Generalized Matrix Algebras and Their Applications

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Abstract

The relations between the radicals of path algebras and connectivity of directed graphs are given. The relations between radicals of generalized matrix rings and Γ-rings are given. All the coquasitriangular structures of group algebra kG are found when G is a finitely generated abelian group.

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0 Introduction

The concepts of generalized matrix algebras (rings) and generalized matrix braided m-Lie algebras were introduced in [16] [17]. The radicals of generalized matrix rings were studied in [13] [16], [5]. The algebraic construction of path algebras was studied in [1] [9] [10].

Generalized matrix algebras (rings) generalize matrix rings, infinite matrix rings, (generalized) path algebras, simple Artin algebras, semisimple Artin algebras and the direct sum of algebras. They can be applied in matrix theory, graph theory, network, braided m-Lie algebras, graded algebras and small preadditive categories.

In this paper we give the relations between radicals of generalized matrix rings and Γ-rings. We obtain the explicit formulas for generalized matrix ring A and radical properties $r = r_b, r_l, r_j, r_n$:

$$r(A) = g.m.r(A) = \sum\{r(A_{ij}) \mid i, j \in I\}.$$

We give the relations between the radicals of path algebras and connectivity of directed graphs. That is, we obtain that every weak component of directed path D is a strong component if and only if every unilateral component of D is a strong component if and only if the Jacobson radical of path algebra A(D) is zero if and only if the Baer radical of path algebra A(D) is zero. We give some examples to establish that generalized matrix algebras are applied in networks. We find all the coquasitriangular structures of group algebra kG when G is a finitely generated abelian group.

Throughout this paper all algebras are over a field k.

1 The radicals of generalized matrix rings

We first recall the concept of generalized matrix rings (algebras). Let I be a set. For any $i, j, l, k \in I$, $(A_{ij}, +)$ is an additive abelian group and there exists a map μ_{ijl} from $A_{ij} \times A_{jl}$ into A_{il} (written $\mu_{ijl}(x, y) = xy$) such that the following conditions are satisfied:

(i) $(x + y)z = xz + yz, \quad w(x + y) = wx + wy.$

(ii) $w(xz) = (wx)z,$

for any $x, y \in A_{ij}$, $z \in A_{jl}, w \in A_{ki}$. Let A be the external direct sum of $\{A_{ij} \mid i, j \in I\}$. We define the multiplication in A as

$$xy = \{\sum_k x_{ik} y_{kj}\}$$

for any $x = \{x_{ij}\}, y = \{y_{ij}\} \in A$. It is easy to check that A is a ring (possibly without unit). We call A a generalized matrix ring, or a gm ring in short, written as $A = \sum\{A_{ij} \mid i, j \in I\}$. I is called the index set of A. Every element in A is called a generalized matrix. We can easily define the gm ideals and gm sub-rings. Similarly. we can get the definition of generalized matrix algebras, in this case, A_{ij} is an algebra over field k for any $i, j \in I$. Let $xE(i, j)$ denote the generalized matrix having a lone x as its (i, j)-entry and all other entries are zero. If B is a non empty subset of generalized matrix ring A and $s, t \in I$, we call the set $\{x \in A_{st} \mid$ there exists $y \in B$ such that $y_{st} = x\}$ the projection on (s, t) of B, written as B_{st}. If $A = \sum\{A_{ij} \mid i, j \in I\}$ is a generalized matrix ring and $I = G$ is an abelian group, then A is a ring graded by G with $A_g = \sum_{i=j+g} A_{ij}$ for any $g \in G$ (see [17, Lemma 2.1]).

Let $g.m.r(A)$ denote the maximal gm ideal of $r(A)$ for a radical property r of rings (see [16]). Let r_b, r_l, r_k, r_j, r_n denote the Baer radical, Levitzki radical, nil radical, Jacobson radical and von Neumann regular radical of rings and Γ -rings, respectively. Let $r(A_{ij})$ denote r radical of A_{ji}-ring A_{ij} for any $i, j \in I$.

Now we study the von Neumann radical $r_n(A)$ of generalized matrix ring $A = \sum\{A_{ij} \mid i, j \in I\}$.

Definition 1.1 *If for all $s, t \in I$, there exists $0 \neq d_{st} \in A_{st}$ such that $x_{is}d_{st} \neq 0$ and $d_{st}y_{tj} \neq 0$ for any $i, j \in I, x_{is} \in A_{is}, y_{tj} \in A_{tj}$, then we say that A has a left gm non-zero divisor.*

Similarly, we can define the right gm non-zero divisor of A.

Lemma 1.2 *(i) If B is an ideal of A, then $\bar{B} = \sum\{B_{ij} \mid i, j \in I\}$ is the gm ideal generated by B in A.*

(ii) If D is a gm ideal of A and $D \subseteq \sum\{r_n(A_{ij}) \mid i, j \in I\}$, then D is an r_n -gm ideal of A.

(iii) Let B_{st} be an r_n-ideal of A_{ts}-ring A_{st} and $D_{ij} = A_{is}B_{st}A_{tj}$ for any $i, j \in I$. If A has left and right gm non-zero divisors, then D is an r_n-gm ideal of A.

