A New Method to Solve Fuzzy Linear Programming Problem Using Duality

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ABSTRACT
The concept of duality plays an important role in optimization theory. In this paper, a new method is proposed to find the fuzzy optimal solution of fully fuzzy linear programming problems using breaking point and duality theory. By using the proposed method the fuzzy optimal solution of fully fuzzy linear programming problem with inequality constraints has been obtained. To illustrate the proposed method numerical examples are solved.

Keywords-Triangular Fuzzy Number, Ranking Fuzzy Number, Revised Tsao’s Method, Fuzzy Linear Programming Problem, Duality in fuzzy linear programming problem, Breaking Point, Fuzzy duality Algorithm

1. INTRODUCTION
In a general linear programming problem, if uncertainty appears in the form of fuzzy inequality or fuzzy numbers in the coefficients or fuzzy valued functions, either in the objective function or in the constraints or in both then that problem is treated as a fuzzy linear programming problem. Several duals for general crisp linear and nonlinear programming problems exist in literature. Duality for the fuzzy linear programming was first introduced by Rodder and Zimmerman [1]. Wu [2] studied weak and strong duality results for fuzzy linear programming problem where the coefficients are fuzzy numbers. Linear programming with fuzzy constraints have been considered by Tanaka and Asai [3]. Fang and Hu [4] consider linear programming with fuzzy constraint coefficients. Allahviranloo et al. [5] solved the fuzzy integer linear programming problem by reducing it into a crisp integer linear programming problem. Allahviranloo et al. [6] proposed a new method for solving FFLP problems by the use of ranking function. Nasseri [7] proposed a method for solving fuzzy linear programming problems by solving the classical linear programming. Lotfii et al. [8] discussed FFLP problems by representing all parameters and variables as triangular fuzzy numbers. Ebrahimnejad and Nasseri[9] used the complementary slackness theorem to solve fuzzy linear programming problem with fuzzy parameters without the need of a simplex tableau. Ebrahimnejad et al. [10] proposed a new primal-dual algorithm for solving linear programming problems with fuzzy variables by using duality results. Mahdavi and Nasseri [11] explain some duality properties in fuzzy number linear programming problems.

T. Beaula and S. Rajalakshmi [12] explained the concept of breaking points and solved the Linear Programming Problems. Here, we provide some important results for Fuzzy linear programming problems and we propose the breaking points of satisfactory level depending on constraints and optimal solutions are obtained successfully.

2. PRELIMINARIES
2.1 Definition
A fuzzy set F in a referential U is characterized by a membership function which associates with each element u in U a real number in the interval [0, 1]. The value of the membership function at element u represents the "grade of membership" of u in F. A fuzzy set F is thus defined as a mapping F: U→[0, 1].

2.2 Definition
The α-level cut of a fuzzy set F is the set of all u ∈ U such that F (u) ≥ α, for 1 ≥ α > 0.

2.3 Definition
A fuzzy set A in X is convex if μ_A(λx + (1-λ)y) ≥ min{ μ_A(x), μ_A(y) }, for all x, y ∈ A and λ ∈ [0,1].

2.4 Definition
A fuzzy linear programming problem is defined as [see Ganesan and Veeramani (2006)]:
Maximize \( \tilde{z} = \tilde{c} \tilde{x} \)
subject to \( A \tilde{x} \leq b \)
\( \tilde{x} \geq 0 \)
where \( \bar{A} \in (R)^{mxn}, \bar{B} \in (F(R))^m \) and \( \bar{c} \in (F(R))^n \) and \( \bar{x} \in F(R)^n \) is to be determined.

2.5 Definition

Any \( \bar{x} = (\bar{x}_1, \bar{x}_2, ..., \bar{x}_n) \in T(R) \) where each \( x_i \in T(R) \), which satisfies the constraints and non negative restrictions of fuzzy Linear Programming Problem is said to be fuzzy feasible solution to fuzzy Linear Programming Problem.

