

The modified tetrahedron equation and its solutions

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Abstract

A large class of 3-dimensional integrable lattice spin models is constructed. The starting point is an invertible canonical mapping operator $\mathcal{R}_{1,2,3}$ in the space of a triple Weyl algebra. $\mathcal{R}_{1,2,3}$ is derived postulating a current branching principle together with a Baxter Z-invariance. The tetrahedron equation for $\mathcal{R}_{1,2,3}$ follows without further calculation. If the Weyl parameter is taken to be a root of unity, $\mathcal{R}_{1,2,3}$ decomposes into a matrix conjugation operator $\mathbf{R}_{1,2,3}$ and a c-number functional mapping $\mathcal{R}_{1,2,3}^{(f)}$. The operator $\mathbf{R}_{1,2,3}$ satisfies a modified tetrahedron equation (MTE) in which the "rapidities" are solutions of a classical integrable Hirota-type equations. $\mathbf{R}_{1,2,3}$ can be represented in terms of the Bazhanov-Baxter Fermat curve cyclic functions, or alternatively in terms of Gauss functions. The paper summarizes several recent publications on the subject.

AMS subject classes: 82B23, 70H06; *keywords:* 3D integrable models

PACS: 05.45-a, 05.50+q

Introduction

The subject of constructing and solving two-dimensional integrable lattice models is a well-established branch of mathematical physics. It has boosted important developments like conformal quantum field theory, quantum groups and the quantum inverse scattering method. But the analogous three-dimensional integrable lattice models have received little attention: the few known cases are complicated and direct physical applications seem to be remote. Many tools which are powerful for dealing with two-dimensional lattice models seem to have little relevance for the higher-dimensional cases. However, the Yang-Baxter equation can easily be generalized to tetrahedron- and D-simplex equations [2, 3] which still guarantee the existence of commuting families of transfer matrices, and concepts from the quantum inverse scattering method like L-operators can be generalized to dimensions $D > 2$ [16]. Since many statistical systems require a truly three-dimensional treatment, exploring the possibilities of constructing and solving 3D-integrable models should not be useless.

The first solution to the tetrahedron equation (TE) was found by Zamolodchikov in 1981 [1], and the partition function of this $N = 2$ -state model was calculated by Baxter [4]

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using only symmetry properties, some factorization and commutativity. Later, Bazhanov and Baxter [5] showed how to generalize the Zamolodchikov model to $N > 2$, using the interaction-round-a-cube language (IRC-ZBB-model). Then, Sergeev, Mangazeev and Stroganov found the vertex-formulation of the ZBB-model [10]. These models share the feature the ZBB-Boltzmann weights are not positive definite, but at least in the thermodynamic limit the partition function becomes real. Since the partition function is a kind of dilogarithm function, there is no temperature-like parameter and the model is critical [7]. There have been several attempts to construct more physical 3D integrable models, avoiding the too restrictive Zamolodchikov TE, using "Modified TEs" [8, 9, 14, 15].

A different starting point to find a large class of integrable 3D-models has been used by Sergeev [16, 17]: First an automorphism of a triple Weyl algebra is constructed, which satisfies a Kirchhoff-like auxiliary linear problem and a Baxter Z-invariance principle. This invertible canonical automorphism implies the TE, so this approach avoids the need of a tedious proof of the TE [20]. If the Weyl parameter is taken to be the N -th primitive root of unity, a very interesting mechanism noticed before in [22, 19, 21, 12, 16, 33] becomes operative: The mapping operator becomes a matrix conjugation (which is the quantum operator of an euclidian time evolution, or the Boltzmann weight) which depends on c -number parameters, which satisfy a set of Hirota-type classical integrable equations of motion. So we have a quantum integrable system which is parameterized by solutions of a classical integrable system.

The different solutions of the classical integrable equations define various 3D-integrable quantum models. This framework contains the ZBB-model which is obtained choosing the trivial classical solution. The general solution to the Hirota-type equations for given global boundary conditions can formally be written down in terms of high-genus theta functions, using the Fay-identity [26, 27, 28, 29].

What is the use of these results? It is clear that the construction of a versatile integrable 3D-statistical model with nice physical properties and its analytic solution is not yet round the corner, but the mathematical structures which such a model should incorporate become more clear. Whether this class of models is sufficiently wide to describe 3D phase transitions, will be seen. Another application which can be made right now is to use these 3D models in an asymmetric geometry and to reduce them to new 2D integrable models [23]. Since Fermat curves are involved, these will be useful relatives of the chiral Potts model and of the relativistic Toda model [33]. The large variety of parameterizations which the MTE allows (in contrast to the Yang-Baxter equation) should help to overcome the serious parameterization problems which have prevented progress to finding more analytic results for the chiral Potts model.

We shall not try to touch all interesting aspects of this program. We concentrate on explaining the construction of the basic triple Weyl automorphism and how it gives rise to the modified tetrahedron equation (MTE) and the classical Hirota-type equations of motion. We focus on the local properties and do not discuss boundary conditions and large systems. For the reader's convenience, we sketch some details of the derivations even if these are rather straightforward. The interested reader may consult references [16, 35, 34] which give more details on recent results on 3D integrable models satisfying MTEs.

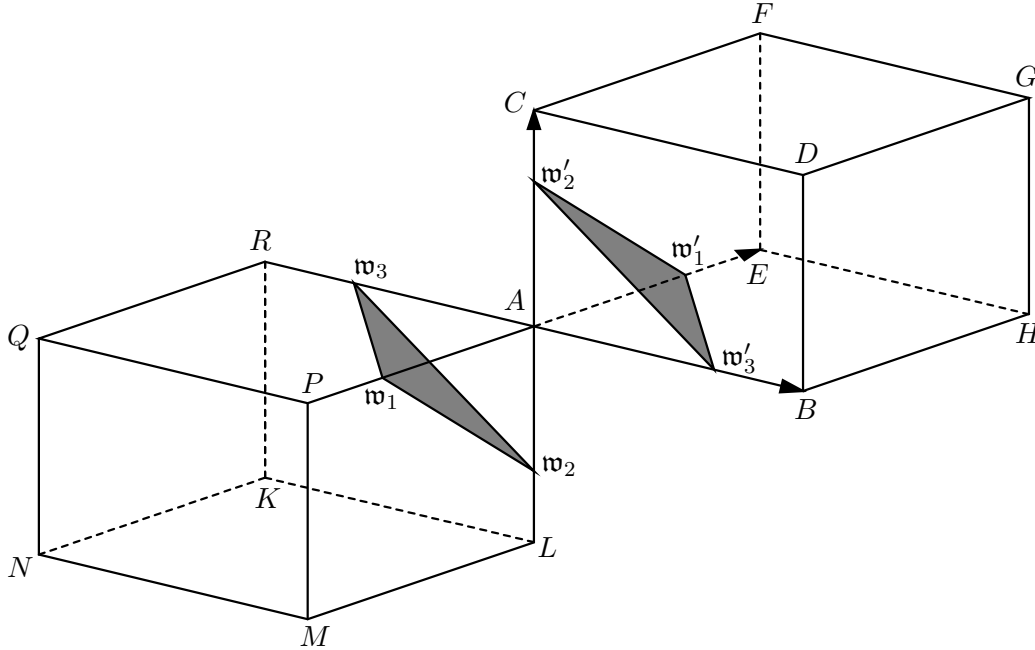


Figure 1: Two cubes of the basic lattice intersected by the auxiliary plane (shaded) in two different positions: first passing through $\mathfrak{w}_1, \mathfrak{w}_2, \mathfrak{w}_3$ and second through $\mathfrak{w}'_1, \mathfrak{w}'_2, \mathfrak{w}'_3$. The second position is obtained from the first by moving the auxiliary plane parallel through the vertex A . The Weyl variables $\mathfrak{w}_i, \mathfrak{w}'_i$ live on the links of the basic lattice.

1 The canonical invertible mapping \mathcal{R}_{123}

The models to be discussed in this paper are defined on a 3-dimensional oriented lattice, which can be imagined as a (possibly distorted) cubic oriented lattice. We use the vertex language: the dynamic variables \mathfrak{w}_i live on the oriented links i of the lattice and the Boltzmann weights are associated to the vertices. We assume that the lattice is non-degenerate so that at all vertices exactly three lines meet. The Boltzmann weights can be considered as operators mapping the three dynamic variables on the incoming links to the three variables on the outgoing links. Fig.1 shows a perspective view of two adjacent cubes of the lattice (to keep the picture simple, we do not show the other vertices and lines). Only the dynamic variables on the links surrounding the vertex A are indicated. The links (or edges) of the lattice arise from the intersection of two planes each, the vertices from the intersection of three planes. We introduce auxiliary planes (shaded in Fig.1) and locate the dynamic variables at the vertices of the lines where the lattice planes intersect the auxiliary planes. The mapping of the variables \mathfrak{w}_i onto the \mathfrak{w}'_i can be regarded as moving the auxiliary plane through the vertex A . This way we may interpret the 3-dimensional statistical model as the euclidian time evolution of the 2-dimensional lattice which arises when we move the auxiliary plane.

The next step is to specify the mapping, or, equivalently, the Boltzmann weights. We assume that the dynamic variables \mathfrak{w}_i are elements of the ultralocal Weyl algebra $\mathfrak{W}_i \in \mathfrak{W}^{\otimes \Delta}$, defined as the tensor product of Δ copies of Weyl pairs (Δ is the number of

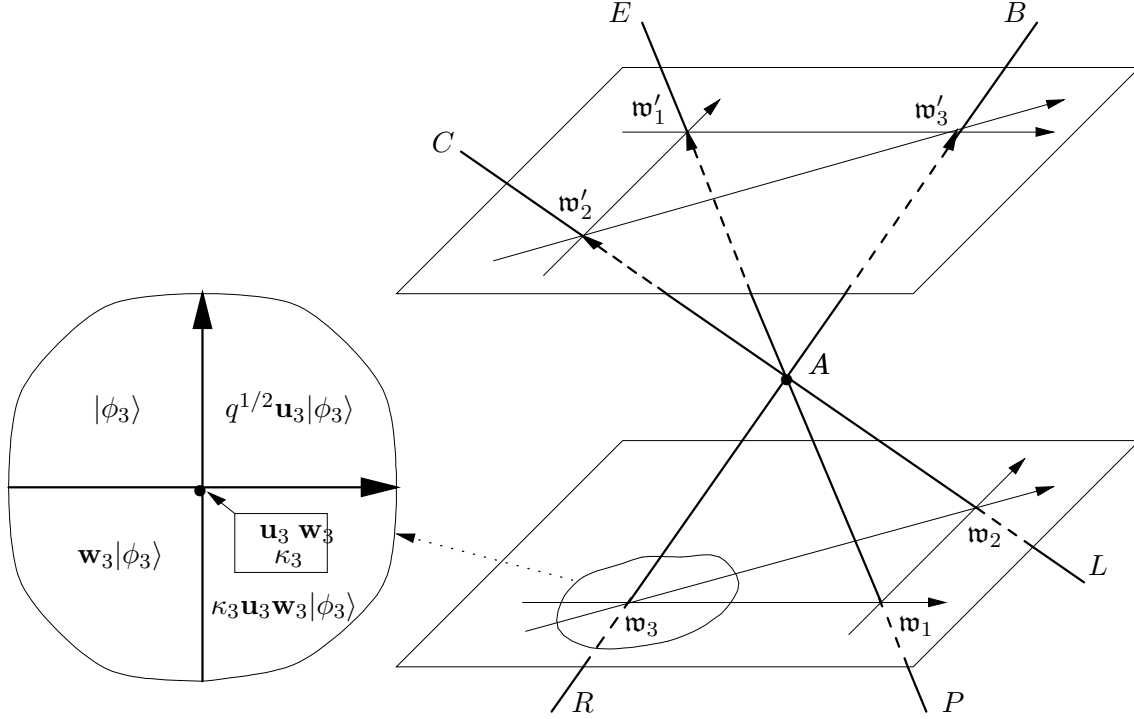


Figure 2: Right hand side: magnified view of the vicinity of point A in Fig.1. Left: magnified view of the auxiliary plane in the vicinity of \mathfrak{w}_3 showing the four sectors formed by the lines $\overline{\mathfrak{w}_3 \mathfrak{w}_2}$ and $\overline{\mathfrak{w}_3 \mathfrak{w}_1}$. : We imagine a current $|\phi_3\rangle$ flowing out of the intersection point into the four 2-dimensional sectors with a distribution steered by the Weyl variable \mathfrak{w}_3 and the coupling κ_3 .

vertices in the auxiliary plane):

$$\mathfrak{W}^{\otimes \Delta} = \{ \mathbf{u}_i = 1 \otimes 1 \otimes \dots \otimes \underbrace{\mathbf{u}}_{i\text{-th place}} \otimes \dots, \mathbf{w}_i = 1 \otimes 1 \otimes \dots \otimes \underbrace{\mathbf{w}}_{i\text{-th place}} \otimes \dots \mid i = 1, \dots, \Delta \} \quad (1)$$

$$\text{with} \quad \mathbf{u}_i \mathbf{w}_j = q^{\delta_{i,j}} \mathbf{w}_j \mathbf{u}_i, \quad q \in \mathbb{C}. \quad (2)$$