(iv) If A has left and right gm non-zero divisors and g.m. $r_n(A) = 0$, then $r_n(A_{ij}) = 0$ for any $i, j \in I$.

Proof. (i) It is trivial.

(ii) For any $x \in D$, there exists a finite subset J of I such that $x_{ij} = 0$ for any $i, j \notin J$. Without lost the generality, we can assume that $J = \{1, 2, \cdots, n\}$ and $J' = \{1, 2, \cdots, n, n+1\} \subseteq I$. Let $J' \times J' = \{(u, v) \mid u, v = 1, 2, \cdots, n+1\}$ with the dictionary order. We now show that there exist two sequences $\{y_{t_2,t_1} \in A_{t_2,t_1} \mid (t_1, t_2) \in J' \times J'\}$ and $\{x^{(t)} \in D \mid (t_1, t_2) \in J' \times J'\}$ with $x^{(1,1)} = x$ and

$$x^{(t+1)} = x^{(t)} - x^{(t)}(y_{(t_2,t_1)}E(t_2, t_1))x^{(t)} \tag{1}$$

such that $x_s^{(t)} = 0$ for any $s, t \in J' \times J'$ with $s \prec t$ by induction. Since $x_{1,1}^{(1,1)} = x_{1,1}$ is a von Neumann regular element, there exists $y_{1,1} \in A_{1,1}$ such that $x_{1,1} = x_{1,1}y_{1,1}x_{1,1}$. See that $x_{1,1}^{(1,2)} = x_{1,1}^{(1,1)} - x_{1,1}^{(1,1)}y_{1,1}x_{1,1}^{(1,1)} = 0$. For $t = (t_1, t_2) \in B$, we assume that there exists $y_{s_2,s_1} \in A_{s_2,s_1}$ and $x_s^{(t)} = 0$ for any $s = (s_1, s_2) \prec (t_1, t_2)$. Since $x_{t_1,t_2}^{(t_1,t_2)}$ is a von Neumann regular element, there exists $y_{t_2,t_1} \in A_{t_2,t_1}$ such that $x_{t_1,t_2}^{(t_1,t_2)} = x_{t_1,t_2}^{(t_1,t_2)}y_{t_2,t_1}x_{t_1,t_2}^{(t_1,t_2)}$. By (1), we have $x_{t_1,t_2}^{(t+1)} = 0$. For $s = (s_1, s_2) \prec t = (t_1, t_2)$, we have either $s_1 = t_1$, $s_2 < t_2$ or $s_1 < t_1$. This implies that $(t_1, s_2) \prec (t_1, t_2)$ or $(s_1, t_2) \prec (t_1, t_2)$. Thus $x_{s_1,s_2}^{(t+1)} = 0$ by (1). Since $x^{(n,n)+1} = 0 \in r_n(A)$, we have that x is von Neumann regular by [Lemma 1][6].

(iii) Let d_{ij} and d'_{ij} in A_{ij} denote the left and right gm non-zero divisors of A for any $i, j \in I$, respectively. For any $x_{ij} \in D_{ij}$, there exists $u_{ts} \in A_{ts}$ such that $d_{st}d_{ti}x_{ij}d'_{js}d'_{st} = d_{st}d_{ti}x_{ij}d'_{js}d'_{st}u_{ts}d_{st}d_{ti}x_{ij}d'_{js}d'_{st}$ and $x_{ij} = x_{ij}d'_{js}d'_{st}u_{ts}d_{st}d_{ti}x_{ij}$ since $d_{st}d_{ti}x_{ij}d'_{js}d'_{st} \in B_{st}$. This implies $D_{ij} \subseteq r_n(A_{ij})$. Considering part (ii), we complete the proof.

(iv) If there exist $s, t \in I$ such that $r_n(A_{st}) \neq 0$, let $B_{st} = r_n(A_{st})$ and $D_{ij} = A_{is}B_{st}A_{tj}$ for any $i, j \in I$. By part (iii), we have that $D = 0$ and $B_{st} = 0$. This is a contradiction. □.

Theorem 1.3 *If A has left and right gm non-zero divisors, then $r_n(A) = g.m.r_n(A) = \sum\{r_n(A_{ij}) \mid i,j \in I\}$.*

Proof. Let $B = r_n(A)$. For any $i,j \in I$ and $x_{ij} \in B_{ij}$, there exists $y \in B$ such that $y_{ij} = x_{ij}$. Let d_{ii} and d'_{jj} be left and right non-zero divisors in A_{ii} and A_{jj}, respectively. Since $(d_{ii}E(i,i))y(d'_{jj}E(j,j)) = (d_{ii}x_{ij}d'_{jj})E(i,j) \in B$, we have that there exists $z \in B$ such that $(d_{ii}x_{ij}d'_{jj})E(i,j) = (d_{ii}x_{ij}d'_{jj})E(i,j)z(d_{ii}x_{ij}d'_{jj})E(i,j)$. By simple computation, we have $d_{ii}x_{ij}d'_{jj} = (d_{ii}x_{ij}d'_{jj})z_{ji}(d_{ii}x_{ij}d'_{jj})$ and $x_{ij} = x_{ij}d'_{jj}z_{ji}d_{ii}x_{ij}$. Thus x_{ij} is von Neumann regular . This implies $B_{ij} \subseteq r_n(A_{ij})$ and $r_n(A) \subseteq \sum\{r_n(A_{ij}) \mid i,j \in I\}$.

