

Operator-valued Semicircular Elements: Solving A Quadratic Matrix Equation with Positivity Constraints

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We show that the quadratic matrix equation $VW + \eta(W)W = I$, for given V with positive real part and given analytic mapping η with some positivity preserving properties, has exactly one solution W with positive real part. Also we provide and compare numerical algorithms based on the iteration underlying our proofs.

This work bears on operator-valued free probability theory, in particular on the determination of the asymptotic eigenvalue distribution of band or block random matrices.

1 Introduction

This paper does two things. One concerns free probability and random matrix theory. The other concerns iterative methods for solving matrix equations of the form

$$VW + \eta(W)W = I, \tag{1.1}$$

where V is a given matrix with positive real part, η is a 'positivity preserving' linear or analytic mapping, and we are looking for a solution W with positive real part.

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The second topic bears directly on the first. Our main motivation for considering this kind of equation comes from free probability theory. We will now briefly review the relevance of this equation in the free probabilistic context of operator-valued semicircular elements. This will only serve as a motivation and is not essential for our main statements about solving Equation (1.1) in Section 2. For a general introduction to free probability theory, see, e.g., [6].

Operator-valued semicircular elements [12, 11] play an important role in free probability and random matrix theory. In particular, a big class of random matrices (band matrices, block matrices) can asymptotically be described by such elements. This fundamental observation was made by Shlyakhtenko in [10]; extensions and the relevance of this from the point of view of electrical engineering problems were treated in [9].

The main problem in a random matrix context is the determination of the asymptotic eigenvalue distribution. Let us denote by $H(z)$ the Cauchy transform of the asymptotic eigenvalue distribution μ , given by

$$H(z) := \int_{\mathbb{R}} \frac{1}{z-t} d\mu(t).$$

This is an analytic function in the upper complex half plane

$$\mathbb{C}^+ := \{z \in \mathbb{C} \mid \text{Im } z > 0\}$$

and allows recovery of μ via the Stieltjes inversion formula

$$d\mu(t) = -\frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \text{Im } H(t + i\varepsilon).$$

The theory of operator-valued semicircular elements tells us that one can get this Cauchy transform as

$$H(z) = \varphi(G(z)),$$

where $G(z)$ is an operator-valued function, i.e. $G(z) \in \mathcal{A}$ for some operator algebra \mathcal{A} (with involution $*$, containing the unit I), and where $\varphi : \mathcal{A} \rightarrow \mathbb{C}$ is a given state on this algebra. (The data \mathcal{A} and φ are determined by the form of the considered matrices; e.g., for block matrices as in [9], $\mathcal{A} = M_d(\mathbb{C})$ are just the $d \times d$ -matrices, for some fixed d , and $\varphi = \text{tr}_d$ is the normalized trace on those matrices.) The operator-valued Cauchy

transform $G(z)$ is determined by the equation

$$zG(z) = I + \eta(G(z))G(z) \quad (z \in \mathbb{C}^+), \quad (1.2)$$

together with its asymptotic behaviour $G(z) \sim \frac{1}{z}I$ for $z \rightarrow \infty$. Here we are given the completely positive linear map $\eta : \mathcal{A} \rightarrow \mathcal{A}$; it contains the information about the covariance of the considered random matrix or operator-valued semicircular element.

One of the main problems, both from a conceptual and a numerical point of view, is that, for a fixed $z \in \mathbb{C}^+$, the Equation (1.2) has not just one, but many solutions. To isolate the correct root is not obvious at all.

We shall use the common notation: the real and imaginary parts of an operator W are defined as

$$\operatorname{Re} W := \frac{1}{2}(W + W^*) \quad \operatorname{Im} W := \frac{1}{2i}(W - W^*).$$

Since G is also the operator-valued Cauchy transform of a semicircular element s ,

$$G(z) = E \left[\frac{1}{z - s} \right]$$

(for some C^* -algebra \mathcal{B} with $s \in \mathcal{B}$, $s = s^*$, and a conditional expectation $E : \mathcal{B} \rightarrow \mathcal{A}$), we have for $\operatorname{Im} z \geq 0$ that

$$\begin{aligned} \operatorname{Im} G(z) &= \frac{1}{2i}(G(z) - G(z)^*) = \frac{1}{2i}E \left[\frac{1}{z - s} - \frac{1}{\bar{z} - s} \right] \\ &= -\operatorname{Im} z \cdot E \left[\frac{1}{z - s} \frac{1}{\bar{z} - s} \right] \leq -\frac{\operatorname{Im} z}{(|z| + \|s\|)^2} \cdot I; \end{aligned}$$

since

$$\frac{1}{z - s} \frac{1}{\bar{z} - s} \geq \frac{1}{(|z| + \|s\|)^2}$$

and a conditional expectation is positive.

This means that we are looking, for any $z \in \mathbb{C}^+$, for a solution $G(z)$ of (1.2) which has the property that its imaginary part is negative and stays away from zero (which is the same as saying that its imaginary part is negative and invertible). Here we will show that actually there exists exactly one solution of (1.2) with this property, and that one can get it by an appropriately chosen iteration procedure.