2.6 Definition

Consider the system \( \bar{A} \bar{x} = \bar{b} \) with \( \bar{x} \geq 0 \), where \( A \) is an \( m \times n \) matrix and \( b \) is an \( m \) vector. Suppose that rank \( (A) = m \). Arranging the column of \( A \) as \( [B, N] \) that \( B \) is an \( m \times m \) matrix. The vector \( \bar{x} = (\bar{x}_B, \bar{x}_N) \) where \( \bar{x}_B = B^{-1}b \) is called a basic feasible solution (BFS) of system. Here \( B \) is called the basic matrix and \( N \) is called the non basic matrix. The components of \( B \) are called the basic variables. If \( \bar{x}_B > 0 \), then \( \bar{x} \) is called a non degenerate basic feasible solution. If \( \bar{x}_B = 0 \), then \( \bar{x} \) is called a degenerate basic feasible solution.

2.7 Definition

A fuzzy basic feasible solution \( \bar{x}_B \) to the fuzzy linear programming problem is called an optimum fuzzy basic feasible solution if \( \bar{Z}_0 = \bar{c}_B \bar{x}_B \geq \bar{Z}_* \), where \( \bar{Z}_* \) is the value of objective function for any fuzzy feasible solution.

2.8 Definition

For the primal FLP problem

Maximize \( \bar{z} = \bar{c} \bar{x} \)

subject to \( \bar{A} \bar{x} \leq \bar{b} \)

\( \bar{x} \geq 0 \)

define the dual problem (DFLP problem) as follows:

Minimize \( \bar{u} = \bar{w} \bar{b} \)

subject to \( \bar{w} A \geq \bar{c} \)

\( \bar{w} \geq 0 \)

Relationships between FLP and DFLP problems

1. The dual of the DFLP problem is the FLP problem.
2. Weak duality: If \( \bar{x}_0 \) and \( \bar{w}_0 \) are feasible solutions to FLP and DFLP problems, respectively, then \( \bar{c}_0 \bar{x}_0 \leq \bar{b} \bar{w}_0 \).
3. Strong duality: If any one of the FLP or DFLP problem has an optimal solution, then the other problem has an optimal solution and the two optimal fuzzy objective values are equal.
4. For any FLP problem and its corresponding DFLP problem, exactly one of the following statements is true:
   - Both have optimal solutions \( \bar{x}_0 \) and \( \bar{w}_0 \) with \( \bar{b} \bar{w}_0 = \bar{c} \bar{x}_0 \)
   - One problem is unbounded and the other is infeasible.
   - Both problems are infeasible.

2.9 Definition

Let us define breaking points of constraints at satisfactory level \( \gamma \) as follow

That is, \( a_{ij} = c - \gamma(c - b) \). By means of the twice intersection of \( a_{ij} = c - \gamma(c - b) \) lines, the \( N = m.n \) piece values of \( \gamma \) (\( 0 \leq \gamma \leq 1 \)) are obtained and then these values change the order of \( a_{ij} \). These values of \( \gamma \) are called the breaking points. The order of \( a_{ij} \)’s in each subinterval that are formed between the iterative breaking points do not change. Therefore, the representative point can be selected in this interval. This selected point can be any point of the interval, but the optimal solution of linear programming problem does not change.
2.10 Definition

A triangular fuzzy number $A(x)=(a_1, a_2, a_3)$ has in general a linear representation as shown in the following figure.

![Triangular Fuzzy Number](image)

Any triangular fuzzy number $A(x) = (a_1, a_2, a_3)$, can be written as follows:

$$A(x) = \begin{cases} 
\frac{(x-a_1)}{(a_2-a_1)}, & x \in [a_1, a_2] \\
\frac{(a_3-x)}{(a_3-a_2)}, & x \in [a_2, a_3] \\
0, & x \leq a_1 \text{ or } x \geq a_3
\end{cases}$$

The interval level of confidence is defined as:

$$A_{\alpha} = [A_{\alpha}^L, A_{\alpha}^R] = [a_1 + \alpha(a_2-a_1), a_3 - \alpha(a_3-a_2)]$$

where $a_1, a_2, a_3, x \in \mathbb{R}$ and $\alpha \in [0, 1]$.