In order to determine the mapping uniquely, we postulate a linear current branching principle (a kind of Kirchhoff law), together with a Baxter Z-invariance. Consider Fig.2. On the right hand side we show again the vicinity of the vertex A of Fig.1: on top and on the bottom there are the two auxiliary planes. The three links of the basic lattice which connect the \mathfrak{w}_i and the \mathfrak{w}'_i , intersect in A . We now postulate currents $|\phi_i\rangle$ flowing out of the vertices into the surrounding four sectors of the auxiliary plane. The distribution of the current $|\phi_i\rangle$ flowing out of \mathfrak{w}_i into the sectors is steered by the value of the variable \mathfrak{w}_i and one coupling constant κ_i , as shown in the left hand picture of Fig.2: The vertex i sends the current $|\phi_i\rangle$ into the sector left to the arrows, into the sector between the two outgoing arrows it sends $q^{1/2} \mathbf{u}_i |\phi_i\rangle$, to the right of the arrows $\mathbf{w}_i |\phi_i\rangle$, and between the incoming arrows $\kappa_i \mathbf{u}_i \mathbf{w}_i |\phi_i\rangle$. The total current in an internal sector is required to be zero and the total current reaching an external sector should be independent of the internal structure. In Fig.3 we show the lower and upper auxiliary planes of the previous two Figures. According to the rules just stated, the total currents flowing into the internal sectors $|\phi_a\rangle$ and $|\phi_h\rangle$ must both vanish. $|\phi_a\rangle$ receives contributions from $|\phi_1\rangle, |\phi_2\rangle$

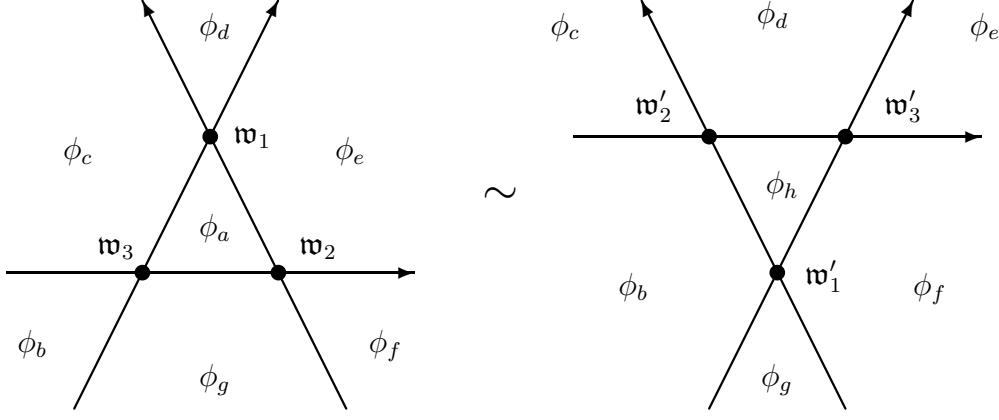


Figure 3: The triangles on the lower and upper auxiliary plane.

and $|\phi_3\rangle$. Since both structures differ just by a shift of one line through one vertex, there is a unique correspondence between the respective sectors. So these rules give:

$$\begin{aligned}
|\phi_a\rangle &= \mathbf{w}_1|\phi_1\rangle + |\phi_2\rangle + q^{1/2}\mathbf{u}_3|\phi_3\rangle = 0 \\
|\phi_b\rangle &= |\phi'_1\rangle + \mathbf{w}'_2|\phi'_2\rangle = \mathbf{w}_3|\phi_3\rangle \\
|\phi_c\rangle &= |\phi'_2\rangle = |\phi_1\rangle + |\phi_3\rangle \\
|\phi_d\rangle &= |\phi'_3\rangle + q^{1/2}\mathbf{u}'_2|\phi'_2\rangle = q^{1/2}\mathbf{u}_1|\phi_1\rangle \\
|\phi_e\rangle &= q^{1/2}\mathbf{u}'_3|\phi'_3\rangle = \kappa_1\mathbf{u}_1\mathbf{w}_1|\phi_1\rangle + q^{1/2}\mathbf{u}_2|\phi_2\rangle \\
|\phi_f\rangle &= \kappa_1\mathbf{u}'_1\mathbf{w}'_1|\phi'_1\rangle + \kappa_3\mathbf{u}'_3\mathbf{w}'_3|\phi'_3\rangle = \kappa_2\mathbf{u}_2\mathbf{w}_2|\phi_2\rangle \\
|\phi_g\rangle &= \mathbf{w}'_1|\phi'_1\rangle = \mathbf{w}_2|\phi_2\rangle + \kappa_3\mathbf{u}_3\mathbf{w}_3|\phi_3\rangle \\
|\phi_h\rangle &= q^{1/2}\mathbf{u}'_1|\phi'_1\rangle + \kappa_2\mathbf{u}'_2\mathbf{w}'_2|\phi'_2\rangle + \mathbf{w}'_3|\phi'_3\rangle = 0
\end{aligned} \tag{3}$$

These eight equations determine the \mathbf{u}'_i and \mathbf{w}'_i uniquely in terms of the \mathbf{u}_i and \mathbf{w}_i and vice versa (we give details of the calculation in the Appendix). The resulting rational transformation can be written as:

$$\begin{aligned}
\mathbf{w}'_1 &= \mathbf{w}_2 \Lambda_3, & \mathbf{w}'_2 &= \Lambda_3^{-1}\mathbf{w}_1, & \mathbf{w}'_3 &= \Lambda_2^{-1}\mathbf{u}_1^{-1}, \\
\mathbf{u}'_1 &= \Lambda_2^{-1}\mathbf{w}_3^{-1}, & \mathbf{u}'_2 &= \Lambda_1^{-1}\mathbf{u}_3, & \mathbf{u}'_3 &= \mathbf{u}_2 \Lambda_1,
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
\Lambda_1 &= \mathbf{u}_1^{-1}\mathbf{u}_3 - q^{1/2}\mathbf{u}_1^{-1}\mathbf{w}_1 + \kappa_1\mathbf{w}_1\mathbf{u}_2^{-1}, \\
\Lambda_2 &= \frac{\kappa_1}{\kappa_2}\mathbf{u}_2^{-1}\mathbf{w}_3^{-1} + \frac{\kappa_3}{\kappa_2}\mathbf{u}_1^{-1}\mathbf{w}_2^{-1} - \frac{\kappa_1\kappa_3}{q^{1/2}\kappa_2}\mathbf{u}_2^{-1}\mathbf{w}_2^{-1}, \\
\Lambda_3 &= \mathbf{w}_1\mathbf{w}_3^{-1} - q^{1/2}\mathbf{u}_3\mathbf{w}_3^{-1} + \kappa_3\mathbf{w}_2^{-1}\mathbf{u}_3.
\end{aligned} \tag{5}$$

In (4) the order of the factors on the right hand side is irrelevant since e.g. Λ_1 contains only operators which commute with both \mathbf{u}_2 and \mathbf{u}_3 , in particular, it contains no \mathbf{w}_2 , \mathbf{w}_3 . The six eqs.(4) are written with only three operators Λ_i because the mapping conserves three centers:

$$\mathbf{w}'_1\mathbf{w}'_2 = \mathbf{w}_2\mathbf{w}_1; \quad \mathbf{u}'_3\mathbf{u}'_2 = \mathbf{u}_2\mathbf{u}_3; \quad \mathbf{w}'_3{}^{-1}\mathbf{u}'_1 = \mathbf{u}_1\mathbf{w}_3{}^{-1}. \tag{6}$$

This rational mapping conserves the Weyl structure, i.e. it is canonical: from (2) it follows that

$$\mathbf{u}'_i \mathbf{w}'_j = q^{\delta_{i,j}} \mathbf{w}'_j \mathbf{u}'_i, \quad (7)$$

as may be verified by explicit use of (4) and (5).

Definition 1 For any rational function Φ of the $\mathbf{u}_1, \dots, \mathbf{w}_3$, the relations (4), (5) define the invertible and canonical mapping $\mathcal{R}_{1,2,3}$

$$(\mathcal{R}_{1,2,3} \circ \Phi)(\mathbf{u}_1, \mathbf{w}_1, \mathbf{u}_2, \dots, \mathbf{w}_3) \stackrel{def}{=} \Phi(\mathbf{u}'_1, \mathbf{w}'_1, \mathbf{u}'_2, \dots, \mathbf{w}'_3). \quad (8)$$

$\mathcal{R}_{1,2,3}$ is an automorphism of $\mathfrak{W}^{\otimes \Delta}$ and conserves the three centers (6).

The derivation of (4), (5) from (3) given in the Appendix does not use that $\mathbf{u}_i, \mathbf{w}_j$ are Weyl operators. However, the canonical property of $\mathcal{R}_{1,2,3}$ is obtained for ultralocal Weyl dynamic variables only. The inverse transformation has a similar form and we summarize it giving Λ_i^{-1} in terms of the primed variables:

$$\begin{aligned} \Lambda_1^{-1} &= \frac{\kappa_1}{\kappa_2} \mathbf{u}'_1 \mathbf{u}'_3^{-1} - q^{1/2} \frac{\kappa_3}{\kappa_2} \mathbf{u}'_1 \mathbf{w}'_1^{-1} + \kappa_3 \mathbf{u}'_2 \mathbf{w}'_1^{-1}, \\ \Lambda_2^{-1} &= \mathbf{u}'_2 \mathbf{w}'_3 - q^{-1/2} \kappa_2 \mathbf{u}'_2 \mathbf{w}'_2 + \mathbf{u}'_1 \mathbf{w}'_2, \\ \Lambda_3^{-1} &= \frac{\kappa_3}{\kappa_2} \mathbf{w}'_1^{-1} \mathbf{w}'_3 - q^{1/2} \frac{\kappa_1}{\kappa_2} \mathbf{u}'_3^{-1} \mathbf{w}'_3 + \kappa_1 \mathbf{w}'_2 \mathbf{u}'_3^{-1}. \end{aligned} \quad (9)$$

There is a way dual to (3) to formulate the current branching principle as "Linear problem". This is more convenient if one is dealing with a large lattice with definite boundary conditions. The resulting mapping is the same (4), (5), we give this alternative derivation in the Appendix. We refer to [35, 32] for a detailed exposition of the linear problem equations for large cubic lattices.

2 Weyl parameter q a root of unity: decomposition of \mathcal{R} into a matrix conjugation and a functional mapping.

Up to now, q was in general position. In all the following we fix q to be a primitive root of unity:

$$q = \omega \stackrel{def}{=} e^{2\pi i/N}, \quad N \in \mathbb{Z}, \quad N \geq 2. \quad (10)$$

Then \mathbf{u}^N and \mathbf{w}^N are centers of the Weyl algebra and we can represent the canonical pair $(\mathbf{u}_i, \mathbf{w}_i)$ by its action on a cyclic basis as unitary $N \times N$ matrices multiplied by complex parameters u_i, w_i . The algebra $\mathfrak{W}^{\otimes \Delta}$ will then be represented in a N^Δ -dimensional product space, see (1). Because of the ultra-locality, the Weyl operators $(\mathbf{u}_i, \mathbf{w}_i) \in \mathfrak{w}_i$ will be represented trivially in all spaces $j \neq i$, acting non-trivially only in the space with label i , where (now omitting the index i):

$$\begin{aligned} \mathbf{u} &= u \mathbf{x}; & \mathbf{w} &= w \mathbf{z}; \\ |\sigma\rangle &\equiv |\sigma \bmod N\rangle; & \langle \sigma | \sigma' \rangle &= \delta_{\sigma, \sigma'}; & \mathbf{x} | \sigma \rangle &= | \sigma \rangle \omega^\sigma; & \mathbf{z} | \sigma \rangle &= | \sigma + 1 \rangle. \end{aligned} \quad (11)$$

In this representation the centers are represented by numbers:

$$\mathbf{u}_i^N = u_i^N, \quad \mathbf{w}_i^N = w_i^N. \quad (12)$$

We shall see that for q a root of unity, the rather complicated looking transformation (4),(5) can be represented in a rather simple way.

2.1 The functional mapping $\mathcal{R}_{1,2,3}^{(f)}$

For q a root of unity, not only the N -th powers of the Weyl variables are numbers, but also the N -th powers of the operators Λ_i will be numbers. This is because in calculating the N -th powers the cross terms drop out, e.g. $(a\mathbf{u} + b\mathbf{w})^N = (au)^N + (bw)^N$ due to $\sum_{j=0}^{N-1} \omega^j = 0$, etc. We get

$$\begin{aligned}\Lambda_1^N &= u_1^{-N} u_3^N + u_1^{-N} w_1^N + \kappa_1^N w_1^N u_2^{-N}, \\ \Lambda_2^N &= \frac{\kappa_1^N}{\kappa_2^N} u_2^{-N} w_3^{-N} + \frac{\kappa_3^N}{\kappa_2^N} u_1^{-N} w_2^{-N} + \frac{\kappa_1^N \kappa_3^N}{\kappa_2^N} u_2^{-N} w_2^{-N}, \\ \Lambda_3^N &= w_1^N w_3^{-N} + u_3^N w_3^{-N} + \kappa_3^N w_2^{-N} u_3^N.\end{aligned}\tag{13}$$

So (4) implies an analogous purely functional mapping of the Weyl centers:

$$\begin{aligned}w_1'^N &= \Lambda_3^N w_2^N, & w_2'^N &= \Lambda_3^{-N} w_1^N, & w_3'^N &= \Lambda_2^{-N} u_1^{-N}, \\ u_1'^N &= \Lambda_2^{-N} w_3^{-N}, & u_2'^N &= \Lambda_1^{-N} u_3^N, & u_3'^N &= \Lambda_1^N u_2^N.\end{aligned}\tag{14}$$

Later we shall need not only the mapping of the centers (14) but also the mapping of the u_i, w_i itself onto the u'_i, w'_i . For this we have to take N -th roots and get a freedom to choose phases. If we demand the number products $w_1 w_2, u_2 u_3, u_1 w_3^{-1}$ to be centers of this mapping, then there are three free phases.