Let $N = g.m.r_n(A)$. Since $g.m.r_n(A//N) = 0$, we have that A_{ij}/N_{ij} is an r_n-semisimple A_{ji}/N_{ji}-ring for any $i,j \in I$ by Lemma 1.2 (iv). It is clear that A_{ij}/N_{ij} is a r_n-semisimple A_{ji}-ring . This implies $r_n(A_{ij}) \subseteq N_{ij}$ for any $i,j \in I$. Consequently, $\sum\{r_n(A_{ij}) \mid i,j \in I\} \subseteq g.m.r_n(A)$. □

If for any $s \in I$, there exists $u_{ss} \in A_{ss}$ such that $xe_{ss} = x$ for any $i \in I$ and $x \in A_{is}$, then we say that A has a right gm unit and u_{ss} is a right gm unit in A_{ss}. Similarly, we can define a left gm unit of A and gm unit of A. In fact, if A has left and right gm units, then every ideal of A is a gm ideal, so $r_n(A) = g.m.r_n(A) \subseteq \sum\{r_n(A_{ij}) \mid i,j \in I\}$ by the proof of Theorem 1.3.

It is clear that if R is a ring and M is a Γ- ring with $R = M = \Gamma$, then $r_n(R) = r_n(M)$. We also have that $r(R) = r(M)$ for $r = r_b, r_k, r_l, r_j$ (see [2, Theorem 5.2], [3, Theorem 10.1] and [16, Theorem 3.3], [5, Theorem 5.1]).

Theorem 1.4 *Let $r = r_b, r_l, r_j, r_n$. Then*

(i) $r(A) = g.m.r(A) = \sum\{r(A_{ij}) \mid i,j \in I\}$.

(ii) $r(A) = \sum\{r(A_{ii}) \mid i \in I\}$ when $A_{ij} = 0$ for any $i \neq j$, i.e. r radical of the direct sum of rings is equal to the direct sum of r radicals of these rings.

(iii) $r(A)$ is graded by G when the index set I of A is an abelian group G. Moreover the grading is canonical.

Here A has left and right gm non-zero divisors when $r = r_n$ in (i), (ii) and (iii).

Proof. (i)

$$\begin{aligned} r_b(A) &= g.m.r_b(A) = \sum\{r_b(A_{ij}) \mid i,j \in I\} \text{ (by [16, Theorem 3.7])} \\ r_l(A) &= g.m.r_l(A) = \sum\{r_l(A_{ij}) \mid i,j \in I\} \quad \text{(by [5, Theorem 1.3 and Theorem 2.5])} \\ r_j(A) &= g.m.r_j(A) = \sum\{r_j(A_{ij}) \mid i,j \in I\} \quad \text{(by [5, Theorem 3.10 and Theorem 1.3])} . \end{aligned}$$

(ii) Since the radicals of ring A_{ii} and A_{ii}-ring A_{ii} are the same, we have (ii).

(iii) It follows from (i) and [17, Lemma 2.1]. □

Let $M_I^f(R)$ denote the generalized matrix ring $A = \sum\{A_{ij} \mid A_{ij} = R, i, j \in I\}$ with infinite index set I, which is called an infinite matrix ring over ring R. In this case, $M_I^f(k)$ is called an infinite matrix algebra over field k. Let $M_{m\times n}(R)$ denote the ring of all $(m \times n)$ matrices over ring R.

Example 1.5 *(i) Let $V = \sum_{g\in G} \oplus V_g$ be a vector space over field k graded by abelian group G with dim $V_g = n_g < \infty$. Let I denote a basis of V. Then $\sum\{A_{ij} \mid A_{ij} = Hom(V_j, V_i), i, j \in G\} = \sum\{A_{ij} \mid A_{ij} = M_{n_i\times n_j}(k), i, j \in G\}$ as generalized matrix algebras. However, $\sum\{A_{ij} \mid A_{ij} = Hom(V_j, V_i), i, j \in G\} = \sum\{A_{ij} \mid A_{ij} = M_{n_i\times n_j}(k), i, j \in G\} = \{f \in End_k V \mid ker f$ has finite codimension $\} = M_I^f(k) = \sum\{A_{ij} \mid A_{ij} = k, i, j \in I\}$ as algebras. Then $r(M_I^f(k)) = \sum\{r(A_{ij}) \mid A_{ij} = k, i, j \in I\} = r(\{f \in End_k V \mid ker f$ has finite codimension $\}) = 0$ for $r = r_b, r_l, r_j$. It is clear that generalized matrix algebra $A = \sum\{A_{ij} \mid A_{ij} = Hom(V_j, V_i), i, j \in G\}$ has left and right gm non-zero divisors if and only if $n_i = n_j$ for any $i, j \in G$. Consequently, $r_n(M_I^f(k)) = \sum\{r_n(A_{ij}) \mid A_{ij} = k, i, j \in I\} = M_I^f(k)$. That is, $\{f \in End_k V \mid ker f$ has finite codimension $\}$ is a von Neumann regular algebra.*

(ii) By Theorem 1.4, $r(M_I^f(R)) = M_I^f(r(R))$ for $r = r_b, r_l, r_j$. If R is a ring with left and right non-zero divisors, then $r_n(M_I^f(R)) = M_I^f(r_n(R))$. Obviously, if R has left and right units, then R is a ring with left and right non-zero divisors ($R \neq 0$), so $r_n(M_I^f(R)) = M_I^f(r_n(R))$.

