On the complexity of Temporal Equilibrium Logic

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Temporal Equilibrium Logic (TEL)

[Cabalar and Vega 2007]

- Answering Set Programming (ASP) capabilities + temporal features of standard LTL.
- For temporal reasoning not representable in ASP.
- Temporal extension of propositional Equilibrium Logic [Pearce 1996], the latter
  - well-known logical foundation of ASP;
  - generalizes stables models of ASP for arbitrary propositional theories.
- Non-monotonic semantics: selection among the models of the monotonic Temporal logic of Here-and-There (THT).

$$THT = LTL + \text{intuitionistic logic of Here-and-There (HT)}$$
Temporal logic of Here and There (THT)

\[ \varphi ::= \bot \mid p \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \varphi \rightarrow \varphi \mid X\varphi \mid \varphi U \varphi \mid \varphi R \varphi \quad p \in P \]
Temporal logic of Here and There (THT)

\[ \varphi ::= \bot | p | \varphi \land \varphi | \varphi \lor \varphi | \varphi \rightarrow \varphi | X\varphi | \varphi U \varphi | \varphi R \varphi \quad p \in P \]

Derived modalities:

\[ \neg \varphi ::= \varphi \rightarrow \bot \quad \text{(negation expressed in terms of implication)} \]
\[ T ::= \neg \bot \]
\[ F \varphi ::= T U \varphi \quad \text{(eventually)} \]
\[ G \varphi ::= \bot R \varphi \quad \text{(always)} \]
THT semantics

LTL interpretation: infinite word over $2^P$

THT interpretation: $(H, T)$ such that $H \subseteq T$

‘There’ LTL-interpretation

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$H \subseteq T$ means $H(i) \subseteq T(i)$ for all $i \geq 0$
**THT semantics**

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‘There’ LTL-interpretation

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$H \subseteq T$ means $H(i) \subseteq T(i)$ for all $i \geq 0$

$(H, T)$ is total if $H = T$
THT semantics

\[ M = (H, T) \]

\( M, i \not\models \bot \)

\( M, i \models p \iff p \in H(i) \)

\( M, i \models \varphi \lor \psi \iff \text{either } M, i \models \varphi \text{ or } M, i \models \psi \)

\( M, i \models \varphi \land \psi \iff M, i \models \varphi \text{ and } M, i \models \psi \)

\( M, i \models \varphi \rightarrow \psi \iff \forall H' \in \{H, T\}, \text{either } (H', T), i \not\models \varphi \text{ or } (H', T), i \models \psi \)

\( M, i \models X\varphi \iff M, i + 1 \models \varphi \)

\( M, i \models \varphi \cup \psi \iff \exists j \geq i, M, j \models \psi \text{ and } \forall i \leq k < j, M, k \models \varphi \)

\( M, i \models \varphi \land \psi \iff \forall j \geq i, \text{either } M, j \models \psi \text{ or } \exists i \leq k < j, M, k \models \varphi \)

\( M \) is a THT model of \( \varphi \) if \( M, 0 \models \varphi \)
THT basic properties

\[(H, T), i \not\models \varphi \iff (H, T), i \models \neg \varphi\]

\[(H, T), i \models \varphi \implies (T, T), i \models \varphi\]

\[(T, T) \models \varphi \iff T \models_{\text{LTL}} \varphi\]

THT satisfiability is PSPACE-complete [Cabalar and Demri 2011] (the same complexity as LTL satisfiability [Sistla and Clarke 1985]).
THT basic properties

\[(H, T), i \not\models \varphi \not\iff (H, T), i \models \neg \varphi\]

\[(H, T), i \models \varphi \implies (T, T), i \models \varphi\]

\[(T, T) \models \varphi \iff T \models_{\text{LTL}} \varphi\]

- Dual temporal modalities independent one from the other one

\[(H, T), i \models F \varphi \not\iff (H, T), i \models \neg G \neg \varphi\]

\[(H, T), i \models 
\psi \text{U} \varphi \not\iff (H, T), i \models \neg (\neg \psi \text{R} \neg \varphi)\]
**THT basic properties**

\[(H, T), i \not\models \varphi \Leftrightarrow (H, T), i \models \neg \varphi\]

\[(H, T), i \models \varphi \Rightarrow (T, T), i \models \varphi\]

\[(T, T) \models \varphi \Leftrightarrow T \models_{LTL} \varphi\]