Definition 2 *The functional counterpart of the mapping $\mathcal{R}_{1,2,3}$ is the mapping $\mathcal{R}_{1,2,3}^{(f)}$, acting on the space of functions of the parameters u_j, w_j ($j = 1, 2, 3$)*

$$\left(\mathcal{R}_{1,2,3}^{(f)} \circ \phi\right)(u_1, w_1, u_2, w_2, u_3, w_3) \stackrel{\text{def}}{=} \phi(u_1', w_1', u_2', w_2', u_3', w_3'),\tag{15}$$

where the primed variables are functions of the unprimed ones, defined via

$$u_1'^N = \mathbf{u}_1^N, \quad w_1'^N = \mathbf{w}_1^N, \quad \text{etc.},\tag{16}$$

such that $w_1' w_2' = w_1 w_2$, $u_2' u_3' = u_2 u_3$, $u_1' w_3'^{-1} = u_1 w_3^{-1}$. The three free phases of the N -th roots are extra discrete parameters of $\mathcal{R}_{1,2,3}^{(f)}$.

In obvious generalization:

Remark 1 *To any rational automorphism of the ultra-local Weyl algebra $\mathfrak{W}^{\otimes \Delta}$ (1) there is a rational mapping in the space of the N -th powers of the parameters of the representation: u_j^N, w_j^N .*

2.2 Matrix structure of the mapping $\mathcal{R}_{1,2,3}$

We now look at the matrix form of $\mathcal{R}_{1,2,3}$ in the representation (11). We separate the numerical factors from the constant diagonal resp. cyclic raising matrices $\mathbf{x}_i, \mathbf{z}_i$. Understanding that the matrices $\mathbf{x}_i, \mathbf{w}_i$ act trivially in the spaces with $j \neq i$, from (4) we immediately get the following complicated looking $N^3 \times N^3$ -matrix equations:

$$\left(\mathbf{x}'_1\right)^{-1} = \frac{\kappa_1 u_1'}{\kappa_2 u_2} \mathbf{x}_2^{-1} + \frac{\kappa_3 u_1' w_3}{\kappa_2 u_1 w_2} \mathbf{x}_1^{-1} \mathbf{z}_2^{-1} \mathbf{z}_3 - \omega^{1/2} \frac{\kappa_1 \kappa_3 u_1' w_3}{\kappa_2 u_2 w_2} \mathbf{x}_2^{-1} \mathbf{z}_2^{-1} \mathbf{z}_3,$$

$$\begin{aligned}
\mathbf{z}'_1 &= \frac{w_2 w_1}{w'_1 w_3} \mathbf{z}_1 \mathbf{z}_2 \mathbf{z}_3^{-1} - \omega^{1/2} \frac{w_2 u_3}{w'_1 w_3} \mathbf{z}_2 \mathbf{x}_3 \mathbf{z}_3^{-1} + \frac{\kappa_3}{w'_1} \mathbf{x}_3, \\
(\mathbf{x}'_2)^{-1} &= \frac{u'_2}{u_1} \mathbf{x}_1^{-1} - \omega^{1/2} \frac{w_1 u'_2}{u_1 u_3} \mathbf{x}_1^{-1} \mathbf{z}_1 \mathbf{x}_3^{-1} + \frac{\kappa_2 w_1 u'_2}{u_2 u_3} \mathbf{z}_1 \mathbf{x}_2^{-1} \mathbf{x}_3^{-1}, \\
(\mathbf{z}'_2)^{-1} &= \frac{w'_2}{w_3} \mathbf{z}_3^{-1} - \omega^{1/2} \frac{w'_2 u_3}{w_1 w_3} \mathbf{z}_1^{-1} \mathbf{x}_3 \mathbf{z}_3^{-1} + \frac{\kappa_3 w'_2 u_3}{w_1 w_2} \mathbf{z}_1^{-1} \mathbf{z}_2^{-1} \mathbf{x}_3, \\
\mathbf{x}'_3 &= \frac{u_2 u_3}{u_1 u'_3} \mathbf{x}_1^{-1} \mathbf{x}_2 \mathbf{x}_3 - \omega^{1/2} \frac{w_1 u_2}{u_1 u'_3} \mathbf{x}_1^{-1} \mathbf{z}_1 \mathbf{x}_2 + \frac{\kappa_2 w_1}{u'_3} \mathbf{z}_1, \\
(\mathbf{z}'_3)^{-1} &= \frac{\kappa_1 u_1 w'_3}{\kappa_2 u_2 w_3} \mathbf{x}_1 \mathbf{x}_2^{-1} \mathbf{z}_3^{-1} + \frac{\kappa_3 w'_3}{\kappa_2 w_2} \mathbf{z}_2^{-1} - \omega^{1/2} \frac{\kappa_1 \kappa_3 u_1 w'_3}{\kappa_2 u_2 w_2} \mathbf{x}_1 \mathbf{x}_2^{-1} \mathbf{z}_2^{-1}. \tag{17}
\end{aligned}$$

However, eqs.(17) can be rewritten in a compact way because there exists a (up to a scalar factor) unique $N^3 \times N^3$ conjugation matrix $\mathbf{R}_{1,2,3}$, such that for $i = 1, 2, 3$

$$\mathbf{x}'_i = \mathbf{R}_{1,2,3} \mathbf{x}_i \mathbf{R}_{1,2,3}^{-1}, \quad \mathbf{z}'_i = \mathbf{R}_{1,2,3} \mathbf{z}_i \mathbf{R}_{1,2,3}^{-1}, \tag{18}$$

and there is a closed formula for the matrix elements of $\mathbf{R}_{1,2,3}$ which later will be given explicitly. Let us write the matrix elements of $\mathbf{R}_{1,2,3}$ in our representation (11) as

$$\langle i_1, i_2, i_3 | \mathbf{R}_{1,2,3} | j_1, j_2, j_3 \rangle \stackrel{\text{def}}{=} R_{i_1, i_2, i_3}^{j_1, j_2, j_3}. \tag{19}$$

Then e.g. the from the equation for \mathbf{x}'_3 in (17) we see that we have to find $R_{i_1, i_2, i_3}^{j_1, j_2, j_3}$ such that it satisfies the recursion

$$\left(\omega^{j_3} - \frac{u'_2}{u_1} \omega^{-i_1 + i_2 + i_3} \right) R_{i_1, i_2, i_3}^{j_1, j_2, j_3} + \omega^{-1/2 + i_2 - i_1} \frac{w_1 u_2}{u_1 u'_3} R_{i_1, i_2, i_3}^{j_1, j_2 - 1, j_3} - \frac{\kappa_2 w_1}{u'_3} R_{i_1, i_2, i_3}^{j_1, j_2, j_3 - 1} = 0, \tag{20}$$

together with five other similar relations from (17). The indices are cyclic mod N .

Now, going back to (8), we remark that in our representation also any rational function Φ of the $\mathbf{u}_1, \dots, \mathbf{w}_3$ is represented by a $N^3 \times N^3$ matrix with entries which are rational functions of u_1, \dots, w_3 . The mapping $\mathcal{R}_{1,2,3}$ transforms the matrix entries by $\mathcal{R}_{1,2,3}^{(f)}$ and conjugates the matrix by $\mathbf{R}_{1,2,3}$. So we get the

Remark 2 *At q a root of unity, the mapping $\mathcal{R}_{1,2,3}$ can be represented as the superposition of the pure functional mapping $\mathcal{R}_{1,2,3}^{(f)}$ and the finite dimensional similarity transformation $\mathbf{R}_{1,2,3}$:*

$$\mathcal{R}_{1,2,3} \circ \Phi = \mathbf{R}_{1,2,3} \left(\mathcal{R}_{1,2,3}^{(f)} \circ \Phi \right) \mathbf{R}_{1,2,3}^{-1}. \tag{21}$$

This remarkable property arising in models where the quantum variables are elements of a Weyl algebra at root of unity, has been pointed out in [22] and in [18], [19].

3 Tetrahedron equation and modified tetrahedron equation

We postpone for a moment the discussion of the explicit form of $\mathbf{R}_{1,2,3}$ and of the possible form of $\mathcal{R}_{1,2,3}^{(f)}$. We shall first see that, once we have constructed the invertible canonical mapping of the triple Weyl algebra $\mathcal{R}_{1,2,3}$, no calculation is needed to conclude that $\mathcal{R}_{i,j,k}$

satisfies the Tetrahedron equation. We want to keep the point of view taken in Fig.1 that the mapping $\mathcal{R}_{1,2,3}$ comes about by moving the auxiliary plane through a vertex of the basic lattice. For considering the single mapping $\mathcal{R}_{1,2,3}$ we selected *three* planes of the basic lattice and intersected these by the auxiliary plane. Now we look at *four* (with no pair being parallel) lattice planes intersecting the auxiliary plane. In the auxiliary plane this gives rise to a figure called quadrangle shown in Fig.4. A quadrangle consists of four lines which form two triangles and one four-sided figure. There are six crossing points. Shifting the auxiliary plane within the basic lattice, we can reverse (from Δ to ∇) either of

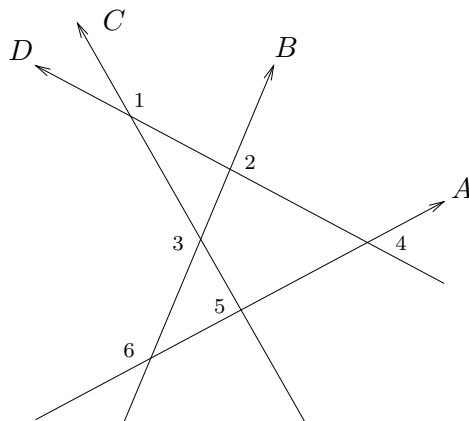


Figure 4: Quadrangle in the auxiliary plane formed by the directed intersection lines of 4 oriented lattice planes.

the two triangles: Shifting such that the plane producing the line B gets to the other side of point 1 (this happens moving the auxiliary plane through the vertex formed by planes B, C, D) we perform the mapping $\mathcal{R}_{1,2,3}$. Alternatively, we may shift the plane C through the point 6 (moving the auxiliary plane through the vertex of A, B, C), so performing the mapping $\mathcal{R}_{3,5,6}$.

As shown in Fig.5, there are two different sequences of mappings which, starting from Q_1 , lead to the same quadrangle Q_5 : either $Q_1 \rightarrow Q_2 \rightarrow Q_3 \rightarrow Q_4 \rightarrow Q_5$ or $Q_1 \rightarrow Q_8 \rightarrow Q_7 \rightarrow Q_6 \rightarrow Q_5$. Since the mapping $\mathcal{R}_{i,j,k}$ is invertible and canonical, we conclude that the two corresponding products of the \mathcal{R} must be the same mapping:

$$\mathcal{R}_{123} \cdot \mathcal{R}_{145} \cdot \mathcal{R}_{246} \cdot \mathcal{R}_{356} \sim \mathcal{R}_{356} \cdot \mathcal{R}_{246} \cdot \mathcal{R}_{145} \cdot \mathcal{R}_{123}, \quad (22)$$

i.e. that $\mathcal{R}_{i,j,k}$ satisfies the Tetrahedron equation². (22) is an operator equation in the space of the six Weyl variables. An equivalent interpretation is that the six dynamic variables live on the six edges of a tetrahedron, and the right hand side of (22) is obtained from the left hand side by moving one corner of the tetrahedron through the plane defined by the other three corners. Inserting the decomposition (21) of $\mathcal{R}_{i,j,k}$ into a matrix- and

²We use the following notation for the superposition of two mappings \mathcal{A} and \mathcal{B} :

$$((\mathcal{A} \cdot \mathcal{B}) \circ \Phi) \stackrel{def}{=} (\mathcal{A} \circ (\mathcal{B} \circ \Phi)). \quad (23)$$