2 Application in Hopf algebras

Example 2.1 *Recall the duality theorem (see [8, Corollary 6.5.6 and Theorem 6.5.11]) for co-Frobenius Hopf algebra H:*

$$(R\#H^{*rat})\#H \cong M_H^f(R) \text{ and } (R\#H)\#H^{*rat} \cong M_H^f(R) \qquad (\text{as algebras}),$$

where $M_H^f(R) = \sum\{B_{ij} \mid i, j \in I\}$ is a gm algebra and I is the basis of H with $B_{ij} = R$ for $i, j \in I$. Note for the Baer radical, Levitzki radical, Jacobson radical and von Neumann regular radical $r(M_H^f(R)) = M_H^f(r(R))$ of $M_H^f(R)$. Consequently, $r((R\#H^{*rat})\#H) \cong M_H^f(r(R))$ and $r((R\#H)\#H^{*rat}) \cong M_H^f(r(R))$. In particular, the Heseberg algebra $H\#H^{*rat} \cong M_H^f(k)$ for infinite co-Frobenius Hopf algebra H. Therefore

$$r(H\#H^{*rat}) \cong r(M_H^f(k)) = M_H^f(r(k)) = \begin{cases} 0 & \text{when } r = r_b, r_l, r_j \\ M_H^f(k) & \text{when } r = r_n. \end{cases}$$

3 Application in path algebras

We first consider the radicals of path algebras.

Assume that D is a directed graph (D is possibly an infinite directed graph and also possibly not a simple graph) or a quiver. Let I denote the vertex set of D, x_{ij} an arrow from i to j and $x = (x_{i_1i_2}x_{i_2i_3}\cdots x_{i_{n-1}i_n})$ a path from i_1 to i_n via arrows $x_{i_1i_2}, x_{i_2i_3}, \cdots, x_{i_{n-1}i_n}$. For two paths $x = (x_{i_1i_2}x_{i_2i_3}\cdots x_{i_{n-1}i_n})$ and $y = (y_{j_1j_2}y_{j_2j_3}\cdots y_{j_{m-1}j_m})$ of D with $i_n = j_1$, we define the multiplication of x and y as

$$xy = (x_{i_1i_2}x_{i_2i_3}\cdots x_{i_{n-1}i_n}y_{j_1j_2}y_{j_2j_3}\cdots y_{j_{m-1}j_m}).$$

Let A_{ij} denote the vector space over field k with basis being all paths from i to j, where $i, j \in I$. Notice that we view every vertex i of D as a path from i to i, written e_{ii} and $e_{ii}x_{ij} = x_{ij}e_{jj} = x_{ij}$. We can naturally define a linear map from $A_{ij} \otimes A_{jk}$ to A_{ik} as $x \otimes y = xy$ for any two paths $x \in A_{ij}, y \in A_{jk}$. We easily show that $\sum\{A_{ij} \mid i, j \in I\}$ is a generalized matrix algebra, which is called a path algebra, written as $A(D)$ (see, [1, Chapter 3]). If $x = (x_{i_1i_2}x_{i_2i_3}\cdots x_{i_{n-1}i_n})$ a path from i_1 to i_n via arrows $x_{i_1i_2}, x_{i_2i_3}, \cdots, x_{i_{n-1}i_n}$, then $n-1$ is called the length of path x, written $l(x) = n-1$. We can define that $l(e_{ii})$ is zero. If $u = \sum_{s=1}^{n} k_sp_s \in A_{ij}$, then the length $l(u)$ is defined as the maximal length $l(p_s)$ for $s = 1, 2, \cdots, n$, where $p_1, p_2, \cdots p_s$ are different paths in A_{ij}. If path $x \neq e_{ii}$ in A_{ii} then x is called a cycle. A path is called regular if it is not contained in any cycles. Let $R(D)$ denote the set of all regular paths. Directed graph D is strong connected if there exists a path from i to j for any two different vertexes $i, j \in I$. Directed graph D is unilateral connected if there exists either a path from i to j or a path from j to i for any two different vertexes $i, j \in I$. Directed graph D is weak connected if there exists a semi-path (that is, a path thrown off the direction) from i to j for any two different vertexes $i, j \in I$. A maximal strong connected subgraph of D is called a strong component of D. A maximal unilateral connected subgraph of D is called a unilateral component of D. A maximal weak connected subgraph of D is called a week component of D.