- **Dual temporal modalities independent one from the other one**

\[(H, T), i \models F \varphi \Leftrightarrow (H, T), i \not\models \neg G \neg \varphi\]

\[(H, T), i \models \psi U \varphi \Leftrightarrow (H, T), i \not\models \neg (\neg \psi R \neg \varphi)\]

- **THT satisfiability is PSPACE-complete** [Cabalar and Demri 2011] (the same complexity as LTL satisfiability [Sistla and Clarke 1985]).
Temporal Equilibrium Logic (TEL)

Non-monotonic semantics: restriction of THT to a subclass of models

A TEL model of $\varphi$ is a total THT model $(T, T)$ of $\varphi$ such that

$H \sqsubseteq T$ implies $(H, T) \not\models \varphi$
Temporal Equilibrium Logic (TEL)

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A TEL model of \( \varphi \) is a total THT model \((T, T)\) of \( \varphi \) such that
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H \sqsubseteq T \text{ implies } (H, T) \not\models \varphi
\]

- TEL models: temporal generalization of stable models in propositional ASP.
  Negation interpreted as default negation in logic programs.
Temporal Equilibrium Logic (TEL)

Non-monotonic semantics: restriction of THT to a subclass of models

A **TEL** model of $\varphi$ is a total THT model $(T, T)$ of $\varphi$

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**TEL** models: temporal generalization of stable models in propositional ASP.

Negation interpreted as default negation in logic programs.

$G(\neg p \rightarrow Xp)$

Time 0, $\neg p \rightarrow Xp$: $p$ false by default, $Xp$ holds.

Time 1, $p$ and $\neg p \rightarrow Xp$: $p$ true.

Time 2, $\neg p \rightarrow Xp$: ...

The unique **TEL** model is $(T, T)$ where $T = \emptyset, \{p\}, \emptyset, \{p\}, \ldots$
Non-existence of TEL models

\[ \text{LTL satisfiability } \not\Rightarrow \text{ TEL satisfiability} \]
Non-existence of TEL models

\[ \text{LTL satisfiability} \not\Rightarrow \text{TEL satisfiability} \]

- **Use of nested implication:**
  (necessary for non-existence of stable models in Equilibrium Logic)

\[ G(\neg p \to p) \]

\[ T = \{p\}^\omega \text{ unique LTL model, but } (\emptyset^\omega, \{p\}^\omega) \text{ is a THT model.} \]
Non-existence of TEL models

\[ \text{LTL satisfiability } \nL \text{ TEL satisfiability} \]

- Use of nested implication:
  (necessary for non-existence of stable models in Equilibrium Logic)

\[ G(\neg p \rightarrow p) \]

\[ T = \{p\}^\omega \text{ unique LTL model, but } (\emptyset^\omega, \{p\}^\omega) \text{ is a THT model.} \]

- No finite justification for minimal knowledge:

\[ GF p \]

LTL/THT satisfiable but no TEL model.
Investigated problems

- Complexity of TEL satisfiability.
  - Systematic analysis of natural THT fragments:
    \[ \text{THT}^m_k (O_1, O_2, \ldots) \]
    - bound on implication nesting depth
    - allowed temporal operators
    - bound on temporal nesting depth
Investigated problems

- Complexity of TEL satisfiability.
  - Systematic analysis of natural THT fragments:
    - bound on implication nesting depth
    - allowed temporal operators
    - bound on temporal nesting depth

\[
\text{THT}_k^m(O_1, O_2, \ldots)
\]

- Complexity of minimal LTL satisfiability.
  An LTL model $T$ of $\varphi$ is **minimal** if $H \not\models_{\text{LTL}} \varphi$ for all $H \subseteq T$. 
EXPSPACE-completeness for TEL satisfiability

TEL satisfiability is known to be in EXPSPACE [Cabalar and Demri 2011].
EXPSPACE-completeness for TEL satisfiability

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**Theorem (EXPSPACE lower bounds)**

TEL satisfiability is EXPSPACE-complete even for the fragments

\[ THT^1_k(F, G, \ldots) \]

\[ THT^m_k(G, \ldots) \]

\[ THT^m_k(U, \ldots) \]

\( m \geq 2 \) (implication nesting depth) and \( k \geq 2 \) (temporal nesting depth)

EXPSPACE-hardness for \( THT^1_2(F, G) \) is surprising because

- LTL/THT satisfiability of \( THT(F, G) \) is NP-complete [Sistla and Clarke 1985, Cabalar and Demri 2011]

- Checking equilibrium models for \( HT^1 \) formulas is NP-complete.
EXPSPACE-hardness for $\text{THT}_{2}^{1}(F, G)$ (no nesting of implication)

Polynomial-time reduction from a domino tiling problem for grids with exponential number of columns.