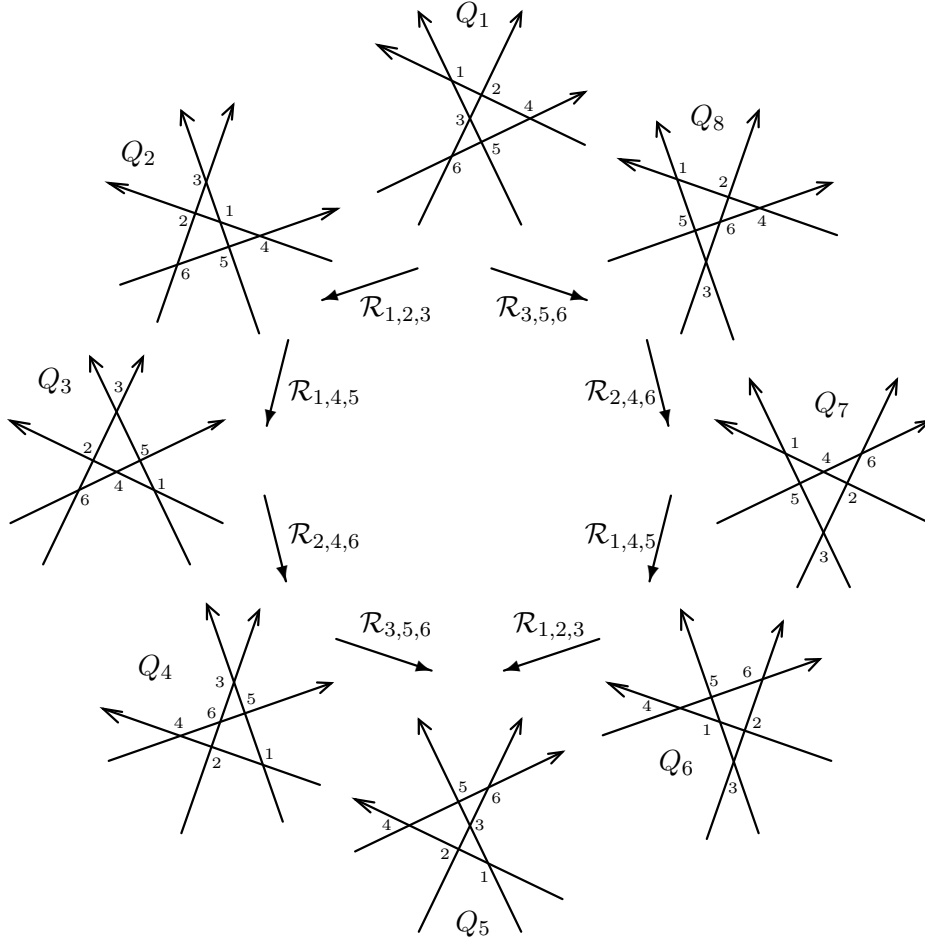


Figure 5: Graphical image of the two equivalent ways of transforming the four-line-graph (“quadrilateral”) Q_1 into graph Q_5 , which leads to tetrahedron equation. Observe that each graph contains only two triangles which can be transformed by a mapping \mathcal{R} : In graph Q_1 either the line 124 can be moved downward through the point 3 (leading to graph Q_2), or the line 456 can be moved upward through point 3 (leading to Q_8). Both the left hand and right hand sequences of four transformations lead to the same graph Q_5 .

a functional part into (22), we get

$$\begin{aligned}
 \mathbf{R}_{123} & \left[\mathcal{R}_{123}^{(f)} \left\{ \mathbf{R}_{145} \left(\mathcal{R}_{145}^{(f)} \left[\mathbf{R}_{246} \left\{ \mathcal{R}_{246}^{(f)} \left(\mathbf{R}_{356} \left(\mathcal{R}_{356}^{(f)} \circ \Phi \right) \mathbf{R}_{356}^{-1} \right) \right\} \mathbf{R}_{246}^{-1} \right) \right] \mathbf{R}_{145}^{-1} \right\} \right] \mathbf{R}_{123}^{-1} \\
 & = \mathbf{R}_{356} \left[\mathcal{R}_{356}^{(f)} \left\{ \mathbf{R}_{246} \left(\mathcal{R}_{246}^{(f)} \left[\mathbf{R}_{145} \left\{ \mathcal{R}_{145}^{(f)} \left(\mathbf{R}_{123} \left(\mathcal{R}_{123}^{(f)} \circ \Phi \right) \mathbf{R}_{123}^{-1} \right) \right\} \mathbf{R}_{145}^{-1} \right) \right] \mathbf{R}_{246}^{-1} \right\} \right] \mathbf{R}_{356}^{-1}.
 \end{aligned} \tag{24}$$

In (24) we have to apply the functional transformation repeatedly to matrices $\mathbf{R}_{i,j,k}$ and to $\mathcal{R}_{i,j,k}^{(f)}$ and Φ . Let us write shorthand

$$\begin{aligned}
 \mathbf{R}^{(1)} & = \mathbf{R}_{123}; & \mathbf{R}^{(2)} & = \mathcal{R}_{1,2,3}^{(f)} \circ \mathbf{R}_{145}; & \mathbf{R}^{(3)} & = \mathcal{R}_{1,2,3}^{(f)} \mathcal{R}_{1,4,5}^{(f)} \circ \mathbf{R}_{246}; \\
 \mathbf{R}^{(4)} & = \mathcal{R}_{1,2,3}^{(f)} \mathcal{R}_{1,4,5}^{(f)} \mathcal{R}_{2,4,6}^{(f)} \circ \mathbf{R}_{356}; & \mathbf{R}^{(5)} & = \mathcal{R}_{3,5,6}^{(f)} \mathcal{R}_{2,4,6}^{(f)} \mathcal{R}_{1,4,5}^{(f)} \circ \mathbf{R}_{123}; \\
 \mathbf{R}^{(6)} & = \mathcal{R}_{3,5,6}^{(f)} \mathcal{R}_{2,4,6}^{(f)} \circ \mathbf{R}_{145}; & \mathbf{R}^{(7)} & = \mathcal{R}_{3,5,6}^{(f)} \circ \mathbf{R}_{246}; & \mathbf{R}^{(8)} & = \mathbf{R}_{356}.
 \end{aligned} \tag{25}$$

Then (24) can be rewritten as

$$\begin{aligned} & \left(\mathbf{R}^{(1)} \mathbf{R}^{(2)} \mathbf{R}^{(3)} \mathbf{R}^{(4)} \right) \left\{ \mathcal{R}_{123}^{(f)} \cdot \mathcal{R}_{145}^{(f)} \cdot \mathcal{R}_{246}^{(f)} \cdot \mathcal{R}_{356}^{(f)} \Phi \right\} \left(\mathbf{R}^{(1)} \mathbf{R}^{(2)} \mathbf{R}^{(3)} \mathbf{R}^{(4)} \right)^{-1} \\ &= \left(\mathbf{R}^{(8)} \mathbf{R}^{(7)} \mathbf{R}^{(6)} \mathbf{R}^{(5)} \right) \left\{ \mathcal{R}_{356}^{(f)} \cdot \mathcal{R}_{246}^{(f)} \cdot \mathcal{R}_{145}^{(f)} \cdot \mathcal{R}_{123}^{(f)} \Phi \right\} \left(\mathbf{R}^{(8)} \mathbf{R}^{(7)} \mathbf{R}^{(6)} \mathbf{R}^{(5)} \right)^{-1}. \end{aligned} \quad (26)$$

Since, as has been discussed in Sec. 2.1, any rational automorphism of the ultra-local Weyl algebra implies a rational mapping in the space of the N -th powers of the parameters of the representation, it is a direct consequence of (22) that the $\mathcal{R}_{i,j,k}^{(f)}$ of (15) solve the tetrahedron equation with the variables $u_j^N, w_j^N, j = 1, \dots, 6$:

$$\left(\mathcal{R}_{123}^{(f)} \cdot \mathcal{R}_{145}^{(f)} \cdot \mathcal{R}_{246}^{(f)} \cdot \mathcal{R}_{356}^{(f)} - \mathcal{R}_{356}^{(f)} \cdot \mathcal{R}_{246}^{(f)} \cdot \mathcal{R}_{145}^{(f)} \cdot \mathcal{R}_{123}^{(f)} \right) \phi(u_1^N, \dots, w_6^N) = 0. \quad (27)$$

Using Maple, by a ten-line-half-minute-program one can easily check the validity of this functional equation straight from the definitions (14), (13).

Now the \mathbf{R}^j depend not on the u_i^N, w_i^N but on the u_i, w_i , see (17). So we have to choose the phases, which are at our disposal in defining $\mathcal{R}_{i,j,k}^{(f)}$ such that it solves the *functional* tetrahedron equation for rational functions of the variables u_j and w_j directly. However, not all phases of the u_j, w_j can be chosen independently because there are various centers which should be conserved. We shall not enter into the discussion of these phase choices here, in [34] it is shown that 16 phases can be chosen arbitrarily. Once a consistent choice of phases has been made, we obtain (27) in the variables u_j, w_j and can cancel the central functional terms in (26). So we arrive at the

Proposition 1 *The finite dimensional matrices \mathbf{R} satisfy the Modified Tetrahedron Equation (MTE):*

$$\begin{aligned} & \mathbf{R}_{1,2,3} \cdot \left(\mathcal{R}_{1,2,3}^{(f)} \circ \mathbf{R}_{1,4,5} \right) \cdot \left(\mathcal{R}_{1,2,3}^{(f)} \mathcal{R}_{1,4,5}^{(f)} \circ \mathbf{R}_{2,4,6} \right) \cdot \left(\mathcal{R}_{1,2,3}^{(f)} \mathcal{R}_{1,4,5}^{(f)} \mathcal{R}_{2,4,6}^{(f)} \circ \mathbf{R}_{3,5,6} \right) \\ & \sim \mathbf{R}_{3,5,6} \cdot \left(\mathcal{R}_{3,5,6}^{(f)} \circ \mathbf{R}_{2,4,6} \right) \cdot \left(\mathcal{R}_{3,5,6}^{(f)} \mathcal{R}_{2,4,6}^{(f)} \circ \mathbf{R}_{1,4,5} \right) \cdot \left(\mathcal{R}_{3,5,6}^{(f)} \mathcal{R}_{2,4,6}^{(f)} \mathcal{R}_{1,4,5}^{(f)} \circ \mathbf{R}_{1,2,3} \right). \end{aligned} \quad (28)$$

The left and right hand sides of (28) may differ by a scalar factor, which arises when we pass from the equivalence of the mappings to the equality of the matrices. So, writing all matrix indices and using again the abbreviations (25), the MTE reads

$$\begin{aligned} & \sum_{j_1 \dots j_6} \left(R^{(1)} \right)_{i_1, i_2, i_3}^{j_1, j_2, j_3} \left(R^{(2)} \right)_{j_1, i_4, i_5}^{k_1, j_4, j_5} \left(R^{(3)} \right)_{j_2, j_4, i_6}^{k_2, k_4, j_6} \left(R^{(4)} \right)_{j_3, j_5, j_6}^{k_3, k_5, k_6} \\ &= \rho \sum_{j_1 \dots j_6} \left(R^{(8)} \right)_{i_3, i_5, i_6}^{j_3, j_5, j_6} \left(R^{(7)} \right)_{i_2, i_4, j_6}^{j_2, j_4, k_6} \left(R^{(6)} \right)_{i_1, j_4, j_5}^{j_1, k_4, k_5} \left(R^{(5)} \right)_{j_1, j_2, j_3}^{k_1, k_2, k_3}. \end{aligned} \quad (29)$$

Here ρ is the scalar density factor. Taking determinants in (29) we can express the N^3 -th power of the scalar factor as

$$\rho^{N^3} = \frac{\det \mathbf{R}^{(1)} \det \mathbf{R}^{(2)} \det \mathbf{R}^{(3)} \det \mathbf{R}^{(4)}}{\det \mathbf{R}^{(8)} \det \mathbf{R}^{(7)} \det \mathbf{R}^{(6)} \det \mathbf{R}^{(5)}}, \quad (30)$$

This can be obtained from the determinant of one single matrix $\mathbf{R}_{1,2,3}$ just by substituting the respective arguments. The components of the eight $\mathbf{R}^{(j)}$ may be considered as

Boltzmann weights of the models. However, in general these matrix elements will not be positive. We call eqs. (28) or (29) *modified* tetrahedron equations because in (28) the $\mathbf{R}^{(j)}$ depend on several "rapidity" variables u_1, \dots, w_6 which are not the same on the left- and right hand sides of the equation, but rather are related by various functional mappings. We shall later see that there is much freedom in choosing $\mathcal{R}_{1,2,3}^{(f)}$, accordingly leading to different models.

4 Details on the matrix- and the functional transformations

4.1 Matrix part of $\mathcal{R}_{1,2,3}$ at root of unity in terms of Fermat curve cyclic functions $W_p(n)$

In Sec.2.2 we introduced the conjugation operator $\mathbf{R}_{1,2,3}$ and its representation $R_{i_1, i_2, i_3}^{j_1, j_2, j_3}$, which allowed to write eqs.(17) in the form (18). We just have seen that the matrix elements of the $\mathbf{R}_{1,2,3}$ play the role of Boltzmann weights and solve the MTE.

In (20) we had remarked that $R_{i_1, i_2, i_3}^{j_1, j_2, j_3}$ has to satisfy six recursion relations with respect to its modulo N defined indices, but have not yet seen its explicit form.

We now show that the $R_{i_1, i_2, i_3}^{j_1, j_2, j_3}$ can be constructed from Fermat curve cyclic functions $W_p(n)$ which were introduced by Bazhanov and Baxter [5] for building up the Boltzmann weights of their model. The form (33) which we shall use has first been given in [10] several years ago.

