for $i=1,2$.

F2. $R_F(x,y,z)$ with $z << x,y$. Let $a=(x+y)/2$ and $g=\sqrt{xy}$. Then

$$R_F(x,y,z) = R_F(x,y,0) - \frac{\sqrt{z}}{g} + \frac{\pi z\theta}{4g^{3/2}}, \tag{32}$$

where $1/(1+\sqrt{z/g}) < \theta < a/g$. Note that $R_F(x,y,0) = \pi/2AGM(\sqrt{x},\sqrt{y})$, where AGM denotes Gauss's arithmetic-geometric mean [7, (6.10-6)(9.2-3)], and hence

$$\frac{1}{\sqrt{a}} \le \sqrt{\frac{2}{a+g}} \le \frac{2}{\sqrt{(a+g)/2}+\sqrt{g}} \le \frac{2}{\pi}R_F(x,y,0) \le \left(\frac{2}{ag+g^2}\right)^{1/4} \le \frac{1}{\sqrt{g}}\,, \tag{33}$$

with equalities iff $x=y$.

D1. $R_D(x,y,z)$ with $x,y << z$. Let $a=(x+y)/2$ and $g=\sqrt{xy}$. Then

$$R_D(x,y,z) = \frac{3}{2z^{3/2}}\left(\ln\frac{8z}{a+g} - 2 + \frac{\theta}{z}\ln\frac{2z}{a+g}\right), \tag{34}$$

where

$$\frac{g}{1-g/z} < \theta < \frac{3a}{2(1-a/z)}\,.$$

D2. $R_D(x,y,z)$ with $z << x,y$. Let $a=(x+y)/2$ and $g=\sqrt{xy}$. Then

$$R_D(x,y,z) = \frac{3}{\sqrt{xyz}}\left(1-\frac{\pi\theta}{2}\sqrt{\frac{z}{g}}\right), \tag{35}$$

where

$$1 - \frac{4}{\pi}\sqrt{\frac{z}{g}} < \theta < \frac{a}{g}.$$

A higher-order approximation is

$$R_D(x,y,z) = \frac{3}{\sqrt{xyz}} - R_D(0,x,y) - R_D(0,y,x) + \frac{3\pi\theta\sqrt{z}}{2g^2(1+\sqrt{z/g})}, \tag{36}$$

where

$$\frac{1}{\sqrt{2/3+\sqrt{z/g}}} < \theta < \frac{3a}{2g(1+\sqrt{z/g})}.$$

An approximation of still higher order is

$$R_D(x,y,z) = \frac{3}{\sqrt{xyz}} - \frac{6}{xy}R_G(x,y,0) + \frac{6a\sqrt{z}}{g^3}\left(1 - \frac{\pi\theta}{4}\sqrt{\frac{z}{a}}\right), \tag{37}$$

where we have used (11) and where

$$\frac{1}{1+\sqrt{z/a}} < \theta < \left(\frac{a}{g}\right)^{3/2}\left(3 - \frac{g^2}{a^2}\right),$$

D3. $R_D(x,y,z)$ with $y, z << x$. Let $a = (y+z)/2$ and $g = \sqrt{yz}$. Then

$$R_D(x,y,z) = \frac{3}{\sqrt{x}}\left(\frac{1}{g+z} - \frac{r}{4x}\right), \tag{38}$$

where

$$\frac{1}{1-g/x}\ln\frac{2x}{a+g} - \frac{2z}{g+z} < r < \frac{1}{1-a/2x}\ln\frac{8x}{a+g}.$$

D4. $R_D(x,y,z)$ with $x << y, z$. Let $a = (y+z)/2$ and $g = \sqrt{yz}$. Then

$$R_D(x,y,z) = R_D(0,y,z) + \frac{3\sqrt{x}}{gz}\left(-1 + \frac{\pi\theta}{4}\sqrt{\frac{x}{a}}\right), \tag{39}$$

where

$$\frac{1}{1+\sqrt{x/a}} < \theta < \left(\frac{a}{g}\right)^{3/2}\left(1 + \frac{y}{a}\right).$$