2.11 Definition

If $A=(a_1,a_2,a_3)$ and $B=(b_1,b_2,b_3)$ with $a_1,a_2,a_3,b_1,b_2,b_3 \in \mathbb{R}$. The arithmetic operations are defined as follows:

$A + B = (a_1+b_1,a_2+b_2,a_3+b_3)$

$A - B = (a_1-b_1,a_2-b_2,a_3-b_3)$

$A \cdot B = (a_1b_1,a_2b_2,a_3b_3)$

$A / B = (a_1/b_3,a_2/b_2,a_3/b_1)$

Remark

We consider $\tilde{0} = (0, 0, 0)$ as the zero triangular fuzzy number.

Let $\tilde{A}, \tilde{B}$ are two fuzzy numbers of LR type:

$$\tilde{A} = (a, \alpha, \beta) \text{ LR}, \quad \tilde{B} = (b, \lambda, \delta) \text{ LR}$$

Then

$$(a, \alpha, \beta) \text{ LR} \oplus (b, \lambda, \delta) \text{ LR} = (a+b, \alpha+\lambda, \beta+\delta) \text{ LR}$$

$$(b, \lambda, \delta) = (-b, \delta, \lambda) \text{ LR}$$

$$(a, \alpha, \beta) \text{ LR} - (b, \lambda, \delta) \text{ LR} = (a-b, \alpha+\delta, \beta+\lambda) \text{ LR}$$

3. RANKING FUNCTION

Ranking fuzzy number with an area between the centroid and the original points using the revised version of Chu and Tsao. Reference [14] relinquished the area proposed by Chu and Tsao but instead consider centroid-point. However, the revised version seems escalate in the complexity of computation. Moreover, validation of the revised method was solely banked on several hypothetical examples and far too little attention has been paid to test it into real case study. Therefore the four-step algorithm is proposed. Computational complexity especially in real case study can be relaxed by executing the following steps and ultimately reduce the computational costs.

3.1 Algorithm

Step 1: Define triangular fuzzy numbers and its respective linguistic variables

The triangular fuzzy numbers is based on a three-value judgment of a linguistic variable. The minimum possible value is denoted as $a_1$, the most possible value denoted as $b$ and the maximum possible value denoted as $c$.

Step 2: Delineate Inverse Function
\( f_L^A : [a, b] \rightarrow [0, w] \) and \( f_R^A : [c, d] \rightarrow [0, w] \). Since \( f_L^A : [a, b] \rightarrow [0, w] \) is continuous and strictly increasing, the inverse function of \( f_L^A \) exists. Similarly, since \( f_R^A : [c, d] \rightarrow [0, w] \) is continuous and strictly decreasing, the inverse function of \( f_R^A \) also exists. The inverse functions of \( f_L^A \) and \( f_R^A \) can be denoted by \( g_L^A \) and \( g_R^A \) respectively. Since \( f_L^A : [a, b] \rightarrow [0, w] \) is continuous and strictly increasing, \( g_L^A : [0, w] \rightarrow [a, b] \) is continuous and strictly increasing. Similarly, since \( f_R^A : [c, d] \rightarrow [0, w] \) is continuous and strictly decreasing, \( g_R^A : [0, w] \rightarrow [c, d] \) is continuous and strictly decreasing, the inverse function of \( f_R^A \) also exists. \( g_L^A \) and \( g_R^A \) are continuous on [0,w]. They are integrable on [0,w]. That is, both \( \int_0^L g_A^L dx \) and \( \int_0^R g_A^R dy \) exist.

**Step 3:** Establish Centroid-Point \((\bar{x}, \bar{y})\)

The centroid point of a fuzzy number corresponds to an \( x \) value on the horizontal axis and a \( y \) value on the vertical axis. The centroid point \((\bar{x}, \bar{y})\) for a fuzzy number \( A \):

\[
\bar{x}(A) = \int_a^b (xf_L^A)dx + \int_c^d (xf_R^A)dx \\
\bar{y}(A) = \int_0^w (yg_L^A)dy + \int_0^w (yg_R^A)dy
\]

where \( f_L^A \) and \( f_R^A \) are the left and right membership functions of fuzzy number \( A \), respectively and \( g_L^A \) are \( g_R^A \) the inverse functions of \( f_L^A \) and \( f_R^A \) respectively. The area between the centroid point \((\bar{x}, \bar{y})\) and original point \((0,0)\) of the fuzzy number \( A \) is then defined as \( S(A) = \bar{x} \bar{y} \) where \( \bar{x} \) and \( \bar{y} \) are the centroid points of fuzzy number \( A \). To rank fuzzy numbers, we know that the importance of the degree of representative location is higher than average height. Based on this concept for any two fuzzy numbers \( A \) and \( B \), we have following situations:

- if \( \bar{x}(A) > \bar{x}(B) \) \( \Rightarrow \bar{A} > \bar{B} \)
- if \( \bar{x}(A) \prec \bar{x}(B) \) \( \Rightarrow \bar{A} \prec \bar{B} \)
- if \( \bar{y}(A) > \bar{y}(B) \) \( \Rightarrow \bar{A} > \bar{B} \)
- if \( \bar{y}(A) \prec \bar{y}(B) \) \( \Rightarrow \bar{A} \prec \bar{B} \)
- if \( \bar{y}(A) = \bar{y}(B) \) \( \Rightarrow \bar{A} = \bar{B} \)

4. **Solving Fully Fuzzy Linear Programming Using Principle of Duality**

1. First find the dual of the given linear programming problem.
2. Rewrite the given fuzzy dual linear programming problem using LR triangular number and Standardizing the given fuzzy linear programming problem by introducing slack or surplus variables.
3. Obtain basic feasible solution to the problem by using \( \bar{w}_B = B^{-1} \bar{b} \)
4. Using breaking points, rearrange the coefficient of constraints as \( a_{ij} = c \gamma (c-b) \)
5. Find the entering variable using Revised Tsao's method.
6. Find the leaving variable using \( \min \left\{ \frac{\bar{w}_{hi}}{\bar{y}_{ir}}, y_i > 0, i = 1,2,\ldots,m \right\} \) and hence find the pivotal element which is common to both entering variable column and leaving variable row.
7. Convert the leading element to unity by dividing its row by pivotal element and all elements in its column to zero.
8. If all \( \bar{w}_j - \bar{c}_j \geq 0 \), then the optimal solution is obtained. Otherwise go to step(4) and repeat the procedure till the solution is obtained.
4.1 Numerical Example

Solve by duality

Minimize $\tilde{z} = (19,20,21)\tilde{x}_1 + (13,15,17)\tilde{x}_2$

Subject to constraints

$(2,5,7)\tilde{x}_1 + (2,3,4)\tilde{x}_2 \geq (15,20,25)$

$(3,4,5)\tilde{x}_1 + (2,5,7)\tilde{x}_2 \geq (45,50,55)$

$\tilde{x}_1, \tilde{x}_2 \geq 0$

Its dual is Maximize $\tilde{z} = (15,20,25)\tilde{w}_1 + (45,50,55)\tilde{w}_2$

Subject to constraints

$(2,5,7)\tilde{w}_1 + (3,4,5)\tilde{w}_2 \leq (19,20,21)$

$(2,3,4)\tilde{w}_1 + (2,5,7)\tilde{w}_2 \leq (13,15,17)$

$\tilde{w}_1, \tilde{w}_2 \geq 0$

LR Representation is

Maximize $\tilde{z} = (20,5,5)\tilde{w}_1 + (50,5,5)\tilde{w}_2$

Subject to constraints

$(5,3,2)\tilde{w}_1 + (4,1,1)\tilde{w}_2 \leq (20,1,1)$

$(3,1,1)\tilde{w}_1 + (5,3,2)\tilde{w}_2 \leq (15,2,2)$

$\tilde{w}_1, \tilde{w}_2 \geq 0$

Standardize the linear programming problem by introducing slack variables $\tilde{w}_3, \tilde{w}_4 \geq 0$

The breaking points of $\gamma$ are intersection points of lines $A_{ij} - A_{ij} (\gamma)$. Breaking values of $\gamma$ in $(0,1)$ interval are $0, 1/3, 1/2, 1$. Intervals among these iterative values are $[0, 1/3], [1/3, 1/2], [1/2, 1]$.