$I = \langle C, \Delta, n, d_{\text{init}}, d_{\text{final}} \rangle$

$\mathcal{I}$ is a set of domino types: tuples of four colors

$\mathcal{I}$ is a set of colors

Tilings of $\mathcal{I}$: grids with $2^n$ columns and $k$ rows (for some $k$) such that

- Each cell contains a domino type;
- the first cell contains $d_{\text{init}}$;
- the last cell is the unique one containing $d_{\text{final}}$;
- adjacent cells have the same color on the shared edge.
EXPSPACE-hardness for $\text{THT}_{2}^{1}(F, G)$ (no nesting of implication)

Polynomial-time reduction from a domino tiling problem for grids with exponential number of columns.

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- Each cell contains a domino type;
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- adjacent cells have the same color on the shared edge.

We construct $\varphi_{\mathcal{I}} \in \text{THT}_{2}^{1}(F, G)$ such that

$$\varphi_{\mathcal{I}} \text{ is TEL satisfiable } \iff \text{ there is a tiling of } \mathcal{I}$$
EXPSPACE-hardness for $\text{THT}^1_{2}(F, G)$ (no nesting of implication)

Encoding of tilings of $\mathcal{I}$:

$$P_{\text{MAIN}} = \Delta \cup [1, n] \times \{0, 1\} \cup \{$$

- Cells with content $d \in \Delta$ and column number $i \in [0, 2^n - 1]$ encoded by finite words in
  $$\{d\}^+ \{(1, b_1)\}^+ \ldots \{(n, b_n)\}^+$$
  $b_1, \ldots, b_n$ is the binary encoding of column number $i$.

- Tilings encoded by finite words over $P_{\text{MAIN}}$ listing the encodings of rows from left to right, separated by occurrences of $\$$. 

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EXPSPACE-hardness for $\text{THT}^1_2(F, G)$ (no nesting of implication)

We construct $\varphi_I$ over $P = P_{\text{MAIN}} \cup P_{\text{TAG}} \cup \{u\}$

$$\varphi_I = \varphi_{\text{PTC}} \land (u \lor \varphi_{\text{bad}})$$

$\varphi_{\text{PTC}}$ captures the pseudo-tiling codes (PTC) $(H, T)$:

- $T$ and $H$ agree on $P_{\text{MAIN}}$ and for all $i$, $T(i) \cap P_{\text{MAIN}}$ is a singleton;
- either $T(i) \supseteq P_{\text{TAG}} \cup \{u\}$ for all $i$ ($(H, T)$ is good),
  or $u \notin T(0)$ and $T(i) \cap P_{\text{TAG}}$ is a singleton;
- if $H \neq T$, $u \notin H(0)$ and $H(i) \cap P_{\text{TAG}}$ is a singleton.
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- if $H \neq T$, $u \notin H(0)$ and $H(i) \cap P_{\text{TAG}}$ is a singleton.
- Unboundness: for infinitely many $i$, $u \in H(i)$. 

Remark: every TEL model of $\varphi_I$ is a good PTC.
EXPSPACE-hardness for $\text{THT}^1_2(F, G)$ (no nesting of implication)

We construct $\varphi_\mathcal{I}$ over $P = P_{\text{MAIN}} \cup P_{\text{TAG}} \cup \{u\}$

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$(T, T)$ is non-good: there is non-total PTC $(H, T)$ s.t. $H$ and $T$ agree on $P \setminus \{u\}$. 
EXPSPACE-hardness for $\text{THT}_{2}^{1}(F, G)$ (no nesting of implication)

We construct $\varphi_{I}$ over $P = P_{\text{MAIN}} \cup P_{\text{TAG}} \cup \{u\}$

$$\varphi_{I} = \varphi_{\text{PTC}} \land (u \lor \varphi_{\text{bad}})$$

$\varphi_{\text{PTC}}$ captures the pseudo-tiling codes (PTC) $(H, T)$:

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Since $\varphi_{\text{bad}}$ is over $P \setminus \{u\}$.