Lemma 3.1 *Let r denote r_b, r_k, r_1, r_j and $s, t \in I$. If $A_{st} \neq 0$, then*
(i) $r(A_{st}) = 0$ if and only if $A_{ts} \neq 0$.
(ii) $r(A_{st}) = A_{st}$ if and only if $A_{ts} = 0$.

Proof. If $r(A_{st}) = 0$, then $r_b(A_{st}) = 0$ and $A_{ts} \neq 0$. Conversely, if $A_{ts} \neq 0$ and $r_j(A_{st}) \neq 0$, then there exists $0 \neq y \in r_j(A_{st})$, i.e. y is a right quasi-regular element of A_{ts}-ring A_{st}. Thus there exists $u \in A_{ts}A_{st}$ such that $y(xy)u = -y(xy) - yu$ for $x \in A_{ts}$. Since the right hand side is shorter than the left hand side, we get a contradiction. This implies that $r_j(A_{st}) = 0$ and $r(A_{st}) = 0$. □

Lemma 3.2 *$r_n(A_{st}) = 0$ for any $s \neq t$.*

Proof. For any $0 \neq x \in A_{st}$, if x is a von Neumann regular element, then there exists $y \in A_{ts}$ such that $x = xyx$. Considering the length of both sides we get a contradiction. Consequently, $r_n(A_{st}) = 0$. □

Theorem 3.3 *(i)* $r(A) = g.m.r(A) = \sum\{r(A_{ij}) \mid i, j \in I\} = kR(D)$, *where* r *denotes* r_b, r_1, r_k *and* r_j.

(ii) $r_n(A) = g.m.r_n(A) = \oplus\{N_{ii} \mid i \in I\}$,

where $N_{ii} = \begin{cases} ke_{ii} = r_n(A_{ii}) & \text{when } A_{si} = A_{is} = 0 \text{ for any } s \in I \text{ with } i \neq s. \\ 0 & \text{otherwise .} \end{cases}$

Proof. (i) By Lemma 3.1, $\sum\{r(A_{ij}) \mid i, j \in I\} \subseteq kR(D)$. Let $x \in kR(D)$ be a regular path from i to j. Then $A_{ij} \neq 0$ and $A_{ji} = 0$, which implies $r(A_{ij}) = A_{ij}$ and $x \in r(A_{ij}$. This has proved $\sum\{r(A_{ij}) \mid i, j \in I\} = kR(D)$. It follows that $r_b(A) = r_j(A) = kR(D)$ from [12, Proposition 5] or Theorem 1.4. Thus $r(A) = kR(D)$. By [5, Theorem 1.3] or Theorem 1.4, $r(A) = g.m.r(A)$. We complete the proof.

(ii) Since A has a gm unit $e_{ss} \in A_{ss}$ for any $s \in I$, we have that $r_n(A) = g.m.r_n(A)$. By Lemma 3.2 and the proof of Theorem 1.3, we have $r_n(A) = g.m.r_n(A) \subseteq \sum\{r_n(A_{ij}) \mid i, j \in I\} = \oplus\{r_n(A_{ii}) \mid i \in I\}$. Let $N = r_n(A)$. If $N_{ss} \neq 0$, then $N_{ss} = r_n(A_{ss}) = Fe_{ss}$ by the proof of Lemma 3.2. For any $t \in I$ with $t \neq s$, since $A_{ts}N_{ss} = A_{ts} \subseteq N_{ts} = 0$, we have $A_{ts} = 0$. Similarly, $A_{st} = 0$. □

Next we give the relations between the radicals of path algebras and connectivity of directed graphs.

Theorem 3.4 *Directed graph* D *is strong connected if and only if* $A = A(D)$ *is a prime algebra.*

Proof. If D is strong connected and A is not prime, then there exist $u, v, s, t \in I$ and $0 \neq x \in A_{uv}, 0 \neq y \in A_{st}$ such that $xAy = 0$, i.e. $xA_{vs}y = 0$, which contradicts with the strong connectivity of D. Consequently A is prime. Conversely, if A is prime, then $e_{ii}A_{ij}e_{jj} \neq 0$ for any $i, j \in I$, which implies that D is strong connected. □

Theorem 3.5 *Every weak component of* D *has at least two vertexes if and only if* $r_n(A) = 0$.

Proof. It follows from Theorem 3.3 (ii). □

Lemma 3.6 *Every directed graph* D *is the union of all unilateral components.*

Proof. For any path $x \in A_{st}$, set

$$\mathcal{K} = \{E \mid E \text{ is a unilateral connected subgraph of } D \text{ with } x \in E\}.$$

By Zorn's Lemma, we have that there exists a maximal Q in $\mathcal{K}$. □

Theorem 3.7 *Let r denote r_b, r_k, r_l and r_j, respectively. The following conditions are equivalent.*

(i) Every weak component of D is a strong component.

(ii) Every unilateral component of D is a strong component.

(iii) Weak component, unilateral component and strong component of D are the same.

(iv) D is the union of strong components of D.

(v) D has no any regular path.

(vi) $A_{ij} = 0$ if and only if $A_{ji} = 0$ for any $i, j \in I$.

(vii) A is a direct sum of prime algebras.

(viii) A is semiprime.

