*Representation of Lie groups and special functions*, by N. Ja. Vilenkin and A. U. Klimyk (translated from the Russian by V. A. Groza and A. A. Groza), Math. Appl., Kluwer Acad. Publ., Dordrecht, \$804.50 (set). Vol. 1: *Simplest Lie groups, special functions and integral transforms*, vol. 72, 1991, xxiv+608 pp., \$408.00, ISBN 0-7923-1466-2; Vol. 2: *Class I representations, special functions, and integral transforms*, vol. 74, 1992, xviii+607 pp., \$397.00, ISBN 0-7923-1492-1; Vol. 3: *Classical and quantum groups and special functions*, vol. 75, 1992, xx+634 pp., \$397.00, ISBN 0-7923-1493-X

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The book under review deals with the interplay between two branches of mathematics, namely representation theory of groups and the theory of special functions. Both fields go back over a century, and their intimate connection has been observed since the forties and fifties. Pioneering work was done by Bargmann [2], Gel'fand & Šapiro [6] and Wigner [23]. They linked the repesentation theory of the Lorentz group and of the rotation group in three-dimensional space to hypergeometric functions and Jacobi polynomials. Since then an enormous amount of work has been done on this subject, also motivated by physical models. The state of the art in the sixties has been given in the books by Talman [17] and, in particular, Vilenkin [19], while Miller's book [14] exposed a very different approach. Some less comprehensive books or edited volumes have appeared afterwards in which newer developments are discussed, for instance [1], [3], [4], [8], [9], [15], [18], [22]. In 1990 (English translation in 1995) Vilenkin and Klimyk wrote a pleasant, relatively short introduction [20] to the subject, taking in account modern developments. But none of these books has the wide scope of the three-volume set under review, which is the successor of Vilenkin's influential book [19].

In the preface the authors state that their aim is "to summarize the development of the theory and to outline its future development." This is certainly a challenging task. On the one hand, if one stays within the scope of the paradigms of the pioneers, the technical complexity of cases being studied has enormously increased. On the other hand there have been many developments in the interaction between special functions and algebraic structures which do not easily fit into the old paradigms. We mention special functions related to any of the following structures: Jordan algebras, symmetric groups and Chevalley groups, $p$-adic groups, discrete subgroups (relation with number theory), infinite dimensional limits of classical groups, affine Lie algebras, root systems, quantum groups, Hecke algebras, association schemes, hypergroups, combinatorics. Some of these new interpretations of special functions are in the volumes under review, as we will indicate below. In our possibly biased view, developments on special functions related to root systems (Heckman-Opdam polynomials, Macdonald polynomials) and their interpretations on Hecke algebras (via Dunkl-Cherednik operators) and on quantum groups have been in particular spectacular, see for instance Macdonald's Bourbaki lecture [13] and Noumi & Sugitani [16]. The authors of the present volumes have treated some of these last topics in a volume 4, called "Recent Advances" [21].

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The notion of Special Functions is not precisely defined, but we tend to think of special functions as functions that occur in solutions of specific problems and satisfy many explicit properties, in particular a rich collection of formulas. Turán and Askey have suggested to call them Useful Functions. As a typical example one might think of the Bessel function, which arises as a solution of the Laplace operator in cylindrical coordinates. Another example is the Jacobi polynomial, which also yields a solution of the Laplace operator but now in spherical coordinates. The Bessel function and the Jacobi polynomial can both be expressed in terms of (generalized) hypergeometric functions, and this feature holds for almost all special functions considered in this book. Jacobi polynomials form a system of orthogonal polynomials, and from a Bessel function one can build a generalized orthogonal system described by the Hankel transform pair. Such orthogonalities, which occur for many special functions, give rise to generalized Fourier analysis and suggest a link with harmonic analysis on groups.