J1. $\boldsymbol{R_J(x,y,z,p)}$ **with** $\boldsymbol{x,y,z << p}$**.** Let $a = (x+y+z)/3$ and $b = (\sqrt{3}/2)(xy + xz + yz)^{1/2}$. Then

$$R_J(x,y,z,p) = \frac{3}{p} R_F(x,y,z) + \frac{3\pi}{2p^{3/2}}(-1+r), \tag{40}$$

where

$$\frac{\sqrt{b/p}}{1+\sqrt{b/p}} < r < \frac{3}{2}\,\frac{\sqrt{a/p}}{1+\sqrt{a/p}}.$$

In the complete case a sharper result is

$$R_J(x,y,0,p) = \frac{3}{p}\left(R_F(x,y,0) - \frac{\pi}{2\sqrt{p}}\right)\left(1 + \frac{\theta/p}{1-\theta/p}\right), \tag{41}$$

where $\sqrt{xy} \le \theta \le (x+y)/2$ with equalities iff $x = y$.

J2. $\boldsymbol{R_J(x,y,z,p)}$ **with** $\boldsymbol{p << x,y,z}$**.** Let $g = (xyz)^{1/3}$, $3h^{-1} = x^{-1} + y^{-1} + z^{-1}$, and $\lambda = \sqrt{xy} + \sqrt{xz} + \sqrt{yz}$. Note that g is the geometric mean and h is the harmonic mean, whence $g \ge h$ with equality iff $x = y = z$. Then

$$R_J(x,y,z,p) = \frac{3}{2\sqrt{xyz}}\left(\ln\frac{4g}{p} - 2 + r\right), \tag{42}$$

where

$$-\ln\frac{g}{h} < r < \frac{3p}{2(g-p)}\ln\frac{g}{p}.$$

A higher-order approximation is

$$R_J(x,y,z,p) = \frac{3}{2\sqrt{xyz}}\ln\frac{4xyz}{p\lambda^2} + 2R_J(x+\lambda, y+\lambda, z+\lambda, \lambda) + \frac{3pr}{4\sqrt{xyz}}, \tag{43}$$

where

$$\frac{2}{g-p}\ln\frac{g}{p} < r < \frac{3}{h-p}\ln\frac{h}{p}.$$

The second term in the approximation is independent of p but is otherwise as complicated as the function being approximated. The same is true of an even more accurate approximation [16, Thm. 11] in which the error is of order p instead of $p \ln p$ and the leading term involves R_C.

J3. $\boldsymbol{R_J(x,y,z,p)}$ **with** $\boldsymbol{x,y << z,p.}$ Let $a=(x+y)/2$ and $g=\sqrt{xy}$. Then

$$R_J(x,y,z,p)=\frac{3}{2\sqrt{z}p}\left[\ln\frac{8z}{a+g}-2R_C\left(1,\frac{p}{z}\right)+\frac{\theta}{p}\ln\frac{2p}{a+g}\right], \tag{44}$$

where

$$\frac{g}{1-g/p}<\theta<\frac{a}{1-a/p}\left(1+\frac{p}{2z}\right).$$

J4. $\boldsymbol{R_J(x,y,z,p)}$ **with** $\boldsymbol{z,p << x,y.}$ Let $a=(x+y)/2$, $g=\sqrt{xy}$, $b=\sqrt{3p(p+2z)}/2$, and $d=(z+2p)/3$. Then

$$R_J(x,y,z,p)=\frac{3}{g}R_C(z,p)-\frac{3\theta}{g-p}\left[R_C(z,g)-\frac{p}{g}R_C(z,p)\right], \tag{45}$$

where $1\le\theta\le a/g$ with equalities iff $x=y$. Since $z<<g$, $R_C(z,g)$ can be estimated from (22). In the complete case (45) reduces to

$$R_J(x,y,0,p)=\frac{3\pi}{2\sqrt{xyp}}\left(1-\frac{\theta\sqrt{p}}{\sqrt{g}+\sqrt{p}}\right) \tag{46}$$

with θ as before. A higher-order approximation is

$$R_J(x,y,z,p)=\frac{3}{g}R_C(z,p)-\frac{6}{xy}R_G(x,y,0)+\frac{3\pi\theta}{2xy}, \tag{47}$$

where we have used (11) and where

$$\frac{\sqrt{b}}{1+\sqrt{b/g}}<\theta<\frac{3a}{2g}\,\frac{\sqrt{d}}{1+\sqrt{d/g}}.$$