**Remark:** every TEL model of $\varphi_{I}$ is a good PTC.
EXPSPACE-hardness for $\text{THT}_2^1(F, G)$ (no nesting of implication)

\[ \varphi_I = \varphi_{\text{PTC}} \land (u \lor \varphi_{\text{bad}}) \]

- for a good total PTC $(T, T)$, no prefix of $T$ encodes a tiling $\iff$ there is non-total PTC $(H, T)$ satisfying $\varphi_{\text{bad}}$. 
EXPSPACE-hardness for $\text{THT}^1_2(F, G)$ (no nesting of implication)

$$\varphi_I = \varphi_{PTC} \land (u \lor \varphi_{bad})$$

- for a good total PTC $(T, T)$, no prefix of $T$ encodes a tiling $\iff$ there is non-total PTC $(H, T)$ satisfying $\varphi_{bad}$.
  - tag propositions mark local portions of $H$: for checking that a bad condition is satisfied.
  - goodness of $(T, T)$ is crucial for ensuring the for each bad condition $B$ in $T$, there is a non-total PTC $(H, T)$ witnessing $B$. 
EXPSPACE-hardness for $\text{THT}_{2}^{1}(F, G)$ (no nesting of implication)

\[ \varphi_{I} = \varphi_{PTC} \land (u \lor \varphi_{bad}) \]

- for a good total PTC $(T, T)$, no prefix of $T$ encodes a tiling $\iff$ there is non-total PTC $(H, T)$ satisfying $\varphi_{bad}$.
  - tag propositions mark local portions of $H$: for checking that a bad condition is satisfied.
  - goodness of $(T, T)$ is crucial for ensuring the for each bad condition $B$ in $T$, there is a non-total PTC $(H, T)$ witnessing $B$.

**Lemma**

The TEL models of $\varphi_{I}$ are the total good PTC $(T, T)$ such that some prefix of $T$ encodes a tiling of $I$.

\[ \varphi_{I} \text{ is TEL satisfiable } \iff \text{ there is a tiling of } I \]
Remaining main THT fragments

- Using only temporal modalities in \( \{X, F\} \).
- No nesting of temporal modalities.
- No nesting of implication.
Using only temporal modalities $X$ and $F$
Using only temporal modalities $X$ and $F$

$(T, T)$ is almost-empty if (*) for some $i$ and for all $k \geq i$, $T(k) = \emptyset$.
The size of $(T, T)$ is the smallest $i$ satisfying (*).
Using only temporal modalities $X$ and $F$

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**Theorem (Small size model property for $THT(X, F)$)**

$\varphi \in THT(X, F)$, $\varphi$ is TEL satisfiable $\Rightarrow$ $\varphi$ has an almost-empty TEL model of size at most $|\varphi|^3$. 

© 2021 [Sistla et Clarke 1985, Cabalar et Demri 2011]
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**Corollary**

TEL satisfiability of $THT(X, F)$ is $\Sigma_2$-complete.
Using only temporal modalities X and F

\((T, T)\) is almost-empty if (*) for some \(i\) and for all \(k \geq i\), \(T(k) = \emptyset\). The size of \((T, T)\) is the smallest \(i\) satisfying (*).

**Theorem (Small size model property for \(THT(X, F)\))**

\(\varphi \in THT(X, F), \ \varphi \ \text{is TEL satisfiable} \Rightarrow \varphi \ \text{has an almost-empty TEL model of size at most } |\varphi|^3.\)

**Corollary**

TEL satisfiability of \(THT(X, F)\) is \(\Sigma_2\)-complete.