<table>
<thead>
<tr>
<th>$\tilde{C}_B$</th>
<th>$\tilde{w}_B$</th>
<th>$V\tilde{w}_B$</th>
<th>$\tilde{w}_1$</th>
<th>$\tilde{w}_2$</th>
<th>$\tilde{w}_3$</th>
<th>$\tilde{w}_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{0}$</td>
<td>$\tilde{w}_3$</td>
<td>21 - $\gamma$</td>
<td>7 - 2$\gamma$</td>
<td>5 - $\gamma$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\tilde{0}$</td>
<td>$\tilde{w}_4$</td>
<td>17 - 2$\gamma$</td>
<td>4 - $\gamma$</td>
<td>7 - 2$\gamma$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Max $\tilde{w}$</td>
<td>$\tilde{0}$</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\tilde{w}_j - \tilde{c}_j$</td>
<td>(-20,5,5)</td>
<td>0</td>
<td>(-50,5,5)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Here $\tilde{y}_{01} = \tilde{w}_1 - \tilde{c}_1 = (20,5,5)$ and $\tilde{y}_{02} = \tilde{w}_2 - \tilde{c}_2 = (-50,5,5)$. Using revised Tsao’s method for ordering $\tilde{y}_{01}$ and $\tilde{y}_{02}$, we have $\tilde{y}_{02} > \tilde{y}_{01}$ and hence $\tilde{w}_2$ enters the basis and leaving variable is $\tilde{w}_1$. Here 7-2$\gamma$ is a pivotal element.

<table>
<thead>
<tr>
<th>$\tilde{C}_B$</th>
<th>$\tilde{w}_B$</th>
<th>$V\tilde{w}_B$</th>
<th>$\tilde{w}_1$</th>
<th>$\tilde{w}_2$</th>
<th>$\tilde{w}_3$</th>
<th>$\tilde{w}_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{0}$</td>
<td>$\tilde{w}_3$</td>
<td>$2\gamma^2 - 27\gamma + 85$</td>
<td>$2\gamma^2 - 15\gamma + 28$</td>
<td>0</td>
<td>1</td>
<td>$\gamma - 5$</td>
</tr>
<tr>
<td>(50,5,5)</td>
<td>$\tilde{w}_2$</td>
<td>$17 - 2\gamma$</td>
<td>$4 - \gamma$</td>
<td>1</td>
<td>0</td>
<td>$1$</td>
</tr>
<tr>
<td>Max $\tilde{y}$</td>
<td>$\tilde{w}_j - \tilde{c}_j$</td>
<td>$\left(\frac{850 - 100\gamma}{7 - 2\gamma}, \frac{85 - 10\gamma}{7 - 2\gamma}, \frac{85 - 10\gamma}{7 - 2\gamma}\right)$</td>
<td>$\left(\frac{60 - 10\gamma}{7 - 2\gamma}, \frac{55 - 15\gamma}{7 - 2\gamma}, \frac{55 - 15\gamma}{7 - 2\gamma}\right)$</td>
<td>$\tilde{0}$</td>
<td>$\tilde{0}$</td>
<td>$\left(\frac{50}{7 - 2\gamma}, \frac{5}{7 - 2\gamma}, \frac{5}{7 - 2\gamma}\right)$</td>
</tr>
</tbody>
</table>
Table 3 solution

<table>
<thead>
<tr>
<th>w</th>
<th>[0,1/3]</th>
<th>[1/3,1/2]</th>
<th>[1/2,1]</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\tilde{w}_1)</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\tilde{w}_2)</td>
<td>(\frac{17-2\gamma}{7-2\gamma})</td>
<td>(\frac{17-2\gamma}{7-2\gamma})</td>
<td>(\frac{17-2\gamma}{7-2\gamma})</td>
</tr>
</tbody>
</table>

5. CONCLUSION

In this paper we have shown that there is a feasible solution in each interval constituting by the iterative breaking points. After the breaking points of the satisfactory level that are changed, the values of decision variables are determined.

REFERENCES