J5. $\boldsymbol{R_J(x,y,z,p)}$ **with** $\boldsymbol{x << y,z,p.}$ Let $a=(y+z)/2$ and $g=\sqrt{yz}$. Then

$$R_J(x,y,z,p)=R_J(0,y,z,p)+\frac{3\sqrt{x}}{gp}\left(-1+\frac{\pi\theta}{4}\sqrt{\frac{x}{g}}\right), \tag{48}$$

where

$$\frac{\sqrt{g/a}}{1+\sqrt{x/a}} < \theta < \frac{a}{g} + \frac{g}{p}.$$

J6. $\boldsymbol{R_J(x,y,z,p)}$ **with** $\boldsymbol{y,z,p << x}$**.** Let $a = (y+z)/2$ and $g = \sqrt{yz}$. Then

$$R_J(x,y,z,p) = \frac{3}{\sqrt{x}}\left[R_C((g+p)^2, 2(a+g)p) - \frac{r}{4}\right], \tag{49}$$

where

$$\frac{1}{x-g}\ln\frac{2x}{a+g} - \frac{2p}{x}R_C((g+p)^2, 2(a+g)p) < r < \frac{1}{x-a/2}\ln\frac{8x}{a+g}.$$

In the complete case this reduces to

$$R_J(x,0,z,p) = \frac{3}{\sqrt{xp}}R_C(p,z) - \frac{3s}{4x^{3/2}}, \tag{50}$$

where

$$\ln\frac{4x}{z} - 2\sqrt{p}R_C(p,z) < s < \frac{1}{1-z/4x}\ln\frac{16x}{z}.$$

G1. $\boldsymbol{R_G(x,y,z)}$ **with** $\boldsymbol{x,y << z}$**.** Let $a = (x+y)/2$ and $g = \sqrt{xy}$. Then

$$R_G(x,y,z) = \frac{\sqrt{z}}{2}\left(1 + \frac{r}{2z}\right), \tag{51}$$

where

$$\frac{a+g}{2}\ln\frac{2z}{a+g} + 2g - \frac{4a}{3} < r < (3a-g)\ln\frac{2z}{a+g} + 2g - \frac{a}{3}.$$

In the right-hand inequality it is assumed that $5a < z$. A sharper result for the complete case is

$$R_G(0,y,z) = \frac{\sqrt{z}}{2} + \frac{y}{8\sqrt{z}}\left(\ln\frac{16z}{y} - 1 + \frac{ys}{2z}\right), \tag{52}$$

where

$$\frac{3}{4}\ln\frac{z}{y} < s < \frac{1}{1-y/z}\left(\ln\frac{16z}{y} - \frac{13}{6}\right).$$

By (13) this follows from

$$E(k) = 1 + \frac{k'^2}{2}\left(\ln\frac{4}{k'} - \frac{1}{2} + k'^2 r\right), \tag{53}$$

where $0 < k' = \sqrt{1-k^2} << 1$ and

$$\frac{3}{8}\ln\frac{1}{k'} < r < \frac{1}{k(1+k)}\left(\ln\frac{4}{k'} - \frac{13}{12}\right).$$

G2. $\boldsymbol{R_G(x,y,z)}$ **with** $\boldsymbol{z << x, y}$**.** Let $a = (x+y)/2$ and $g = \sqrt{xy}$. Then

$$R_G(x,y,z) = R_G(x,y,0) + \pi\theta z/8, \tag{54}$$

where

$$\frac{1}{\sqrt{a}}\left(1 - \frac{4}{\pi}\sqrt{\frac{z}{a}}\right) < \theta < \left(\frac{2}{ag+g^2}\right)^{1/4} \le \frac{1}{\sqrt{g}}$$

with equality iff x = y.