(ix) A_{ij} is a semiprime A_{ji}-ring for $i, j \in I$.

(x) $r(A_{ij}) = 0$ for any $i, j \in I$.

(xi) $r(A) = 0$.

Proof. By Theorem 3.3, (v), (vi), (viii), (ix), (x) and (xi) are equivalent.

(i) $\Rightarrow$ *(vi)*. If i and j belong to the same weak component, then $A_{ij} \neq 0$ and $A_{ji} \neq 0$. If i and j do not belong to the same weak component, then obviously $A_{ij} = 0$ and $A_{ji} = 0$.

(ii) $\Rightarrow$ *(vi)*. If $A_{ij} \neq 0$, then there exists a path $x \in A_{ij}$. By Lemma 3.6, x belongs to a certain unilateral component of D. Consequently, x belongs to a certain strong component of D. This implies $A_{ji} \neq 0$.

(vi) $\Rightarrow$ *(ii)*. If i and j belong to the same unilateral component of D, then $A_{ij} \neq 0$ or $A_{ji} \neq 0$. Consequently $A_{ij} \neq 0$ and $A_{ji} \neq 0$, which implies that i and j belong to the same strong component of D. Therefore (ii) holds.

(iv) $\Rightarrow$ *(vi)*. If $A_{ij} \neq 0$, then there exists a path $x \in A_{ij}$ and x belongs to a certain strong component of D. This implies $A_{ji} \neq 0$.

(vi) $\Rightarrow$ *(iv)*. For any arrow $x \in A_{ij}$, we only need show that x belongs to a certain strong component of D. By Lemma 3.6, there exists a certain unilateral component C of D such that $x \in C$. Since (ii) and (vi) are equivalent, we have that C is a strong component of D.

(iv) $\Rightarrow$ *(i)*. If i and j belong to the same weak component, then there exists a semi-path $x = x_{ii_1}x_{i_1i_2}\cdots x_{i_nj}$. If i and j belong to different strong components, then we can assume that i_s is the first vertex, which does not belong to the strong component containing i. Consequently, $A_{i_{s-1}i} \neq 0$ and $A_{ii_{s-1}} \neq 0$, and either $A_{ii_s} \neq 0$ or $A_{i_si} \neq 0$. Since (iv) and (vi) are equivalent, we have that $A_{ii_s} \neq 0$ and $A_{i_si} \neq 0$. We get a contradiction. This shows that i and j belong to the same strong components.

(iii) $\Rightarrow$ *(i)* is obvious.

(i) $\Rightarrow$ *(iii)*. Since (i) and (ii) are equivalent, we have (iii).

(vii) $\Rightarrow$ *(viii)*. It follows from Theorem 1.4 (ii).

(iv) ⇒ *(vii)*. Let $\{D^{(\alpha)} \mid \alpha \in \Omega\}$ be all of the strong component of D and $D = \cup\{D^{(\alpha)} \mid \alpha \in \Omega\}$. Thus $A(D) = \oplus\{A(D^{(\alpha)}) \mid \alpha \in \Omega\}$. However, for

any $\alpha \in \Omega$, $A(D^{(\alpha)})$ is a prime algebra by Theorem 3.4. We complete the proof. □

We easily obtain the following by the preceding conclusion for $r = r_b, r_l, r_k, r_j$. D has no cycle if and only if $r(A_{ij}) = A_{ij}$ for any $i \neq j \in I$; s and t $(s \neq t)$ are not contained in the same cycle if and only if $r(A_{st}) = A_{st}$; s and t ($s \neq t$) are contained in the same cycle if and only if $r(A_{st}) = 0$.

We give an example to show whether the condition in Theorem 1.3 is a necessary condition.

Example 3.8 *(i) Let D be a directed graph with vertex set $I = \{1, 2\}$ and only one arrow $x_{12} \in A_{12}$. Obviously, $A_{12} = kx_{12}, A_{11} = ke_{11}, A_{22} = ke_{22}, A_{21} = 0$, $r_n(A_{ii}) = ke_{ii}$ and $r_n(A_{ij}) = 0$ for any $i, j \in I$ with $i \neq j$. By Theorem 3.3 (ii), $r_n(A) = 0 \neq \sum\{r_n(A_{ij}) \mid i, j \in I\}$. It is clear that A has no left gm non-zero divisor since $A_{21} = 0$. Consequently, it is possible that Theorem 1.3 does not hold if its condition is dropped.*

(ii) Let $I = \{1, 2\}$ and $A_{ij} = M_{i\times j}(k)$ for any $i, j \in I$. It is clear that A has no left gm non-zero divisor in A_{12} since, for any non-zero $x \in A_{12}$, there exists a non-zero $y \in A_{21}$ such that $xy = 0$. However, $r_n(A) = \sum\{r_n(A_{ij}) \mid i, j \in I\} = M_{3\times 3}(k)$. Consequently, the condition in Theorem 1.3 is not a necessary condition.

4 Application in networks

We give some examples to establish that generalized matirx algebras are applied in networks.