The groups that play a role are usually Lie groups. Already three-dimensional groups like $SU(2)$, $SO(3)$, $SL(2,\mathbb{R})$, $ISO(2)$ and the Heisenberg group allow interpretations of many familiar one-variable special functions. On higher dimensional analogues of these groups like $SU(n)$ and $SL(n,\mathbb{R})$ one finds interpretations of the same special functions for more general parameter values, and interpretations of special functions of more complex nature. Typically, on such a Lie group $G$, one considers some canonical decomposition of $G$ in terms of certain subgroups and one takes a coordinate system on $G$ corresponding to this decomposition. Then one considers the irreducible representations (irreps) of $G$ in a suitable basis, behaving nicely w.r.t. a subgroup $H$ involved in the decomposition of $G$, one looks at the matrix elements as functions in the coordinates, and one tries to recognize the matrix elements as (products of) special functions. Most commonly, one does this for the spherical functions on $G$, i.e., for matrix elements which are, as functions on $G$, left and right invariant w.r.t. $H$. After having established expressions for matrix elements of irreps one can use the group to find properties of the special function involved. For compact groups one finds orthogonality relations for the special functions from Schur's orthogonality relations. This works also for square integrable representations of non-compact groups, but in general one finds transform pairs in the non-compact case. This is essentially the computation of the Plancherel formula. Also, from the homomorphism property of the representations it is possible to derive addition formulas for the special functions. These are only a few of the many properties which can be obtained from their interpretation on the group. It is important to know that this is not a one-way influence. Specific properties of the special functions involved are sometimes needed to establish theorems on the group level. For instance, Harish-Chandra [7] conjectured the explicit Plancherel measure for the spherical Fourier transform on a non-compact symmetric space by using spectral analysis of the hypergeometric differential operator. Later he found a complete proof.

Another important way to link special functions to representations of Lie groups is the following. If the same representation has two explicit realizations in terms of (generalized) orthonormal bases, then the transition matrix is orthogonal and we obtain orthogonality relations if the matrix elements can be calculated explicitly in terms of special functions. Typical examples of such constructions are the Clebsch-Gordan coefficients and the Racah coefficients connecting different orthonormal bases in tensor product representations.

The interaction between group representations and special functions can be approached from at least three different points of view:

(i) Start with a special group. Consider different kinds of special functions occurring on it. This usually gives rise to relationships between these special functions. For instance, Jacobi polynomials and Hahn polynomials live on $SU(2)$ as matrix elements of irreps and as Clebsch-Gordan coefficients, respectively. As a consequence, products of Hahn polynomials occur as expansion coefficients in the expansion of a product of Jacobi polynomials in terms of other Jacobi polynomials, see §8.3.6 of the volumes under review.

(ii) Start with a special function. Find interpretations for it of various kinds on several groups. Then try to find a conceptual explanation which links these interpretations together. For instance, Krawtchouk polynomials live on $SU(2)$, again as matrix elements (see §6.8.1) and on wreath products of symmetric groups as spherical functions (see §13.1.4). A conceptual link between the two interpretations was given in Koornwinder [11].

(iii) Start with a general structure in the context of group representations. Find properties involving special functions which fit into this structure. For instance, consider (zonal) spherical functions on Gelfand pairs, see §17.2. Spherical functions satisfy many nice properties. Whenever one has an interpretation of a special function as a spherical function, then one should try to rephrase these nice properties in terms of the special function.

In our opinion, the third approach should get the most emphasis. Of course, the cases where elegance in group representations and in special functions happily meet, do not exhaust everything of interest in special functions. Many important formulas for special functions may be derived in a shorter or longer, but not very illuminating way from the group interpretation, but may possibly have a shorter derivation just from the analytic definition of the special function. We think that for these cases one should be pragmatic, and give the shortest derivation.

Let us now discuss the volumes under review in more detail. The authors have succeeded in writing an encyclopaedic treatise containing a wealth of identities on special functions. The main examples fit into the methods sketched above. But there is also a chapter in volume 3 on the quantum $SU(2)$ group, or better on the quantized universal enveloping algebra $U_q(\mathfrak{sl}(2,\mathbb{C}))$. There is also some information on the symmetric group, on groups over finite fields and over the $p$-adics, on affine Lie algebras and on modular forms.

A typical chapter starts with a discussion of the Lie group and Lie algebra involved, while introducing suitable bases for the Lie algebra and corresponding one-parameter subgroups of the Lie group. This then gives suitable coordinates on the Lie group like the Euler angles on $SU(2)$. Next the representation theory is discussed: constructions, irreducibility and intertwiners. The special functions are then brought into play and usually the remainder of such a chapter is on the special functions involved. Then the role of the group is pushed into the background and the special functions take a predominant role. Some special functions identities derived do not involve any group theoretic considerations.