3 Proofs

Most of the results in Section 2 are obtained by replacing an integrand f by an approximation f_a, writing $\int f = \int f_a + \int(f - f_a)$, and finding upper and lower bounds for $\int(f - f_a)$. All integrals are taken over the positive real line. The function f_a is usually chosen to be a uniform approximation $f_a = f_i + f_o - f_m$, where f_i is an approximation in the inner region, f_o in the outer region, and f_m in the overlap region or matching region. For instance, if $f(t) = [(t+x)(t+y)(t+z)]^{-1/2}$ with $x, y << z$, we get f_i by neglecting t compared to z, f_o by neglecting x and y compared to t, and f_m by doing both. A first example of this process is the proof of Lemma 1.

Lemma 1. *If $x \ge 0$, $y \ge 0$, and $0 < x + y << z$, then*

$$\int_0^\infty \frac{dt}{\sqrt{(t+x)(t+y)(t+z)}} = \frac{1}{z-\theta}\ln\frac{2z}{a+g}, \tag{55}$$

where $\sqrt{xy} = g \le \theta \le a = (x+y)/2$ with equalities iff $x = y$.

Proof. Let

$$\begin{aligned} f(t) &= \frac{1}{\sqrt{(t+x)(t+y)(t+z)}}, \quad f_i(t) = \frac{1}{z\sqrt{(t+x)(t+y)}}, \\ f_o(t) &= \frac{1}{t(t+z)}, \qquad\qquad f_m(t) = \frac{1}{zt}\,. \end{aligned}$$

Taking $f_a = f_i + f_o - f_m$, we find

$$\int_0^\infty f_a(t)dt = \frac{1}{z}\ln\frac{2z}{a+g}$$

and

$$f - f_a = \frac{t}{z(t+z)}\left(\frac{1}{t} - \frac{1}{\sqrt{(t+x)(t+y)}}\right).$$

Inequality (64) in the Appendix implies

$$f - f_a = \frac{\theta}{z\sqrt{(t+x)(t+y)(t+z)}}, \qquad g \le \theta \le a,$$

and thus

$$\int f = \int f_a + \int (f - f_a) = \int f_a + \frac{\theta}{z}\int f = \frac{1}{1-\theta/z}\int f_a = \frac{1}{z-\theta}\ln\frac{2z}{a+g}. \qquad \square$$

As a second example, in which Lemma 1 is used, consider $R_F(x, y, z)$ with $x, y << z$. Let

$$\begin{aligned} f(t) &= \frac{1}{\sqrt{(t+x)(t+y)(t+z)}}, \quad f_i(t) = \frac{1}{\sqrt{(t+x)(t+y)z}}, \\ f_o(t) &= \frac{1}{t\sqrt{t+z}}, \qquad\qquad f_m(t) = \frac{1}{\sqrt{z}t}. \end{aligned}$$

Taking $f_a = f_i + f_o - f_m$, we find (with a and g the same as before)

$$\int_0^\infty f_a(t)dt = \frac{1}{\sqrt{z}}\ln\frac{8z}{a+g}$$

and

$$f - f_a = \left(\frac{1}{\sqrt{z}} - \frac{1}{\sqrt{t+z}}\right)\left(\frac{1}{t} - \frac{1}{\sqrt{(t+x)(t+y)}}\right).$$

Inequalities (61) and (64) imply

$$\frac{g}{2\sqrt{z(t+x)(t+y)(t+z)}} < f - f_a < \frac{a}{2z\sqrt{(t+x)(t+y)(t+z)}}.$$

Hence, by Lemma 1,

$$\frac{g}{2\sqrt{z}(z-g)}\ln\frac{2z}{a+g} < \int(f-f_a) < \frac{a}{2z}\int f < \frac{a/2z}{1-a/2z}\int f_a\,,$$

where the last inequality follows from the next to last. We complete the proof of (26) by noting that

$$2R_F(x,y,z) = \int f = \int f_a + \int(f-f_a).$$

Equations (28) and (29) are obtained from [14, (2.15)(3.25)] with $w=\infty$. To derive (32) we construct $f_a = f_i + f_o - f_m$ as usual and find bounds for $\int(f-f_a)$ by using(60) and (65). To simplify the upper bound we note that $R_F(x,y,z) \le R_F(x,y,0)$ and use (33).