Introduce a two component vector $p = (x, y)$ which is restricted to the Fermat curve $x^N + y^N = 1$. Then define the function $W_p(n)$ by

$$W_p(0) = 1, \quad W_p(n) = \prod_{\nu=1}^n \frac{y}{1 - \omega^\nu x} \quad \text{for } n > 0. \quad (31)$$

$W_p(n)$ satisfies the obvious recursion relation

$$W_p(n-1) = (1 - \omega^n x) y^{-1} W_p(n). \quad (32)$$

and is cyclic in n : $W_p(n+N) = W_p(n)$, because of $\prod_{\nu=0}^{N-1} (1 - \omega^\nu x) = y^N$. Observe that we need the Fermat curve restriction in order to get the cyclic property. In order to satisfy the *six* recursion relation required by (17), $R_{i_1, i_2, i_3}^{j_1, j_2, j_3}$ must be taken to be a kind of cross ratio of four functions W_p , depending on four Fermat points p_1, \dots, p_4 :

Proposition 2 *In the basis (11) the mapping relations (17), (18) are solved by the matrix*

$$R_{i_1, i_2, i_3}^{j_1, j_2, j_3} = \delta_{i_2+i_3, j_2+j_3} \omega^{(j_1-i_1)j_3} \frac{W_{p_1}(i_2 - i_1) W_{p_2}(j_2 - j_1)}{W_{p_3}(j_2 - i_1) W_{p_4}(i_2 - j_1)} \quad (33)$$

where the x -coordinates of the four Fermat curve points are connected by

$$x_1 x_2 = \omega x_3 x_4, \quad (34)$$

and the Fermat points are given in terms of the variables $u_j, w_j, \kappa_j, j = 1, 2, 3$ by

$$\begin{aligned} x_1 &= \frac{\omega^{-1/2} u_2}{\kappa_1 u_1}, & x_2 &= \omega^{-1/2} \kappa_2 \frac{u_2'}{u_1'}, & x_3 &= \omega^{-1} \frac{u_2'}{u_1'}, & x_4 &= \omega^{-1} \frac{\kappa_2 u_2}{\kappa_1 u_1'}, \\ \frac{y_3}{y_1} &= \kappa_1 \frac{w_1}{u_3'}, & \frac{y_4}{y_1} &= \omega^{-1/2} \kappa_3 \frac{w_3}{w_2}, & \frac{y_3}{y_2} &= \frac{w_2'}{w_3}, & \frac{y_4}{y_2} &= \omega^{-1/2} \frac{\kappa_3 u_3'}{\kappa_1 w_1'}, \end{aligned} \quad (35)$$

The u'_j , w'_j and u_j , w_j are related by the functional transformation (15):

$$u'_j = \mathcal{R}_{1,2,3}^{(f)} \circ u_j, \quad w'_j = \mathcal{R}_{1,2,3}^{(f)} \circ w_j. \quad (36)$$

In order to prove this proposition, we have to check the six recursion relations implied by (17) and have to compare the Fermat point coefficients arising from (32) with the Weyl coefficients in (17). We give one example:

$R_{i_1, i_2, i_3}^{j_1, j_2, j_3}$ satisfies

$$R_{i_1, i_2+1, i_3-1}^{j_1, j_2, j_3} = R_{i_1, i_2, i_3}^{j_1, j_2, j_3} \cdot \frac{y_1}{y_4} \cdot \frac{1 - \omega^{i_2-j_1+1} x_4}{1 - \omega^{i_2-i_1+1} x_1}$$

which can be rewritten in the form

$$R_{i_1, i_2, i_3}^{j_1, j_2, j_3} \omega^{-j_1} = \frac{1}{\omega^{i_2+1} x_4} R_{i_1, i_2, i_3}^{j_1, j_2, j_3} + \frac{x_1 y_4}{\omega^{i_1} y_1 x_4} R_{i_1, i_2+1, i_3-1}^{j_1, j_2, j_3} - \frac{y_4}{\omega^{i_2+1} x_4 y_1} R_{i_1, i_2+1, i_3-1}^{j_1, j_2, j_3}. \quad (37)$$

Using the identifications (35) we see that this agrees with the matrix elements of the first equation of (17).

In order to find the density factor of the modified tetrahedron equation, in (30) we have seen that we need the determinant of $\mathbf{R}_{1,2,3}$. Its calculation is made nontrivial by the presence of the factor $\delta_{i_2+i_3, j_2+j_3} \omega^{(j_1-i_1)j_3}$ in (33). There is an interesting technology involving the Fermat curve functions $W_p(n)$, details of which can be found in [5, 10, 34], compare also the appearance of $W_p(n)$ in connection with the quantum dilogarithm e.g. in eq.(3.7) of [19]. The calculation of $\det \mathbf{R}$ is given in detail in [34]. The result is:

$$\det \mathbf{R} = N^{N^3} \left(\left(\frac{x_4}{y_1 y_2} \right)^{N(N-1)/2} \frac{V(x_1)V(x_2)}{V(x_3)V(x_4)} \right)^{N^2} \quad (38)$$

with $V(x) = \prod_{\nu=1}^{N-1} (1 - \omega^{\nu+1} x)^\nu$.

4.2 The matrix $R_{i_1, i_2, i_3}^{j_1, j_2, j_3}$ for $N = 2$.

For $N = 2$, using combined indices $i = 1 + i_1 + 2i_2 + 4i_3$; $j = 1 + j_1 + 2j_2 + 4j_3$ we can give $R_{i_1, i_2, i_3}^{j_1, j_2, j_3}$ explicitly in matrix form: We define

$$Y_j = \frac{y_j}{1 + x_j} = \sqrt{\frac{1 - x_j}{1 + x_j}} \quad \text{for } j = 1, 2, 3, 4;$$

$$Z_{ij} = \frac{Y_i}{Y_j} \quad \text{for } ij = 13, 14, 23, 24; \quad Z_{12} = Y_1 Y_2; \quad Z_{34} = \frac{1}{Y_3 Y_4}. \quad (39)$$

and get

$$\mathbf{R}_i^j = \begin{pmatrix} 1 & Z_{24} & 0 & 0 & 0 & 0 & Z_{23} & -Z_{34} \\ Z_{13} & Z_{13}Z_{24} & 0 & 0 & 0 & 0 & -Z_{12} & Z_{14} \\ 0 & 0 & Z_{13}Z_{24} & Z_{13} & Z_{14} & -Z_{12} & 0 & 0 \\ 0 & 0 & Z_{24} & 1 & -Z_{34} & Z_{23} & 0 & 0 \\ 0 & 0 & Z_{23} & Z_{34} & 1 & -Z_{24} & 0 & 0 \\ 0 & 0 & Z_{12} & Z_{14} & -Z_{13} & Z_{13}Z_{24} & 0 & 0 \\ Z_{14} & Z_{12} & 0 & 0 & 0 & 0 & Z_{13}Z_{24} & -Z_{13} \\ Z_{34} & Z_{23} & 0 & 0 & 0 & 0 & -Z_{24} & 1 \end{pmatrix}_{ij} \quad (40)$$

The determinant can be calculated directly:

$$\det \mathbf{R} = (Y_1 Y_2 (1 + Y_3^{-2})(1 + Y_4^{-2}))^4 . \quad (41)$$

4.3 The functional mapping in terms of trilinear Hirota equations

When we wrote the MTE in Sec.3, we did not specify the details of the functional transformations $\mathcal{R}_{i,j,k}^{(f)}$ which act on the scalar parameters of the quantum operators $\mathbf{R}_{i,j,k}$ (by (33) the scalar parameters then determine the Fermat variables). We are interested to write this more explicitly and we like to understand which restrictions the MTE imposes on the $\mathcal{R}_{i,j,k}^{(f)}$.

Slightly rewritten, eqs. (13), (14) which define the functional mapping, are

$$\begin{aligned} \frac{u_3^N}{u_3^N} &= \frac{u_2^N}{u_2^N} = (u_3^{-1} \Lambda_1 u_2)^N = \frac{u_2^N}{u_1^N} + \frac{w_1^N u_2^N}{u_1^N u_3^N} + \kappa_1^N \frac{w_1^N}{u_3^N}, \\ \frac{u_1^N}{u_1^N} &= \frac{w_3^N}{w_3^N} = (w_3 \Lambda_2 u_1)^N = \frac{\kappa_1^N u_1^N}{\kappa_2^N u_2^N} + \frac{\kappa_3^N w_3^N}{\kappa_2^N w_2^N} + \frac{\kappa_1^N \kappa_3^N u_1^N w_3^N}{\kappa_2^N u_2^N w_2^N}; \\ \frac{w_1^N}{w_1^N} &= \frac{w_2^N}{w_2^N} = (w_1^{-1} \Lambda_3 w_2)^N = \frac{w_2^N}{w_3^N} + \frac{w_2^N u_3^N}{w_1^N w_3^N} + \kappa_3^N \frac{u_3^N}{w_1^N}. \end{aligned} \quad (42)$$

If we just use (42) repeatedly to calculate the $\mathbf{R}^{(j)}$ in (25), we get very lengthy expressions. $\mathbf{R}^{(3)}$ will be much more complicated than $\mathbf{R}^{(2)}$ etc. It is not transparent, which variables are independent because the conservation of the centers is hidden in complicated formulas. However, the nice symmetry of Fig.5 suggests that a different parameterization should be possible which treats all operators in (28) on essentially equal footing. Indeed this is achieved applying a Legendre transformation or, equivalently, by the introduction of tau-functions as it is common practice in the theory of classical integrable systems [35]. Let us see in an elementary way how this works.

In Fig.6 we show a magnified view of the two quadrangles Q_1 and Q_2 of Fig.5. Consider Q_1 : The twelve scalar variables u_i, w_i ($i = 1, \dots, 6$) are located at the six vertices labeled $i = 1, \dots, 6$. The vertices cut each directed line A, B, C, D into four line-sections. We label these line-sections as shown in the figure. Ratios of line sections will be our new variables and we explain the rule for the variables u_1, w_1 associated with vertex "1" in the upper left corner:

- $u_1 = c_2/c_3$, i.e. a u_i is expressed as the ratio of the sections *before* to *after* point "1" on the *right* pointing line through the vertex "1".
- $w_1 = d_3/d_2$, i.e. a w_i is expressed as the ratio of the sections *after* to *before* point "1" on the *left* pointing line.

This way the twelve scalar variables $u_1, w_1, \dots, u_6, w_6$ of each quadrangle are expressed in terms of eight "internal" line sections, $a_1, a_2, \dots, d_1, d_2$ for Q_1 , and eight "external" line sections $a_0, a_3, \dots, d_0, d_3$.

Passing from one quadrilateral to an adjacent one, corresponding to a mapping \mathcal{R} , always three of the "internal" lines are changed, and for distinction, we attach to these changed variables dashes (if left-turning in Fig.5 and daggers (if right turning). The

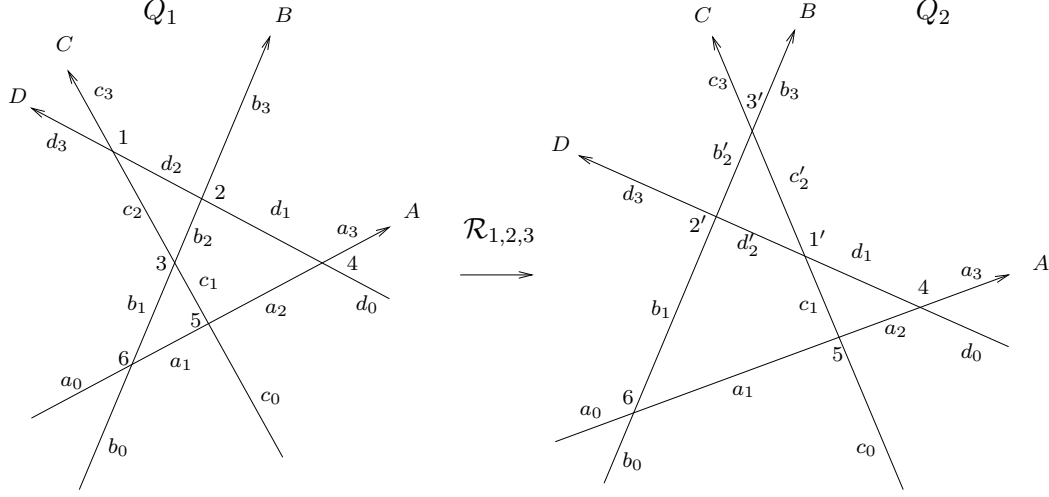


Figure 6: Parameterization of the arguments $u_i^{(j)}$, $w_i^{(j)}$ of the rational mapping $\mathcal{R}_{i,j,k}^{(j)}$ in terms of line-section ratios.

“external“ line sections remain unchanged in all mappings and so are no relevant dynamic variables.