As mentioned previously, the three-volume set is the successor of Vilenkin's 1965 book

[19], so a comparison is in order. Vilenkin's book [19] is about the size of one of these volumes. The subject of the chapters of volume 1 overlaps with Chapters 1–8 of [19], although the material is expanded, which is particularly true for Chapter 8 in volume 1 on Clebsch-Gordan and Racah coefficients, one of the fields of expertise of the second author [9] (see also Chapter 18 in volume 3). Also new compared to [19] is the consideration of the transition matrices for irreps of $SL(2,\mathbb{R})$ in Chapter 7. Volume 2 has some overlap, but much less so, with Chapters 9–11 of [19]. So the new material is mainly contained in volumes 2 and 3.

Chapters 13, 14 and 19 deviate from the others in the sense that the groups considered are not Lie groups. In Chapter 13 discrete groups are considered. First there are two examples of finite groups; the symmetric group (but not any other Weyl group) related to Hahn and Krawtchouk polynomials and finite groups of Lie type related to basic (or $q$-)hypergeometric polynomials. It is a pity that the authors' notation for basic hypergeometric series has a meaning which is different from the usual one, see the standard reference by Gasper and Rahman [5]. Also in Chapter 13 there is a section on the $p$-adic number field and related $\Gamma$ and $B$-functions and on $SL_2$ over the $p$-adics, but not on the spherical functions on a group of $p$-adic type, see Macdonald [12]. Chapter 14 contains a discussion of the quantum $SL(2,\mathbb{C})$ group and its relation to basic hypergeometric orthogonal polynomials. The chapter does not treat the important interpretation of Askey-Wilson polynomials as spherical functions or matrix elements on the quantum $SL(2,\mathbb{C})$ group, see for instance Koelink's survey lectures [10]. Finally, Chapter 19 contains an introduction to affine Lie algebras and modular forms.

The present book does not always follows a clear philosophy about what should be obtained from the group context and what can be derived as well analytically (see our point of view above). For example, the second order differential equation for the Jacobi polynomials is first derived in §6.7.5 by composition of two ladder operators acting on matrix elements of irreps of $SU(2)$. These ladder operator actions are derived in a rather complicated way with the group playing some role, but the opportunity is missed to make a link with the ladder operator action of the Lie algebra in §6.2.2. The most conceptual interpretation of the second order differential equation, from the action of the Casimir operator, only occurs 30 pages later, in a short remark in §6.10.3. For an example of another kind, the first order divided difference formula for Racah polynomials is given (not quite correctly) in §8.5.4. The straightforward and short proof from the ${}_4F_3$ formula for Racah polynomials is not mentioned. Instead a proof is indicated which uses formulas which are obtained from the group context, but which does not look very conceptual and which is tedious in computational details.

We consider this three-volume set as a reference work, rather than a book which one will use for learning the subject. With this in mind there are some comments to be made on the presentation. Apart from a 5-page Chapter 0 in volume 1 there is no motivation given. One would expect that at least for each chapter a short introduction would be in place. The table of contents is extraordinary long, giving the full list of sections and subsections of each chapter. On the other hand, the index is too restricted, while we badly missed a cumulative index in volume 3. There is no hierarchy with important results being formulated as theorems and less important results as propositions, apart from sections

18.6–7, which have been written by A. V. and L. V. Rozenblyum. Almost nowhere in the text are references given to the literature. The lists of references at the end of each of the three volumes altogether contain 845 items. At the end of volume 3 the authors have indicated the primary and secondary references for each chapter, but without specifying the particular section or subsection to which the reference applies. We missed more verbose notes at the end of each chapter. Navigation through the volumes is further complicated because the thousands of formulas have a simple numbering starting by 1 in each new subsection, while the running heads only mention the chapter number, not the number of the section or subsection. Tables and diagrams are almost absent. They would have made the book more accessible. Although a geometric approach is sometimes emphasized, this is not supported by pictures.

The book contains many errors; we think too many. Most of them are harmless, but one should not blindly trust the formulas in the book. Apparently there has been no double or triple checking of the formulas. We have found a few conceptual mathematical errors. For producing a full list of errata one has to read and check computations in all 1800 pages, which we have not done.

In view of the price of the books it is likely that only the wealthiest libraries and individuals can afford to buy them. For this price we would have expected greater care in editing on the part of the publisher than was evident from the books.

The conclusion is that the authors have done a great job in compiling this encyclopaedic treatise and that this three-volume set will become a standard reference for special functions and group representations. Needless to say, these books form a worthwhile addition to any mathematics library. But with some more time and effort, both from the authors and the publisher, the result could have been much better. However, for anybody starting to learn the subject Vilenkin's older book [19], possibly complemented with [20], is still the best buy.

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