Equations (22),(23),(24), and (25) follow from (32),(26),(29), and (27), respectively, by replacing x by y, replacing z by x, and simplifying.

Among the approximations for R_D we need discuss only (35) and (37), since (34), (36), (38), and (39) follow from (44), (47), (49), and (48), respectively, by putting $p=z$ and simplifying. To prove (35) we let

$$f(t) = \frac{1}{\sqrt{(t+x)(t+y)}(t+z)^{3/2}}, \qquad f_i(t) = \frac{1}{g(t+z)^{3/2}}\,,$$

choose $f_a = f_i$, and apply (65) to get

$$\begin{aligned}
\frac{t}{g(t+g)(t+z)^{3/2}} &\le f_a - f \le \frac{at}{g^2\sqrt{(t+x)(t+y)}(t+z)^{3/2}}\,,\\
\frac{1}{g(g-z)\sqrt{t+z}}\left(\frac{g}{t+g}-\frac{z}{t+z}\right) &\le f_a - f < \frac{a}{g^2\sqrt{t+z}(t+g)}\,,\\
\frac{2}{g-z}\left[R_C(z,g)-\frac{\sqrt{z}}{g}\right] &\le \int(f_a-f) < \frac{2a}{g^2}R_C(z,g).
\end{aligned}$$

Use of (22) completes the proof. Approximation (37) follows from applying (39) to two terms on the right side of

$$R_D(x,y,z) = 3(xyz)^{-1/2} - R_D(z,x,y) - R_D(z,y,x), \tag{56}$$

an identity that comes from [7, (5.9-5)(6.8-15)].

In discussing approximations for R_J, we define

$$f(t) = \frac{1}{\sqrt{(t+x)(t+y)(t+z)}(t+p)}$$

and construct f_i, f_o, and f_m for each case in the manner described at the beginning of this Section. For example, if $x, y, z << p$, then f_i is obtained by neglecting t compared to p. Unless otherwise stated, we define $f_a = f_i + f_o - f_m$, take $\int f_a$ as an approximation to $\int f$, and find bounds for $\int (f - f_a)$ by using the inequalities in the Appendix.

To prove (40) we use (69). To prove (41) we use (64) and note that $\int (f - f_a) = (\theta/p) \int f$. Before discussing (42), we consider (43), in which the error bounds are easily found by using (70). Finding $\int f_a$ requires an integration by parts and a formula of which we omit the proof,

$$\int_0^\infty (ln\, t) \frac{d}{dt}[(t+x)(t+y)(t+z)]^{-1/2} dt = \frac{1}{\sqrt{xyz}} \ln \frac{\lambda^2}{4xyz} - \frac{4}{3} R_J(x+\lambda, y+\lambda, z+\lambda, \lambda), \quad (57)$$

where $\lambda = \sqrt{xy} + \sqrt{xz} + \sqrt{yz}$. To have a simpler approximation (42), we define $f_a = f_i + f_o - f_m$ and $\phi_a = f_i + f_s - f_m$, where f_o has been replaced by

$$f_s(t) = \frac{1}{t(t+g)^{3/2}}$$

Then

$$\int \phi_a = \frac{1}{\sqrt{xyz}} \left(\ln \frac{4g}{p} - 2 \right),$$

and an upper bound for $\int (f - \phi_a)$ is found by using $\sqrt{(t+x)(t+y)(t+z)} \geq (t+g)^{3/2}$ and (63). To find a lower bound, we note that $f - f_a > 0$, whence

$$f - \phi_a = f - f_a + f_o - f_s > f_o - f_s.$$

A lower bound for $\int (f_o - f_s)$ follows from (73).