Applied to the transformation $\mathcal{R}_{1,2,3}^{(f)}$ this means that we substitute

$$\begin{aligned}
 u_1 &= \frac{c_2}{c_3}; & u_2 &= \frac{b_2}{b_3}; & u_3 &= \frac{b_1}{b_2}; & w_1 &= \frac{d_3}{d_2}; & w_2 &= \frac{d_2}{d_1}; & w_3 &= \frac{c_2}{c_1}; \\
 u'_1 &= \frac{c_1}{c'_2}; & u'_2 &= \frac{b_1}{b'_2}; & u'_3 &= \frac{b'_2}{b_3}; & w'_1 &= \frac{d'_2}{d_1}; & w'_2 &= \frac{d_3}{d'_2}; & w'_3 &= \frac{c_3}{c'_2},
 \end{aligned} \tag{43}$$

and (42) become tri-linear equations for the three dashed variables:

$$\begin{aligned}
 b_2'^N c_2^N d_2^N &= b_1^N c_3^N d_2^N + b_2^N c_3^N d_3^N + \kappa_1^N b_3^N c_2^N d_3^N; \\
 \kappa_2^N b_2^N c_2^N d_2^N &= \kappa_1^N b_3^N c_1^N d_2^N + \kappa_3^N b_2^N c_3^N d_1^N + \kappa_1^N \kappa_3^N b_3^N c_2^N d_1^N; \\
 b_2^N c_2^N d_2'^N &= b_2^N c_1^N d_3^N + b_1^N c_1^N d_2^N + \kappa_3^N b_1^N c_2^N d_1^N;
 \end{aligned} \tag{44}$$

The merit of this parameterization is that it automatically incorporates the invariance of the centers of the mapping: $w'_1 w'_2 = w_1 w_2$, $u'_2 u'_3 = u_2 u_3$, $u'_1 w'_3{}^{-1} = u_1 w_3{}^{-1}$. It is also simple to express the Fermat variables appearing in (35) in terms of the line-section parameters. For $\mathbf{R}_{1,2,3}$ we label the Fermat coordinates by a superscript (1) (as in (25)), and get:

$$\begin{aligned}
 x_1^{(1)} &= \omega^{-1/2} \frac{b_2 c_3}{\kappa_1 b_3 c_2}; & x_2^{(1)} &= \omega^{-1/2} \frac{\kappa_2 b_1 c'_2}{b'_2 c_1}; & x_3^{(1)} &= \omega^{-1} \frac{b_1 c_3}{b'_2 c_2}; \\
 \frac{y_3^{(1)}}{y_1^{(1)}} &= \frac{\kappa_1 b_3 d_3}{b'_2 d_2}; & \frac{y_4^{(1)}}{y_1^{(1)}} &= \omega^{-1/2} \frac{\kappa_3 c_2 d_1}{c_1 d_2}; & \frac{y_3^{(1)}}{y_2^{(1)}} &= \frac{c_1 d_3}{c_2 d'_2}.
 \end{aligned} \tag{45}$$

Now to find all the functional mappings which appear in the MTE (28), we have to write also the other seven relations corresponding to $Q_2 \rightarrow Q_3, \dots, Q_1 \rightarrow Q_8$, which look all

similar to (44), (45). E.g. $\mathbf{R}_{1,4,5}$ leading from Q_2 to Q_3 is governed by $a_2'', c_1'', d_1'', c_2', d_2'$ etc. which are related by

$$\begin{aligned} a_2''^N c_1^N d_1^N &= a_1^N c_2'^N d_1^N + a_2^N c_2'^N d_2'^N + \kappa_1^N a_3^N c_1^N d_2'^N; \\ \kappa_4^N a_2^N c_1^N d_1^N &= \kappa_1^N a_3^N c_0^N d_1^N + \kappa_5^N a_2^N c_2'^N d_0^N + \kappa_1^N \kappa_5^N a_3^N c_1^N d_0^N; \\ a_2^N c_1^N d_1^N &= a_2^N c_0^N d_2'^N + a_1^N c_0^N d_1^N + \kappa_5^N a_1^N c_1^N d_0^N. \end{aligned} \quad (46)$$

Since we know already that $\mathcal{R}_{i,j,k}^{(f)}$ is a solution to the functional TE, one may easily show that eight of the 8×3 trilinear eqs. of type (44) are superfluous. For the whole MTE we have introduced 16 line sections to replace the 12 scalar variables, but the eight “external“ variables a_0, \dots, d_3 and the six couplings κ_i turn out to be irrelevant, i.e. these can be eliminated by a re-scaling [34].

So the conclusion is that the eight “internal“ variables $a_1, a_2, \dots, d_1, d_2$ together with sixteen phase choices parameterize the MTE eq.(28) or (29). This is much more complicated than the rapidities which appear equally on both sides of a Yang-Baxter-equation. Of course, there is no difference property of the parameters of the MTE.

4.4 Solving the MTE using tools of algebraic geometry

It is well-known [26, 27, 28, 29, 30] that discrete integrable classical equations can be solved by methods of algebraic geometry. The bilinear Hirota equation is equivalent to Fay’s identity for theta-functions on the Jacobian of an algebraic curve [29]. The tri-linear equations (44) are also of the correct type to be solved using Fay’s identity.

We use standard concepts, see e.g. [31]: Let Γ_g be a generic algebraic curve of genus g and ω the canonical g -dimensional vector of the holomorphic differentials. For any two points $X, Y \in \Gamma_g$ let $\mathbf{I}: \Gamma^{\otimes 2} \mapsto \mathbb{C}^g$ be the Jacobi transform:

$$\mathbf{I}_X^Y \equiv \mathbf{I}(X, Y) \stackrel{def}{=} \int_X^Y \omega \in \text{Jac}(\Gamma_g). \quad (47)$$

Let further $E(X, Y) = -E(Y, X)$ be the prime form on $\Gamma^{\otimes 2}$, $\mathbf{v} \in \mathbb{C}^g$, and $\Theta(\mathbf{v})$ the zero characteristic theta-function on $\text{Jac}(\Gamma_g)$. Then, for $A, B, C, D \in \Gamma_g$ Fay’s identity states

$$\begin{aligned} \Theta(\mathbf{v}) \Theta(\mathbf{v} + \mathbf{I}_B^A + \mathbf{I}_D^C) &= \Theta(\mathbf{v} + \mathbf{I}_D^A) \Theta(\mathbf{v} + \mathbf{I}_B^C) \frac{E(A, B) E(D, C)}{E(A, C) E(D, B)} \\ &+ \Theta(\mathbf{v} + \mathbf{I}_B^A) \Theta(\mathbf{v} + \mathbf{I}_D^C) \frac{E(A, D) E(C, B)}{E(A, C) E(D, B)}. \end{aligned} \quad (48)$$

In order to solve (44), (46), etc. we can build a tri-linear identity by applying (48) twice:

$$\begin{aligned} &\Theta(\mathbf{v} + \mathbf{I}_X^Q) \Theta(\mathbf{v} + \mathbf{I}_Y^Q + \mathbf{I}_Z^{Z'}) \Theta(\mathbf{v} + \mathbf{I}_Z^Q + \mathbf{I}_Y^{Y'}) \\ &= \Theta(\mathbf{v} + \mathbf{I}_X^Q + \mathbf{I}_Z^{Z'}) \Theta(\mathbf{v} + \mathbf{I}_Y^Q) \Theta(\mathbf{v} + \mathbf{I}_Z^Q + \mathbf{I}_Y^{Y'}) \frac{E(Z, X) E(Y, Z')}{E(Z', X) E(Y, Z)} \\ &+ \Theta(\mathbf{v} + \mathbf{I}_X^Q + \mathbf{I}_Y^{Y'}) \Theta(\mathbf{v} + \mathbf{I}_Z^Q) \Theta(\mathbf{v} + \mathbf{I}_Y^Q + \mathbf{I}_Z^{Z'}) \frac{E(Y, X) E(Y', Z)}{E(Y', X) E(Y, Z)} \\ &- \Theta(\mathbf{v} + \mathbf{I}_X^Q + \mathbf{I}_Y^{Y'} + \mathbf{I}_Z^{Z'}) \Theta(\mathbf{v} + \mathbf{I}_Y^Q) \Theta(\mathbf{v} + \mathbf{I}_Z^Q) \frac{E(Z, X) E(Y, X) E(Y', Z')}{E(Z', X) E(Y', X) E(Y, Z)}. \end{aligned} \quad (49)$$

Q, X, Y, Y', Z, Z' are arbitrary distinct points on Γ_g . The dependence on Q is trivial since it appears only in a fixed combination with \mathbf{v} . It is not easy to handle the prime forms themselves, but here only cross ratios of prime forms are needed and these are well-defined quasi-periodical functions on $\Gamma^{\otimes 4}$:

$$\frac{E(A, B) E(D, C)}{E(A, C) E(D, B)} = \frac{\Theta_{\epsilon_{odd}}(\mathbf{I}_A^B) \Theta_{\epsilon_{odd}}(\mathbf{I}_D^C)}{\Theta_{\epsilon_{odd}}(\mathbf{I}_A^C) \Theta_{\epsilon_{odd}}(\mathbf{I}_D^B)},$$

where $\Theta_{\epsilon_{odd}}$ is a non-singular odd characteristic theta function such that $\Theta_{\epsilon_{odd}}(\mathbf{0}) = 0$.

Consider eqs.(44) for showing the application of (49) to solve our tri-linear equations. If we identify

$$\begin{aligned} b_1^N &\sim \Theta(\mathbf{v} + \mathbf{I}_X^Q); & b_2^N &\sim \Theta(\mathbf{v} + \mathbf{I}_X^Q + \mathbf{I}_Y^{Y'}); & b_3^N &\sim \Theta(\mathbf{v} + \mathbf{I}_X^Q + \mathbf{I}_Y^{Y'} + \mathbf{I}_Z^{Z'}); \\ b_2'^N &\sim \Theta(\mathbf{v} + \mathbf{I}_X^Q + \mathbf{I}_Z^{Z'}); & c_2^N &\sim \Theta(\mathbf{v} + \mathbf{I}_Y^Q); & c_3^N &\sim \Theta(\mathbf{v} + \mathbf{I}_Y^Q + \mathbf{I}_Z^{Z'}); \\ d_2^N &\sim \Theta(\mathbf{v} + \mathbf{I}_Z^Q + \mathbf{I}_Y^{Y'}); & d_3^N &\sim \Theta(\mathbf{v} + \mathbf{I}_Z^Q); & d_2'^N &\sim \Theta(\mathbf{v} + \mathbf{I}_Z^Q + \mathbf{I}_X^{X'}). \end{aligned} \quad (50)$$

then, not discussing here the prime form cross ratios, the first eq. of (44) takes just the form (49). The third eq. of (44) takes this form if we exchange $X \leftrightarrow Z$ in (49). In order to obtain the second eq. of (44) replace in (49) $X \rightarrow Y', Y \rightarrow X$ and $\mathbf{v} \rightarrow \mathbf{v} + \mathbf{I}_Y^{Y'}$.

Here we shall not give the details [37] of the calculation which shows that all 24 tri-linear equations can be written in the form (49). For parameterizing the line section parameters and so parameterizing the eight $\mathbf{R}^{(j)}$ which appear in the MTE, we have to choose a Γ_g and to introduce just eight points $X, X', Y, Y', Z, Z', U; U' \in \Gamma_g$.

From these formulas simple parameterizations can be obtained taking the rational limit. Then the theta functions become trigonometric functions and the cross ratios of the prime forms become simple rational cross ratios [35].

5 Free bosonic realization of $\mathbf{R}_{1,2,3}$

Together with introducing the ZBB-model in [25], Bazhanov and Baxter also considered a related continuum bosonic model. Since our framework includes the ZBB-model, we expect that such a bosonic representation should also exist for $\mathbf{R}_{1,2,3}$. Since this broadens the class of models, we shall present here the basic formulas. We shall not consider this realization a limiting case of the discrete realizations, but rather a new representation.

Instead of the cyclic weights $W_p(n)$ of (31) we introduce the following Gaussian weights:

$$W_x(\sigma) = \exp\left(\frac{i}{2\hbar} \frac{x}{x-1} \sigma^2\right); \quad \sigma \in \mathbb{R}, \quad x \in \mathbb{C}; \quad \Im m \frac{x}{x-1} > 0. \quad (51)$$

At each vertex j of a graph define a pair of operators $\mathbf{q}_j, \mathbf{p}_j$ satisfying $[\mathbf{q}_{j'}, \mathbf{p}_j] = i\hbar\delta_{j,j'}$, and scalar variables u_j, w_j . We choose a basis $|\sigma_j\rangle$ with $\langle\sigma_{j'}|\sigma_j\rangle = \delta(\sigma_{j'} - \sigma_j)$, such that for $\psi(\sigma) \in L^2$ we have

$$\psi(\sigma_j) \stackrel{def}{=} \langle\sigma_j|\psi\rangle; \quad \langle\sigma_j|\mathbf{q}_j|\psi\rangle = \sigma_j\psi(\sigma_j); \quad \langle\sigma_j|\mathbf{p}_j|\psi\rangle = \frac{\hbar}{i} \frac{\partial\psi(\sigma_j)}{\partial\sigma_j}. \quad (52)$$

For each set of vertices we define the corresponding operators and scalars and a direct product space of the single vertex spaces. Consider now the following mapping $\mathbf{R}_{1,2,3}$ in

the product space of three points $j = 1, 2, 3$ by

$$\begin{aligned}\mathbf{R}_{123} \mathbf{q}_j \mathbf{R}_{123}^{-1} &= \sum_{k=1}^3 \frac{\partial \log u'_j}{\partial \log u_k} \mathbf{q}_k + \frac{\partial \log u'_j}{\partial \log w_k} \mathbf{p}_k \equiv \mathbf{q}'_j \\ \mathbf{R}_{123} \mathbf{p}_j \mathbf{R}_{123}^{-1} &= \sum_{k=1}^3 \frac{\partial \log w'_j}{\partial \log u_k} \mathbf{q}_k + \frac{\partial \log w'_j}{\partial \log w_k} \mathbf{p}_k \equiv \mathbf{p}'_j.\end{aligned}\quad (53)$$

where the relation between the primed and unprimed scalars u_j , w_j is given by (15) with $N = 1$. The κ_j are further parameters (“coupling constants”) at the vertices j . As the following Proposition shows, equations (53) are the bosonic continuum analogs to eqs.(17) of the discrete case.