We first recall some basic concepts. A pair (H, r) is called a coquasitriangular Hopf algebra if H is a Hopf algebra over field k and there exists a convolution- invertible k-bilinear form r: $H \otimes H \to k$ such that the following conditions hold: for any $a, b, c \in H$,

(CQT1) $r(a, bc) = \sum r(a_1, b)r(a_2, c)$;

(CQT2) $r(ab, c) = \sum r(a, c_1)r(b, c_2)$;

(CTQ3) $\sum r(a_1, b_1)a_2b_2 = \sum b_1a_1r(a_2, b_2)$.

Furthermore, if $r^{-1}(a, b) = r(b, a)$ for any $a, b \in H$, then (H, r) is called a cotriangular Hopf algebra.

For coquasitriangular Hopf algebra (H, r), we can define a braiding C^r in the category ${}^H\mathcal{M}$ of H-comodules as follows:

$$C^r_{U,V} : U \otimes V \longrightarrow V \otimes U$$

sending $u \otimes v$ to $\sum r(v_{(-1)}, u_{(-1)})v_{(0)} \otimes u_{(0)}$ for any $u \in U, v \in V$, where (U, ϕ_U) and (V, ϕ_V) are left H-comodules (see [11, Proposition VIII.5.2]). This braided tensor category is called one determined by coquasitriangular structure. Dually, we have the concept of quasitriangular Hopf algebras. We also have that $({}_H\mathcal{M}, C^R)$ is a braided tensor category, which is called one determined by quasitriangular structure.

Let G and H be two abelian groups and $G \times H$ denote the external direct product of G and H. A map $r : G \times H \to k$ is called a bicharacter on $G \times H$ if for any $a, b \in G, c, d \in H$, the following conditions hold

(i) $r(ab, c) = r(a, c)r(b, c)$,

(ii) $r(a, cd) = r(a, c)r(a, d)$,

(iii) r is normal, i.e. $r(a, e) = r(e, c) = 1$,

(iv) r is convolution invertible.

Furthermore, if $G = H$, then r is called a bicharacter on G in short. r is called skew symmetric if $r^{-1}(a, b) = r(b, a)$ for any $a, b \in G$.

In fact, condition (iv) can be obtained from the others, i.e. $r^{-1}(a, b) = r(a, b^{-1}) = r(a^{-1}, b)$ for any $a \in G, b \in H$.

It is clear that for an abelian group G, r is a coquasitriangular structure of group algebra kG if and only if r is a bicharacter on G.

Example 4.1 *Let*

$$\begin{aligned} I_N &= \{x \mid x \textit{ is the name of a country }\}, \\ I_O &= \{x \mid x \textit{ is the name of an organization}\}, \\ I_S &= \{x \mid x \textit{ is the name of a server }\}, \quad \textit{and} \\ I_C &= \{x \mid x \textit{ is the name of a computer }\}. \end{aligned}$$

Let I denote one of I_N, I_O, I_S and I_C. We construct a directed graph D as follows. Let I be the vertex set of D and each piece of information from i to j denote an arrow from i to j for any $i, j \in I$. By the quiver we obtain a generalized matrix algebra.

Example 4.2 *Let V_1, V_2, V_3 and V_4 are vector spaces with basis*

$$\begin{aligned} &\{x \mid x \textit{ is a name of a train }\}, \\ &\{x \mid x \textit{ is the name of an automobile}\}, \\ &\{x \mid x \textit{ is the name of a boat }\} \quad \textit{and} \\ &\{x \mid x \textit{ is the name of an airplane }\}, \textit{respectively.} \end{aligned}$$

Let $G = \mathbf{Z}_4$ and r a bicharacher on G. In this way, we can turn the map about pathes of trains, automobiles, boats and airplanes into a braided diagram. Note the following details:

(i) If the path is parallel to the latitude, then we view that the west is higher than the east.

(ii) A braiding denotes the cross of two pathes.

(iii) If the path is concave, then we view that the path is evaluation.

(iv) If the path is convex, then we view that the path is coevaluation.

5 Coquasitriangular Structures of Group Algebras

It is clear that r is a coquasitriangular structure of group algebra kG if and only if r is a bicharacter on G which is abelian. Thus it is necessary to find all bicharacters of abelian groups.

For convenience, we say the order of a torsion-free elelment in group G is zero.

Proposition 5.1 *Let $G = (a)$ and $H = (b)$ be two cyclic groups. Then r is a bicharacter on $G \times H$ if and only if $r(a^m, b^n) = r(a, b)^{mn}$ for any $m, n \in \mathbf{Z}$ and $r(a, b)^N = 1$, where N denotes the greatest common divisor of the orders of a and b.*

Proof. We first show our claim when both of G and H are finite groups. Assume that r is a bicharacter on $G \times H$. Then
$r(a^m, b^n) = r(a^{m-1}a, b^n) = r(a^{m-1}, b^n)r(a, b^n) = \ldots = r(a, b^n)^m = r(a, bb^{n-1})^m = r(a, b)^m r(a, b^{n-1})^m = \ldots = r(a, b)^{mn}$. Because $N = (o(a), o(b))$, where $o(a)$ is the order of a, and $o(b)$ the order of b, there exist $s, t \in \mathbf{Z}$ such that $N = o(a)s + o(b)t$. Then $r(a, b)^N = r(a, b)^{o(a)s+o(b)t} = r(a, b)^{o(a)s} r(a, b)^{o(b)t} = r(a^{o(a)s}, b^{o(b)t}) = r(e, b)r(a, e) = 1$.