The straightforward proof of (44) uses (64), (67), and Lemma 1. For the elementary approximation (45) we choose $f_a = f_i$ and use (66). For the more accurate approximation (47) we take $f_a = f_i + f_o - f_m$ and evaluate $\int f_a$ by integrating by parts. The error bounds follow from (66) and (69) with two variables equated. To find the error bounds for (48), we use (68), (60), and (71) to prove

$$\frac{\sqrt{t}}{(t+x)(t+a)} < \frac{2gp}{x}(f - f_a) < \left(\frac{a}{g} + \frac{g}{p} \right) \frac{1}{\sqrt{t+z}(t+g)}.$$

After integrating, (22) is used to complete the proof. In the case of (49), where $\int(f_o - f_m)$ is infinite, we choose $f_a = f_i$ and evaluate $\int f_a$ by (20). It follows from (61) that

$$\frac{1}{2\sqrt{x(t+y)(t+z)}}\left(\frac{1}{t+x} - \frac{p}{x(t+p)}\right) < f_a - f < \frac{1}{2x\sqrt{(t+x)(t+y)(t+z)}},$$

where we have replaced $t/(t+p)$ by 1 in the upper bound and $x/(x-p)$ by 1 in the lower bound. We then use (20), (55), and (27).

The function R_G can be expressed in terms of R_F and R_D by (17) and [7, Table 9.3-1]:

$$2R_G(x,y,z) = zR_F(x,y,z) - \frac{1}{3}(z-x)(z-y)R_D(x,y,z) + \sqrt{\frac{xy}{z}}. \qquad (58)$$

Applying (26) and (34), we obtain (51). The error bounds have been substantially simplified by using the numerical value of ln 2 and assuming $5a < z$ in the upper bound. It is not hard to obtain (53) from a well-known infinite series [15, p. 54] for $E(k)$ by using the inequality

$$1 + \frac{3}{8}k'^2 < {}_2F_1(\tfrac{1}{2}, \tfrac{3}{2}; 2; k'^2) < (1-k'^2)^{-1/2} = 1/k, \qquad 0 < k' < 1,$$

for the hypergeometric function ${}_2F_1$. Unfortunately (58) does not lead to simple error bounds for (54). Instead, we define $f(z) = R_G(x,y,z)$ and find from [7, (5.9-9)(6.8-6)] that

$$f'(z) = \frac{1}{8}\int_0^\infty \frac{tdt}{\sqrt{(t+x)(t+y)}(t+z)^{3/2}}.$$

Since this is a strictly decreasing function of z, the mean value theorem yields $f(z) = f(0) + zf'(\zeta)$ where

$$f'(z) < f'(\zeta) < f'(0) = \frac{1}{4}R_F(x,y,0).$$

By (71) and (5) we see that

$$f'(z) \geq \frac{1}{8}\int_0^\infty \frac{tdt}{(t+a)(t+z)^{3/2}} = \frac{1}{4(a-z)}[-\sqrt{z} + aR_C(z,a)].$$

Use of (33) and (22) completes the proof of (54).

4 Application to linear independence

In [7, Thm. 9.2-1] it is shown that $R_F(x,y,z)$, $R_G(x,y,z)$, an integral of the third kind called $R_H(x,y,z,p)$, and the algebraic function $(xyz)^{-1/2}$ are linearly independent with respect to coefficients that are rational functions of x,y,z,p. It then follows [7, §9.2] that every elliptic integral can be expressed in terms of R_F, R_G, R_H, and elementary functions. From (58) and a known relation expressing R_H in terms of R_J and R_F, we may conclude that every elliptic integral can be expressed in terms of R_F, R_D, R_J, and elementary functions. In order to reach the same conclusion without invoking R_G and R_H, we shall use the results of this paper to prove the linear independence of R_F, R_D, R_J, and $(xyz)^{-1/2}$ with respect to coefficients that are rational functions.