Proposition 3 *In the basis (52) the operator $\mathbf{R}_{1,2,3}$ has the following kernel*

$$\begin{aligned}\langle \sigma_1, \sigma_2, \sigma_3 | \mathbf{R}_{1,2,3} | \sigma'_1, \sigma'_2, \sigma'_3 \rangle \\ = \delta(\sigma_2 + \sigma_3 - \sigma'_2 - \sigma'_3) e^{-\frac{i}{\hbar}(\sigma'_1 - \sigma_1)\sigma'_3} \frac{W_{x_1}(\sigma_2 - \sigma_1) W_{x_2}(\sigma'_2 - \sigma'_1)}{W_{x_3}(\sigma'_2 - \sigma_1) W_{x_4}(\sigma_2 - \sigma'_1)},\end{aligned}\quad (54)$$

with the constraint $x_1 x_2 = x_3 x_4$. In terms of the variables u_j , w_j , κ_j , ($j = 1, 2, 3$), the $x_k \in \mathbb{R}$ are defined by (obtained by putting formally $\omega^{-1/2} \rightarrow -1$ in (35)):

$$x_1 = -\frac{1}{\kappa_1} \frac{u_2}{u_1}; \quad x_2 = -\kappa_2 \frac{u'_2}{u'_1}; \quad x_3 = \frac{u'_2}{u_1}; \quad x_4 = \frac{\kappa_2}{\kappa_1} \frac{u_2}{u'_1}, \quad (55)$$

where u'_1 and u'_2 are defined as in (36) with $N = 1$.

Proof: We give the proof for one of the six equations (53), as the other equations follow analogously. Let us write shorthand $|\sigma\rangle$ for $|\sigma_1, \sigma_2, \sigma_3\rangle$ and $d^3\sigma = d\sigma_1 d\sigma_2 d\sigma_3$ etc. We consider:

$$\int d^3\sigma' \langle \sigma | \mathbf{R}_{1,2,3} | \sigma' \rangle \langle \sigma' | \mathbf{q}_3 | \sigma'' \rangle = \int d^3\sigma' \langle \sigma | \mathbf{q}'_3 | \sigma' \rangle \langle \sigma' | \mathbf{R}_{1,2,3} | \sigma'' \rangle \quad (56)$$

which should be satisfied for all $\sigma_1, \sigma_2, \sigma_3, \sigma''_1, \sigma''_2, \sigma''_3$. Written more explicitly, the kernel (54) is:

$$R_{\sigma_1, \sigma_2, \sigma_3}^{\sigma'_1, \sigma'_2, \sigma'_3} = \delta(\sigma_2 + \sigma_3 - \sigma'_2 - \sigma'_3) \exp\left(\frac{i}{2\hbar} \Sigma(\sigma, \sigma')\right);$$

where

$$\begin{aligned}\Sigma(\sigma, \sigma') &= \frac{w_1(\sigma_2 - \sigma_1)^2 + u_3(\sigma_1 - \sigma'_2)^2}{u_2^{-1} w_1 (\kappa_1 u_1 + u_2)} + \frac{(\kappa_1 u_1 w_2 + \kappa_3 u_2 w_3 + \kappa_1 \kappa_3 u_1 w_3)(\sigma_2 - \sigma'_1)^2}{\kappa_3 w_3 (\kappa_1 u_1 + u_2)} \\ &+ \frac{u_3(\kappa_1 u_1 w_2 + \kappa_3 u_2 w_3 + \kappa_1 \kappa_3 u_1 w_3)(\sigma'_2 - \sigma'_1)^2}{(\kappa_3 u_3 w_3 + w_1 w_2 + u_3 w_2)(\kappa_1 u_1 + u_2)} - 2(\sigma'_1 - \sigma_1)\sigma'_3.\end{aligned}\quad (57)$$

From (53) we get:

$$\mathbf{q}'_3 = \frac{(w_1 + u_3)u_2(\mathbf{q}_2 - \mathbf{q}_1) + u_2 u_3 \mathbf{q}_3 + (u_2 + \kappa_1 u_1)w_1 \mathbf{p}_1}{u_2 u_3 + u_2 w_1 + \kappa_1 u_1 w_1}.$$

and (56) becomes:

$$\begin{aligned}
& \int d^3\sigma' \delta(\sigma_2 + \sigma_3 - \sigma'_2 - \sigma'_3) \exp\left(\frac{i}{2\hbar}\Sigma(\sigma, \sigma')\right) \sigma_3'' \delta^3(\sigma' - \sigma'') \\
&= \int d^3\sigma' \delta^3(\sigma - \sigma') \frac{(w_1 + u_3)u_2(\sigma'_2 - \sigma'_1) + u_2u_3\sigma'_3 + (u_2 + \kappa_1u_1)w_1\frac{\hbar}{i}\frac{\partial}{\partial\sigma'_1}}{u_2u_3 + u_2w_1 + \kappa_1u_1w_1} \times \\
&\quad \times \delta(\sigma'_2 + \sigma'_3 - \sigma''_2 - \sigma''_3) \exp\left(\frac{i}{2\hbar}\Sigma(\sigma', \sigma'')\right). \tag{58}
\end{aligned}$$

Since

$$(u_2 + \kappa_1u_1)\frac{w_1}{2} \frac{\partial\Sigma(\sigma', \sigma'')}{\partial\sigma'_1} = (w_1 + u_3)u_2\sigma'_1 - u_2w_1\sigma'_2 - u_2u_3\sigma''_2 + (w_1u_2 + \kappa_1u_1w_1)\sigma_3'',$$

eq.(58) reduces to

$$\sigma_3''\delta(\sigma_2 + \sigma_3 - \sigma''_2 - \sigma''_3) = \frac{u_2u_3(\sigma_2 + \sigma_3 - \sigma''_2) + (u_2w_1 + \kappa_1u_1w_1)\sigma_3''}{u_2u_3 + u_2w_1 + \kappa_1u_1w_1}\delta(\sigma_2 + \sigma_3 - \sigma''_2 - \sigma''_3).$$

□

Certainly, if this bosonic representation is used, the MTE involves integrations over \mathbb{R} instead of summations over \mathbb{Z}_N . The bosonic MTE may be proven directly with help of Gaussian integrations.

There are indications that this bosonic model will not be critical all over and so be physically interesting [36].

6 Conclusions

A large class of integrable 3-dimensional lattice spin models is constructed. The starting point and central object of the construction is an invertible canonical automorphism of a triple ultralocal Weyl algebra which satisfies two physically motivated principles: A current branching rule and a Baxter Z-invariance. To argue that this automorphism operator fulfils the tetrahedron equation requires no calculation. If the Weyl parameter q taken to be a root of unity, one chooses the $N \times N$ unitary representation of the Weyl operators and the N -th powers of the Weyl operators are represented by numbers. Accordingly, in this representation the automorphism operator can be decomposed into a matrix conjugation and a functional mapping. Quite trivially, the functional mapping operator satisfies a tetrahedron equation too. The conjugation matrix which can be expressed in compact fashion in terms of the Bazhanov-Baxter Fermat curve cyclic functions, satisfies the modified tetrahedron equation, (28). So we have a quantum problem with coefficients which are form a classical integrable system themselves. The large class of different solutions to the classical integrable system leads to a large class of quantum 3D-integrable models. The Zamolodchikov-Baxter-Bazhanov model is obtained if we choose the trivial classical solution. There is also a Gaussian continuum representation of the automorphism operator.

Our expectation is that the framework set is broad enough to contain physically interesting 3D-integrable lattice models. An immediate application is to perform an asymmetric limit to construct new 2D-integrable chiral models involving rapidity parameters living on

managable algebraic curves. The vast possibilities to choose convenient parameterization of the 3D-Boltzmann weights should open new possibilities to solve the 2D-descendent models.

Acknowledgements

This work was supported in part by the contract INTAS OPEN 00-00055. S.P.'s work was supported in part by the grants CRDF RM1-2334-MO-02, RFBR 01-01-00539 and the grant for support of scientific schools RFBR 00-15-9655. S.S.'s work was supported in part by the grants CRDF RM1-2334-MO-02 and RFBR 01-01-00201.

7 Appendix

7.1 \mathcal{R} -mapping from eqs.(3)

We give some details of the calculation leading from (3) to (4): We use the first four eqs. of (3) to express the currents $|\phi'_i\rangle$ ($i = 1, 2, 3$) and $|\phi_2\rangle$ in terms of $|\phi_1\rangle$ and $|\phi_3\rangle$. This gives the four equations (not commuting any Weyl variables):

$$\begin{aligned}
|\phi_g\rangle &\longrightarrow (\mathbf{w}_2\mathbf{w}_1 - \mathbf{w}'_1\mathbf{w}'_2)|\phi_1\rangle + (\mathbf{w}'_1\mathbf{w}_3 - \mathbf{w}'_1\mathbf{w}'_2 + q^{1/2}\mathbf{w}_2\mathbf{u}_3 - \kappa_3\mathbf{u}_3\mathbf{w}_3)|\phi_3\rangle = 0 \\
|\phi_e\rangle &\longrightarrow q^{1/2}(\mathbf{u}'_3\mathbf{u}'_2 - \mathbf{u}_2\mathbf{u}_3)|\phi_3\rangle + \left\{ \kappa_1 q^{-1/2}\mathbf{u}_1\mathbf{w}_1 - \mathbf{u}_2\mathbf{w}_1 - q^{1/2}\mathbf{u}'_3(\mathbf{u}_1 - \mathbf{u}'_2) \right\} |\phi_1\rangle = 0 \\
|\phi_h\rangle &\longrightarrow \frac{1}{2}(\mathbf{w}'_3\mathbf{u}_1 - \mathbf{u}'_1\mathbf{w}_3)(|\phi_1\rangle - |\phi_3\rangle) \\
&\quad + \left\{ \kappa_2 q^{-1/2}\mathbf{u}'_2\mathbf{w}'_2 - \mathbf{u}'_1\mathbf{w}'_2 - \mathbf{w}'_3\mathbf{u}'_2 + \frac{1}{2}(\mathbf{w}'_3\mathbf{u}_1 + \mathbf{u}'_1\mathbf{w}_3) \right\} (|\phi_1\rangle + |\phi_3\rangle) = 0 \\
|\phi_f\rangle &\longrightarrow \left\{ \kappa_1\mathbf{u}'_1\mathbf{w}'_1\mathbf{w}'_2 - \kappa_2\mathbf{u}_2\mathbf{w}_2\mathbf{w}_1 + q^{1/2}\kappa_3\mathbf{u}'_3\mathbf{w}'_3(\mathbf{u}'_2 - \mathbf{u}_1) \right\} |\phi_1\rangle \\
&\quad + \left\{ \kappa_1\mathbf{u}'_1\mathbf{w}'_1(\mathbf{w}'_2 - \mathbf{w}_3) + q^{1/2}\kappa_3\mathbf{u}'_3\mathbf{w}'_3\mathbf{u}'_2 - q^{1/2}\kappa_2\mathbf{u}_2\mathbf{w}_2\mathbf{u}_3 \right\} |\phi_3\rangle = 0. \quad (59)
\end{aligned}$$

These homogenous eqs. allow the currents $|\phi_1\rangle$ and $|\phi_3\rangle$ to be chosen freely so that their coefficients must vanish and we have

$$\mathbf{w}'_1\mathbf{w}'_2 = \mathbf{w}_2\mathbf{w}_1; \quad \mathbf{u}'_3\mathbf{u}'_2 = \mathbf{u}_2\mathbf{u}_3; \quad \mathbf{w}'_3{}^{-1}\mathbf{u}'_1 = \mathbf{u}_1\mathbf{w}_3{}^{-1} \quad (60)$$

We see that for ultralocal Weyl algebras the three products $\mathbf{w}_1\mathbf{w}_2$, $\mathbf{u}_2\mathbf{u}_3$, $\mathbf{u}_1\mathbf{w}_3{}^{-1}$ are centers of our mapping. From the second terms of $|\phi_g\rangle$ and $|\phi_e\rangle$ we get

$$\begin{aligned}
\mathbf{w}'_1 &= \left(\mathbf{w}_2\mathbf{w}_1 - q^{1/2}\mathbf{w}_2\mathbf{u}_3 + \kappa_3\mathbf{u}_3\mathbf{w}_3 \right) \mathbf{w}_3{}^{-1} \\
\mathbf{u}'_3 &= \left(\mathbf{u}_2\mathbf{u}_3 - q^{-1/2}\mathbf{u}_2\mathbf{w}_1 + \kappa_1 q^{-1}\mathbf{u}_1\mathbf{w}_1 \right) \mathbf{u}_1{}^{-1}. \quad (61)
\end{aligned}$$

Using the center eqs.(60), we can express immediately also \mathbf{w}'_2 and \mathbf{u}'_2 in terms of the unprimed variables.