Conversely, we show that r is a bicharacter on $G \times H$. It is easy to check that r is well defined. Therefore, r is a bicharacter on $G \times H$.

Next we show our claim when G is an infinite group. In this case, we say $o(a) = 0$. Considering $N = o(b)$, the greatest common divisor of the orders of a and b, we easily complete the proof. $\Box$

Theorem 5.2 *Let $G = \sum_{i \in I} \oplus G_i$ be the direct sum of cycle group $G_i = (a_i)$ for $i \in I$. Set $\mathcal{R} = \{r \mid r \text{ is a bicharacter on } G\}$ and $\mathcal{Q} = \{\{q_{ij}\} \mid q_{ij} \in k, (q_{ij})^{N_{ij}} = 1, \text{ where } N_{ij} \text{ is the greatest common divisor of the orders of } a_i \text{ and } a_j \text{ for } i, j \in I\} \subseteq k^{I \times I}$. Define $\Phi : \begin{cases} \mathcal{R} \longrightarrow \mathcal{Q} \\ r \mapsto \{r(a_i, a_j)\} \end{cases}$. Then Φ is bijective.*

Proof. We first show that Φ is surjective. For any $\{q_{ij}\} \in \mathcal{Q}$, we define a bilinear map r on G as follows:

$$r(a_i, a_j) = q_{ij} \text{ for any } i, j \in I \quad \text{and} \qquad r(x, y) = \Pi_{i,j \in I}(q_{ij})^{m_i n_j}$$

for every $x = \{a_i^{m_i}\}, y = \{a_i^{n_i}\} \in G$. Now we show that r is a bicharacter on G. For any $x = \{a_i^{m_i}\}, y = \{a_i^{n_i}\}$, and $z = \{a_i^{t_i}\} \in G$.

$$r(xy, z) = \prod_{i,j \in I} (q_{ij})^{m_i t_j} (q_{ij})^{n_i t_j} = r(x, z) r(y, z)$$

.

Similarly $r(x, yz) = r(x, y)r(x, z)$.
Let $o(a_j) = N_{ij}s_{ij}$. Then $r(x, e) = \prod_{i,j \in I}(q_{ij})^{m_i N_{ij} s_{ij}} = 1$. Similarly $r(e, x) = 1$. So r is a bicharacter on G and $\phi(r) = \{q_{ij}\}$. Therefore, Φ is Surjective.

Now we show that Φ is injective. Assume $r, r' \in \mathcal{R}$ with $r(a_i, a_j) = r'(a_i, a_j)$ for any $i, j \in I$. That is, $\Phi(r) = \Phi(r')$. It is clear that $r = r'$. We complete the proof. $\Box$

Corollary 5.3 *If G is a finitely generated abelian group with $G = (a_1) \times (a_2) \times \cdots \times (a_n)$. Set $\mathcal{R} = \{r \mid r$ is a bicharacter on $G\}$ and $\mathcal{Q} = \{\{q_{ij}\} \mid q_{ij} \in k, q_{ij}^N = 1$ when the order of (a_i) or (a_j) is a natural number N; $q_{ij} \neq 0$ for $i, j = 1, 2, \cdots, n\}$. Define* $\Phi : \begin{cases} \mathcal{R} \longrightarrow \mathcal{Q} \\ r \mapsto \{r(a_i, a_j)\} \end{cases}$. *Then Φ is bijective.*

Proof. Let $I = \{1, 2, \ldots, n\}$. It follows from Theorem 5.2.$\Box$

Example 5.4 *If $G = \mathbf{Z}$, then $G = (1)$. By Theorem 5.2, for any $0 \neq q \in k$, if we define that $r(m, n) = q^{mn}$ for any $m, n \in \mathbf{Z}$, then r is a bicharacter on $\mathbf{Z}$. Furthermore, these are all bicharacters of $\mathbf{Z}$. Similarly, we can obtain all of the bicharacters of $\mathbf{Z}_n$.*

Proposition 5.5 *If $G = (a)$ is a cyclic group with odd order, then G has no skew-symmetric bicharacter except the trivial one(i.e. $r(a, b) = 1$ for any $a, b \in G$).*

Proof. Assume that there exists a nontrivial skew-symmetric bicharacter r on G. Then $r^{-1}(a, a) = r(a^{-1}, a) = r(a, a), i.e. r(a^{-1}, a)r(a, a) = r(a, a)^2 = 1$. Let $\mid G \mid = 2N+1$, where N is a natural number. Because $r(e, a) = r(a^{2N+1}, a) = r(a, a)^{2N+1} = (r(a, a)^2)^N r(a, a) = r(a, a) = 1$, $r(b, c) = r(a^m, a^n) = r(a, a)^{mn} = 1$ for any $b, c \in G$ with $b = a^m, c = a^n$, where $m, n \in \mathbf{Z}$. This is a contradiction with the fact that r is nontrivial. $\Box$

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