- LTL/THT satisfiability of \(THT(X, F)\) is already PSPACE-complete [Sistla et Clarke 1985, Cabalar et Demri 2011]

For \(THT(X, F)\), LTL/THT satisfiability is harder than TEL satisfiability!
No nesting of temporal modalities

- LTL/THT satisfiability of $\text{THT}_1$ is NP-complete
  [Demri et al. 2002, Cabalar et al. 2007]
- TEL satisfiability of $\text{THT}_1$ is NEXPTIME-complete
No nesting of temporal modalities

- **LTL/THT** satisfiability of $\text{THT}_1$ is NP-complete
  
  [Demri et al. 2002, Cabalar et al. 2007]

- **TEL** satisfiability of $\text{THT}_1$ is NEXPTIME-complete

  - Untractable fragments of $\text{THT}_1$: ☹

    \[
    \begin{aligned}
    \text{THT}_1^m(F,G,\ldots) & \\
    \text{THT}_1^m(U,\ldots) & \\
    \text{THT}_1^m(R,\ldots) & \end{aligned}
    \]

    \[
    \begin{aligned}
    \text{NEXPTIME}-\text{complete}
    \end{aligned}
    \]

    $m \geq 2$ (implication nesting depth)
No nesting of temporal modalities

- **LTL/THT** satisfiability of $\text{THT}_1$ is NP-complete
  [Demri et al. 2002, Cabalar et al. 2007]

- **TEL** satisfiability of $\text{THT}_1$ is NEXPTIME-complete
  - Untractable fragments of $\text{THT}_1$: ☹
    $\text{THT}^m_1(F, G, \ldots)
    \quad THT^m_1(U, \ldots)
    \quad THT^m_1(R, \ldots)$
    \[
    \begin{array}{c}
    \text{NEXPTIME-complete} \\
    \end{array}
    \]
    $m \geq 2$ (implication nesting depth)
  - Tractable fragments of $\text{THT}_1$: ☻
    $\text{THT}_1(X, F)$
    $\text{THT}_1(X, G)$
    $\text{THT}^1_1$
    \[
    \begin{array}{c}
    \Sigma_2\text{-complete} \\
    \text{NP-complete} \\
    \end{array}
    \]
No nesting of implication (negation expressed in terms of implication)

TEL satisfiability of THT$^1$ is EXPSPACE-complete

- Untractable fragments of THT$^1$: 😞

\[ THT^1_m(F, G, \ldots) \text{ is EXPSPACE-complete} \]

\[ m \geq 2 \text{ (temporal nesting depth)} \]
No nesting of implication (negation expressed in terms of implication)

**TEL satisfiability of THT$^1$ is EXPSPACE-complete**

- **Untractable fragments of THT$^1$: 😞**
  
  THT$^1_m$(F, G, ...) is EXPSPACE-complete

  $m \geq 2$ (temporal nesting depth)

- **Tractable fragments of THT$^1$: 😊**
  
  THT$^1_1$

  $\begin{align*}
  \text{THT}^1(X, R) & \quad \text{NP-complete} \\
  \text{THT}^1(X, U) & \quad \text{PSPACE-complete}
  \end{align*}$
No nesting of implication: the fragment $\text{THT}^1(X, R)$

Remark:

$\varphi \in \text{THT}^1, \quad (T, T)$ is a TEL model of $\varphi$
No nesting of implication: the fragment $THT^1(X, R)$

Remark:

$$\varphi \in THT^1, \quad T \text{ minimal LTL model of } \varphi \quad \Rightarrow \quad (T, T) \text{ is a TEL model of } \varphi$$

Lemma (Main result for $THT^1(X, R)$)

An LTL satisfiable $THT^1(X, R)$ formula has a minimal LTL model.
No nesting of implication: the fragment $THT^1(X, R)$

**Remark:**

\[
\varphi \in THT^1, \\
T \text{ minimal LTL model of } \varphi \}
\quad \Rightarrow \quad (T, T) \text{ is a TEL model of } \varphi
\]

**Lemma (Main result for $THT^1(X, R)$)**

An LTL satisfiable $THT^1(X, R)$ formula has a minimal LTL model.