In order to express \mathbf{u}'_1 in terms of the unprimed variables we take the difference of the two curly brackets in $|\phi_f\rangle$ and using (60) obtain

$$\kappa_1\mathbf{u}'_1\mathbf{w}'_1\mathbf{w}_3 - q^{1/2}\kappa_3\mathbf{u}'_3\mathbf{u}'_1\mathbf{w}_3 = \kappa_2\mathbf{u}_2\mathbf{w}_2(\mathbf{w}_1 - q^{1/2}\mathbf{u}_3) \quad (62)$$

Using (61) and $\mathbf{u}'_i\mathbf{u}'_j = q^{\delta_{i,j}}\mathbf{u}'_j\mathbf{u}'_i$, $\mathbf{u}_i\mathbf{u}_j = q^{\delta_{i,j}}\mathbf{u}_j\mathbf{u}_i$ leads to

$$\mathbf{u}'_1 \left\{ \kappa_1\mathbf{w}_2 + q\kappa_3\mathbf{u}_2\mathbf{u}_1{}^{-1}\mathbf{w}_3 + q^{1/2}\kappa_1\kappa_3\mathbf{w}_3 \right\} (\mathbf{w}_1 - q^{1/2}\mathbf{u}_3) = \kappa_2\mathbf{u}_2\mathbf{w}_2(\mathbf{w}_1 - q^{1/2}\mathbf{u}_3). \quad (63)$$

One easily checks that (60), (61), (63) can be written as (4), (5). The inverse relations look similar, we give only one term: To express \mathbf{u}_1 or \mathbf{w}_3 in terms of primed variables, we use the second term of $|\phi_h\rangle$, obtaining immediately

$$\mathbf{u}_1 = \left(\mathbf{u}'_1 \mathbf{w}'_2 + \mathbf{w}'_3 \mathbf{u}'_2 - \kappa_2 q^{-1/2} \mathbf{u}'_2 \mathbf{w}'_2 \right) \mathbf{w}'_3^{-1}. \quad (64)$$

7.2 \mathcal{R} -mapping from “Linear Problem“

It is often useful, instead of requiring (3), to start the derivation of the mapping $\mathcal{R}_{1,2,3}$ postulating six ”linear problem“-equations for ”co-currents” $\langle \phi_i |$.

One may understand the nature of the mapping and the origin of the local Weyl algebra relations if we start from the most general consideration, i.e. introducing at each vertex *three* dynamic variables.

In each sector ϕ_j of the auxiliary plane there shall be a co-current $\langle \phi_j |$. At each vertex i some dynamic elements \mathbf{u}_i , \mathbf{w}_i and \mathbf{v}_i acts on the four co-currents of the surrounding sectors such that

- 1) the interaction with the co-current $\langle \phi_j |$ of the sector *between* the arrows leads to a co-current $\langle \phi_j | q^{1/2} \mathbf{u}_i$ at the vertex; with a co-current $\langle \phi_j |$ in the sector to the *left* of the arrows: to $\langle \phi_j |$. With the $\langle \phi_j |$ of the sector to the *right* of the arrows the interaction produces $\langle \phi_j | \mathbf{v}_i$, *etc.*
- 2) The total co-current at a vertex is obtained by linear superposition of the four contributions and it must vanish,
- 3) These properties are unchanged if one line is shifted through the vertex i ($\nabla - \Delta$ -invariance).

The dynamic elements \mathbf{u}_i , \mathbf{w}_i and \mathbf{v}_i form an associative noncommutative ring (actually a body). Co-currents belong to the formal left module of this ring. Note that the co-currents are not the conjugates of the currents used in eqs.(3).

Applied to the six vertices appearing in Fig.3, these rules lead to the following six equations:

$$\begin{aligned} \langle \phi_1 | &\equiv \langle \phi_c | + \langle \phi_d | q^{1/2} \mathbf{u}_1 + \langle \phi_a | \mathbf{w}_1 + \langle \phi_e | \mathbf{v}_1 = 0, \\ \langle \phi_2 | &\equiv \langle \phi_a | + \langle \phi_e | q^{1/2} \mathbf{u}_2 + \langle \phi_g | \mathbf{w}_2 + \langle \phi_f | \mathbf{v}_2 = 0, \\ \langle \phi_3 | &\equiv \langle \phi_c | + \langle \phi_a | q^{1/2} \mathbf{u}_3 + \langle \phi_b | \mathbf{w}_3 + \langle \phi_g | \mathbf{v}_3 = 0, \\ \langle \phi'_1 | &\equiv \langle \phi_b | + \langle \phi_h | q^{1/2} \mathbf{u}'_1 + \langle \phi_g | \mathbf{w}'_1 + \langle \phi_f | \mathbf{v}'_1 = 0, \\ \langle \phi'_2 | &\equiv \langle \phi_c | + \langle \phi_d | q^{1/2} \mathbf{u}'_2 + \langle \phi_b | \mathbf{w}'_2 + \langle \phi_h | \mathbf{v}'_2 = 0, \\ \langle \phi'_3 | &\equiv \langle \phi_d | + \langle \phi_e | q^{1/2} \mathbf{u}'_3 + \langle \phi_h | \mathbf{w}'_3 + \langle \phi_f | \mathbf{v}'_3 = 0. \end{aligned} \quad (65)$$

Here $q^{1/2} \in \mathbb{C}$ is just a scale factor and plays no role. The mapping is provided by the demand of the linear equivalence of Δ -shape and ∇ -shape triangles: any two equations of (65) must be linear combinations of the rest four. In the other words, any four external co-currents are to be linearly independent.

As an example of the application of this principle consider two linear combinations which are designed such that both do not contain the co-currents $\langle \phi_a |$, $\langle \phi_c |$, $\langle \phi_h |$:

$$\begin{aligned}\langle \psi' | &\equiv \langle \phi'_1 | - \langle \phi'_3 | (\mathbf{w}'_3)^{-1} q^{1/2} \mathbf{u}'_1, \\ \langle \psi | &\equiv \langle \phi_3 | \mathbf{w}_3^{-1} - \langle \phi_1 | \mathbf{w}_3^{-1} + \langle \phi_2 | (\mathbf{w}_1 - q^{1/2} \mathbf{u}_3) \mathbf{w}_3^{-1}.\end{aligned}\quad (66)$$

Explicitly, using (65), we have

$$\begin{aligned}\langle \psi' | &= \langle \phi_b | + \langle \phi_g | \mathbf{w}'_1 - \langle \phi_d | \mathbf{w}'_3^{-1} q^{1/2} \mathbf{u}'_1 \\ &\quad - \langle \phi_e | q \mathbf{u}'_3 \mathbf{w}'_3^{-1} \mathbf{u}'_1 + \langle \phi_f | (\mathbf{v}'_1 - \mathbf{v}'_3 \mathbf{w}'_3^{-1} q^{1/2} \mathbf{u}'_1),\end{aligned}\quad (67)$$

$$\begin{aligned}\langle \psi | &= \langle \phi_b | + \langle \phi_g | (\mathbf{v}_3 \mathbf{w}_3^{-1} + \mathbf{w}_2 \mathbf{w}_1 \mathbf{w}_3^{-1} - q^{1/2} \mathbf{w}_2 \mathbf{u}_3 \mathbf{w}_3^{-1}) \\ &\quad - \langle \phi_d | q^{1/2} \mathbf{u}_1 \mathbf{w}_3^{-1} - \langle \phi_e | (\mathbf{v}_1 \mathbf{w}_3^{-1} + q \mathbf{u}_2 \mathbf{u}_3 \mathbf{w}_3^{-1} - q^{1/2} \mathbf{u}_2 \mathbf{w}_1 \mathbf{w}_3^{-1}) \\ &\quad + \langle \phi_f | \mathbf{v}_2 (\mathbf{w}_1 - q^{1/2} \mathbf{u}_3) \mathbf{w}_3^{-1}.\end{aligned}\quad (68)$$

The difference $\langle \psi' | - \langle \psi |$ contains exactly four external co-currents, therefore due to the rank = 4 demand all the coefficients in $\langle \psi' | - \langle \psi |$ must vanish identically. This gives in particular the expressions for \mathbf{w}'_1 and \mathbf{u}'_3 in the terms of the unprimed elements, and some other relations.

In the same way one may consider all the other combinations of $\langle \phi_i |$, \dots , $\langle \phi'_3 |$ giving linear combinations of four external co-currents. There are exactly eight independent relations for \mathbf{u}_i , \dots , \mathbf{v}'_i . The solution of the linear equivalence problem may be written as follows: First, the six primed elements may be expressed explicitly

$$\begin{aligned}\mathbf{w}'_1 &= \mathbf{w}_2 \Lambda_3, & \mathbf{w}'_2^{-1} &= \mathbf{w}_1^{-1} \Lambda_3, & \mathbf{u}'_3 \mathbf{v}'_3^{-1} &= \mathbf{v}_1 \mathbf{w}_1^{-1} \widetilde{\Lambda}_2, \\ \mathbf{w}'_1 \mathbf{v}'_1^{-1} &= \mathbf{v}_3 \mathbf{u}_3^{-1} \widetilde{\Lambda}_2, & \mathbf{u}'_2^{-1} &= \mathbf{u}_3^{-1} \Lambda_1, & \mathbf{u}'_3 &= \mathbf{u}_2 \Lambda_1,\end{aligned}\quad (69)$$

where

$$\begin{aligned}\Lambda_1 &= \mathbf{u}_3 \mathbf{u}_1^{-1} - q^{-1/2} \mathbf{w}_1 \mathbf{u}_1^{-1} + q^{-1} \mathbf{u}_2^{-1} \mathbf{v}_1 \mathbf{u}_1^{-1}, \\ \widetilde{\Lambda}_2 &= \mathbf{u}_3 \mathbf{v}_3^{-1} \mathbf{w}_2 \mathbf{v}_2^{-1} + \mathbf{w}_1 \mathbf{v}_1^{-1} \mathbf{u}_2 \mathbf{v}_2^{-1} - q^{-1/2} \mathbf{v}_2^{-1}, \\ \Lambda_3 &= \mathbf{w}_1 \mathbf{w}_3^{-1} - q^{1/2} \mathbf{u}_3 \mathbf{w}_3^{-1} + \mathbf{w}_2^{-1} \mathbf{v}_3 \mathbf{w}_3^{-1}.\end{aligned}\quad (70)$$

Also there exist three extra relations written below in the left column

$$\begin{array}{lll}\mathbf{w}'_3^{-1} \mathbf{u}'_1 = \mathbf{u}_1 \mathbf{w}_3^{-1} & \text{vs} & \mathbf{u}'_3 \mathbf{v}'_3^{-1} \mathbf{v}'_1 \mathbf{w}'_1^{-1} = \mathbf{v}_1 \mathbf{w}_1^{-1} \mathbf{u}_3 \mathbf{v}_3^{-1}, \\ \mathbf{v}'_1 \mathbf{u}'_1^{-1} \mathbf{v}'_2 \mathbf{u}'_2^{-1} = \mathbf{v}_2 \mathbf{u}_2^{-1} \mathbf{v}_1 \mathbf{u}_1^{-1} & \text{vs} & \mathbf{w}'_1 \mathbf{w}'_2 = \mathbf{w}_2 \mathbf{w}_1, \\ \mathbf{v}'_3 \mathbf{w}'_3^{-1} \mathbf{v}'_2 \mathbf{w}'_2^{-1} = \mathbf{v}_2 \mathbf{w}_2^{-1} \mathbf{v}_3 \mathbf{w}_3^{-1} & \text{vs} & \mathbf{u}'_3 \mathbf{u}'_2 = \mathbf{u}_2 \mathbf{u}_3.\end{array}\quad (71)$$

The right column in (71) is just an evident consequence of (69).

Two important notes are to be made. Recall, at the first, expressions (69),(70),(71) are obtained in the framework of the associative noncommutative ring, i.e. we never changed the order of the invertible elements \mathbf{u}_i , \mathbf{w}_i , \mathbf{v}_i , \mathbf{u}'_i , \mathbf{w}'_i and \mathbf{v}'_i . Secondly, since the linear equivalence principle implies eight independent relations, there are eight independent relations between nine relations (69),(71). In the framework of the noncommutative ring the proof of this miracle is a quite complicated exercise.

Now the question is how to get rid of one degree of freedom in the primed elements. The answer lies in the comparison of the left and right columns of (71). One may see that it is natural to identify left and right columns. The identification produces two demands

$$\mathbf{v}_i \mathbf{u}_i^{-1} \sim \mathbf{w}_i, \quad \mathbf{v}_i \mathbf{w}_i^{-1} \sim \mathbf{u}_i \quad (72)$$

and the analogous relations for the primed elements. Evidently, eqs. (72) fix the Weyl algebra structure.

Let now the initial elements \mathbf{u}_i , \mathbf{w}_i and \mathbf{v}_i obey the local Weyl algebra relations (2) with $\mathbf{v}_i = \kappa_i \mathbf{u}_i \mathbf{w}_i$, $\kappa_i \in \mathbb{C}$. Then (69),(71) immediately give $\mathbf{v}'_i = \kappa'_i \mathbf{u}'_i \mathbf{w}'_i$, where κ'_i are centers such that $\kappa_1 \kappa_2 = \kappa'_1 \kappa'_2$, $\kappa_2 \kappa_3 = \kappa'_2 \kappa'_3$, and the mapping (69) defines the automorphism of the local Weyl algebra. External “physical” demand is that since κ_i are centers, a well defined automorphism should not change them: $\kappa'_i = \kappa_i$. After the identification of κ_i and κ'_i , eqs. (69),(70) coincide with (4),(5) up to the rescaling $\Lambda_2 = q\kappa_1\kappa_3\widetilde{\Lambda}_2$.

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