Theorem 1. *The functions $R_F(x,y,z)$, $R_D(x,y,z)$, $R_J(x,y,z,p)$, and $(xyz)^{-1/2}$ are linearly independent with respect to coefficients that are rational functions of x,y,z, and p.*

Proof. Let α,β,γ, and δ be rational functions of x,y,z, and p. We need to prove that

$$\alpha R_F(x,y,z) + \beta R_D(x,y,z) + \gamma R_J(x,y,z,p) + \delta\,(xyz)^{-1/2} \equiv 0 \tag{59}$$

iff α,β,γ, and δ are identically 0. We may assume that these coefficients are polynomials since we can multiply all terms by the denominator of any rational function. As $p \to 0$, (42) shows that $R_J(x,y,z,p)$ involves $\ln p$ while all other quantities are polynomials in p, whence $\gamma \equiv 0$. As $z \to \infty$ we have

$$\alpha = az^m(1+O(1/z)), \qquad \beta = bz^n(1+O(1/z)),$$

where m and n are nonnegative integers and a and b are polynomials in x,y, and p. Using (26) and (34) and multiplying all terms by $2z^{3/2}$, we find

$$az^{m+1}\left[\ln\frac{8z}{a+g} + O\left(\frac{\ln z}{z}\right)\right] + 3bz^n\left[\ln\frac{8z}{a+g} - 2 + O\left(\frac{\ln z}{z}\right)\right] + 2\delta\,(xy)^{-1/2}z \equiv 0.$$

Cancellation of the leading terms in $\ln z$ requires $az^{m+1} + 3bz^n \equiv 0$, implying $n = m + 1$ and $a \equiv -3b$ and leaving

$$O(z^m \ln z) - 6bz^{m+1} + 2\delta\,(xy)^{-1/2} z \equiv 0.$$

Because the second term is of different order from the first and does not have a square root in common with the third, it follows that $b \equiv 0$, whence also $a \equiv 0$. Since the leading terms of the polynomials α and β are identically 0, so too are α and β. Finally, with only one term remaining in (59), we have $\delta \equiv 0$. □

It is an open question whether Theorem 1 is still true if the coefficients are algebraic functions instead of rational functions. However, polynomial coefficients suffice (see the first paragraph of [7, § 9.2]) to prove that every elliptic integral can be expressed in terms of R_F, R_D, R_J, and elementary functions.

Appendix

Elementary inequalities

Assuming x, y, z, and t are positive, we list and prove some inequalities that are used in this paper to obtain error bounds:

$$\frac{x}{2\sqrt{t}(t+x)} < \frac{1}{\sqrt{t}} - \frac{1}{\sqrt{t+x}} < \frac{x}{2t\sqrt{t+x}}, \tag{60}$$

$$\frac{t}{2\sqrt{x}(t+x)} < \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{t+x}} < \frac{t}{2x\sqrt{t+x}}, \tag{61}$$

$$\frac{1}{t^{3/2}} - \frac{1}{(t+x)^{3/2}} = \frac{\theta x}{t^{3/2}(t+x)}, \qquad 1 < \theta < \frac{3}{2}, \tag{62}$$

$$\frac{1}{x^{3/2}} - \frac{1}{(t+x)^{3/2}} = \frac{\theta t}{x^{3/2}(t+x)}, \qquad 1 < \theta < \frac{3}{2}. \tag{63}$$

In the next five inequalities let $a = (x+y)/2$ and $g = \sqrt{xy}$. Inequalities become equalities in (64), (65), and (66) iff $x = y$.

$$\frac{1}{t} - \frac{1}{\sqrt{(t+x)(t+y)}} = \frac{\theta}{t\sqrt{(t+x)(t+y)}}, \qquad g \le \theta \le a, \tag{64}$$

$$\frac{t}{g(t+g)} \le \frac{1}{\sqrt{xy}} - \frac{1}{\sqrt{(t+x)(t+y)}} \le \frac{at}{g^2\sqrt{(t+x)(t+y)}}, \tag{65}$$

or alternatively,

$$\frac{1}{\sqrt{xy}} - \frac{1}{\sqrt{(t+x)(t+y)}} = \frac{\theta t}{g(t+g)}, \qquad 1 \le \theta \le \frac{a}{g}, \tag{66}$$

$$\frac{1}{\sqrt{x}y} - \frac{1}{\sqrt{t+x}(t+y)} = \frac{\theta t}{\sqrt{x}y(t+y)}, \qquad 1 < \theta < 1 + \frac{y}{2x}, \tag{67}$$

$$\frac{1}{\sqrt{xy}z} - \frac{1}{\sqrt{(t+x)(t+y)}(t+z)} = \frac{\theta t}{gz\sqrt{(t+x)(t+y)}}, \qquad 1 < \theta < \frac{a}{g} + \frac{g}{z}. \tag{68}$$