**Corollary**

For $THT^1(X, R)$, LTL satisfiability $= TEL$ satisfiability.
No nesting of implication: the fragment $\text{THT}^1(X, U)$

**Remark:**

$\varphi \in \text{THT}^1$, $T$ minimal LTL model of $\varphi$ \quad \Rightarrow \quad (T, T)$ is a TEL model of $\varphi$
No nesting of implication: the fragment $\text{THT}^1(X, U)$

Remark:

$$\varphi \in \text{THT}^1, \quad T \text{ minimal LTL model of } \varphi \quad \Rightarrow \quad (T, T) \text{ is a TEL model of } \varphi$$

**Lemma (Properties of $\text{THT}(X, U)$)**

Every TEL model of a $\text{THT}(X, U)$ formula is *almost-empty*. 
No nesting of implication: the fragment $\text{THT}^1(X, U)$

Remark:
\[
\varphi \in \text{THT}^1, \\
T \text{ minimal LTL model of } \varphi \}
\Rightarrow (T, T) \text{ is a TEL model of } \varphi
\]

Lemma (Properties of $\text{THT}(X, U)$)
Every TEL model of a $\text{THT}(X, U)$ formula is almost-empty.

Corollary (Main result for $\text{THT}^1(X, U)$)
Let $\varphi \in \text{THT}^1(X, U)$ and $\psi = \varphi \land F G \land p \in P(\varphi) \quad \neg p$.
\[
\varphi \text{ is TEL satisfiable } \iff \psi \text{ is LTL satisfiable}
\]

Proof: $\Rightarrow$ by the lemma above.
$\Leftarrow$ if $\psi$ is LTL satisfiable, then $\varphi$ has a minimal LTL model. By the remark above, $\varphi$ has a TEL model.
No use of implication: the fragment $\text{THT}^0$

**Remark:** every $\text{THT}^0$ formula is LTL and THT satisfiable.

**Theorem (Lower bound for $\text{THT}^0$)**

TEL satisfiability of $\text{THT}^0$ is PSPACE-hard.

**Open question:** the exact complexity of TEL satisfiability for $\text{THT}^0$. 
Minimal LTL satisfiability

**Theorem**

Minimal LTL satisfiability is EXPSPACE-complete.

**Proof:** Lower bound: the same reduction for the lower bound of TEL satisfiability of $\text{THT}^1_2(F, G)$.

Upper bound: generalization of automata-theoretic approach for LTL satisfiability.
Minimal LTL satisfiability

Theorem

Minimal LTL satisfiability is EXPSPACE-complete.

Proof: Lower bound: the same reduction for the lower bound of TEL satisfiability of $THT_{1}(F, G)$.

Upper bound: generalization of automata-theoretic approach for LTL satisfiability.

- Minimal LTL satisfiability versus TEL satisfiability: different costs for THT fragments.
  - Example: for $THT_{1}$, minimal LTL satisfiability is NP-complete, while TEL satisfiability is NEXPTIME-complete.
Discussion: wrap up

▶ Systematic analysis of complexity of TEL satisfiability for natural THT fragments.
  ▶ No difference between implication (resp., temporal) nesting depth 2 and $k > 2$.
  ▶ THT($X$, $F$): the unique tractable fragment with both nesting of implication and nesting of temporal modalities.
  ▶ Different computational cost of dual temporal modalities.
Example: for THT($G$), EXPSPACE-completeness; for THT($X$, $F$), $\Sigma_2$-completeness.
Discussion: wrap up

- Systematic analysis of complexity of **TEL** satisfiability for natural **THT** fragments.
  - No difference between implication (resp., temporal) nesting depth 2 and $k > 2$.
  - **THT**(X, F): the unique tractable fragment with both nesting of implication and nesting of temporal modalities.
  - Different computational cost of dual temporal modalities. Example: for **THT**(G), EXPSPACE-completeness; for **THT**(X, F), $\Sigma_2$-completeness.

- Complexity of minimal **LTL** satisfiability.
  - **LTL** over finite words: **LTL** satisfiability = minimal **LTL** satisfiability.
  - **LTL** over infinite words: minimal **LTL** satisfiability exponentially harder than **LTL** satisfiability.
Discussion: perspectives

- **Expressiveness issues for TEL fragments:**
  - Kind of temporal problems expressible in tractable fragments.
  - Is the syntactical hierarchy of considered THT fragments semantically strict w.r.t. THT or TEL semantics?

  Known results: the hierarchy of $\text{THT}_m(U)$ fragments is strict w.r.t. LTL semantics [Etessami et Wilke 1996].
Discussion: perspectives

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- **Characterization of TEL languages:**
  - Known results: TEL languages are $\omega$-regular [Cabalar et Demri 2011].
  - Conjecture: TEL languages are LTL definable!
MANY THANKS 😊