Finally we have

$$\frac{b}{t^{3/2}(t+b)} < \frac{1}{t^{3/2}} - \frac{1}{\sqrt{(t+x)(t+y)(t+z)}} < \frac{3a}{2t^{3/2}(t+a)}, \tag{69}$$

where $a = (x+y+z)/3$ and $b = \sqrt{3(xy+xz+yz)}/2$, and

$$\frac{t}{g^{3/2}(t+g)} < \frac{1}{\sqrt{xyz}} - \frac{1}{\sqrt{(t+x)(t+y)(t+z)}} < \frac{3t}{2g^{3/2}(t+h)}, \tag{70}$$

where $g = (xyz)^{1/3}$ and $3h^{-1} = x^{-1} + y^{-1} + z^{-1}$.

To prove (60) we write

$$\frac{1}{\sqrt{t}} - \frac{1}{\sqrt{t+x}} = \frac{\sqrt{t+x} - \sqrt{t}}{\sqrt{t(t+x)}} = \frac{x}{\sqrt{t(t+x)}(\sqrt{t+x} + \sqrt{t})}$$

and replace the last denominator factor by either $2\sqrt{t}$ or $2\sqrt{t+x}$. Interchange of t and x leads from (60) to (61). To prove (62) let $y = \sqrt{1+x/t}$ and write

$$\frac{t^{3/2}(t+x)}{x}\left(\frac{1}{t^{3/2}} - \frac{1}{(t+x)^{3/2}}\right) = \frac{y^2}{y^2-1}\left(1 - \frac{1}{y^3}\right) = 1 + \frac{1}{y(y+1)},$$

which increases from 1 to 3/2 as t increases from 0 to ∞ and y decreases from ∞ to 1. Interchange of t and x leads from (62) to (63).

If the left side of (64) is put over a common denominator, it suffices to observe that

$$t + g \le \sqrt{(t+x)(t+y)} \le t + a. \tag{71}$$

The left inequality is enough to prove the left inequality in (65). To prove the right inequality in (65), we define

$$\phi(t) = (\sqrt{(t+x)(t+y)} - \sqrt{xy})/t$$

and note that $\phi(t)$ tends to a/g as $t \to 0$ and to 1 as $t \to \infty$. Differentiation shows that ϕ decreases monotonically, because

$$t^2\sqrt{(t+x)(t+y)}\phi' = -(ta+g^2) + [(ta+g^2)^2 - t^2(a^2-g^2)]^{1/2} \leq 0,$$

with equality iff $x = y$. Because of (71), (65) implies (66).

Equation (67) is proved by solving for θ and using (61). Likewise, (68) is proved by solving for

$$\theta = \phi(t) + \frac{t}{t+z}$$

and using the result just established that $1 \leq \phi(t) \leq a/g$.

To prove (69) we use Maclaurin's inequality [17, Thm. 52] to find that

$$t^3 + 2bt^2 + 4b^2t/3 < (t+x)(t+y)(t+z) \leq (t+a)^3,$$

and hence

$$\sqrt{t}(t+b) < \sqrt{(t+x)(t+y)(t+z)} \leq (t+a)^{3/2}. \tag{72}$$

Inequality (69) follows from this and (62).

The proof of (70) uses Maclaurin's inequality and the inequality of arithmetic and geometric means to get

$$\frac{t+g}{g} \leq \left[\frac{(t+x)(t+y)(t+z)}{xyz}\right]^{1/3} = \left[\left(1+\frac{t}{x}\right)\left(1+\frac{t}{y}\right)\left(1+\frac{t}{z}\right)\right]^{1/3} \leq 1 + \frac{t}{h},$$

with equalities iff $x = y = z$, whence

$$(t+g)^{3/2} \leq \sqrt{(t+x)(t+y)(t+z)} \leq \left(\frac{g}{h}\right)^{3/2}(t+h)^{3/2}. \tag{73}$$

Two applications of (63) complete the proof of (70).

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