If we write $G(z) = -iW(z)$, then $\text{Im } G(z) \leq -\varepsilon I$ gets replaced by the nicer condition $\text{Re } W(z) \geq \varepsilon I$. Thus we will work with $W(z)$ instead of $G(z)$ in the following. Equation (1.2) reads in terms of W as

$$-izW(z) + \eta(W(z))W(z) = I \quad (1.3)$$

As it turns out the linearity of η is not crucial (as long as it is analytic, and has certain positivity and boundedness properties). Thus we will treat the problem directly in this more general setting. Furthermore, one can also replace the complex number $z \in \mathbb{C}^+$ by an arbitrary element $Z \in \mathcal{A}$ with positive imaginary part. Again we replace this by the condition of positive real part by going over to $V = -iZ$, ending up with the Equation (1.1).

2 Main Theorem

2.1 Setting of our problem

We work in a C^* -algebra, denoted \mathcal{A} . Interesting examples arise already in the finite-dimensional case; thus the reader might just take \mathcal{A} as a matrix algebra of complex $d \times d$ -matrices, $\mathcal{A} = M_d(\mathbb{C})$ for some $d \in \mathbb{N}$.

For a selfadjoint operator A we mean with $A \geq 0$ that its spectrum is contained in $[0, \infty)$. Let \mathcal{A}_+ denote the strict right half plane of \mathcal{A} ,

$$\mathcal{A}_+ := \{W \in \mathcal{A} \mid \text{Re } W \geq \varepsilon I \text{ for some } \varepsilon > 0\}.$$

Note that $W \in \mathcal{A}_+$ if and only if $\text{Re } W \geq 0$ and $\text{Re } W$ is invertible.

Furthermore, we are given an analytic mapping

$$\eta : \mathcal{A}_+ \rightarrow \mathcal{A}_+,$$

which is bounded on bounded domains in \mathcal{A}_+ , i.e.

$$\sup\{\|\eta(W)\| \mid W \in \mathcal{A}_+, \|W\| \leq r\} < \infty$$

for any $r > 0$.

For a given $V \in \mathcal{A}_+$ we consider our key equation:

$$VW + \eta(W)W = I. \quad (2.1)$$

We are looking for a solution $W \in \mathcal{A}_+$.

Note that the solution of this equation is the same as a fixed point of the map $W \mapsto \mathcal{F}_V(W)$, where

$$\mathcal{F}_V(W) := [V + \eta(W)]^{-1}. \quad (2.2)$$

Here is our main result.

Theorem 2.1. For fixed $V \in \mathcal{A}_+$, there exists exactly one solution $W \in \mathcal{A}_+$ to (2.1); this W is the limit of iterates

$$W_n = \mathcal{F}_V^n(W_0)$$

for any $W_0 \in \mathcal{A}_+$. Furthermore, we have that

$$\|W\| \leq \|(\operatorname{Re} V)^{-1}\|$$

and

$$\operatorname{Re} W \geq \frac{1}{m^2 \cdot \|(\operatorname{Re} V)^{-1}\|} I,$$

where

$$m := \|V\| + \sup\{\|\eta(W)\| \mid W \in \mathcal{A}_+, \|W\| \leq \|(\operatorname{Re} V)^{-1}\|\}. \quad \square$$

Remark 2.2. The uniqueness part of our theorem can be used to give an easy direct proof of Proposition 5.6 in [3]. There two operator-valued Cauchy transforms $G(\lambda)$ and $G_n(\lambda)$ are considered (where λ might also be a matrix, with positive imaginary part), and it is shown that both $G_n(\lambda)$ and $G(\Lambda_n(\lambda))$ fulfill the same equation, which is, after our rescaling, of the form (2.1). Since both $G(\Lambda_n(\lambda))$ and $G_n(\lambda)$ satisfy (as Cauchy transforms at some value of the argument) the right positivity condition, they both must agree with the unique solution, given by our theorem, and thus $G(\Lambda_n(\lambda)) = G_n(\lambda)$.

2.2 Related Topics

Suppose η maps selfadjoint elements to selfadjoint elements; then it maps positive elements \mathcal{SA}_+ of \mathcal{A} to \mathcal{SA}_+ . Suppose further that $V > 0$. Then, if η is linear, the map \mathcal{F}_V restricted to \mathcal{SA}_+ is a monotone decreasing map, that is, if $W_1 \geq W_2 \geq 0$, then $\mathcal{F}_V(W_1) \leq \mathcal{F}_V(W_2)$. There are results in [8] and [4] yielding fixed points in this case with proofs quite different than here. A list of applications and existing results on special cases is in these papers.

3 Contraction maps and proofs

We will prove our theorem by applying Banach's fixed point theorem to the map \mathcal{F}_V . One should note that \mathcal{F}_V is usually not a contraction in the given operator norm on \mathcal{A}_+ . Analyticity of our mapping, however, provides us with another metric in which we have the contraction property.

On \mathcal{A}_+ , since it is a domain in a Banach space, is the well-known Carathéodory metric, one of the biholomorphically invariant metrics on \mathcal{A}_+ . See for [2] for an excellent exposition of this and material germane to our treatment here. The crucial point is that strict holomorphic mappings on such domains are automatically strict contractions in this metric, and thus Banach's fixed point theorem guarantees a unique fixed point of such mappings. For the reader's convenience we recall here the relevant theorem, due to Earle and Hamilton [1].

Theorem 3.1. (Earle-Hamilton) Let \mathcal{D} be a nonempty domain in a complex Banach space X and let $h : \mathcal{D} \rightarrow \mathcal{D}$ be a bounded holomorphic function. If $h(\mathcal{D})$ lies strictly inside \mathcal{D} (i.e. there is some $\epsilon > 0$ such that $B_\epsilon(h(x)) \subset \mathcal{D}$, whenever $x \in \mathcal{D}$, where $B_\epsilon(y)$ is the ball of radius ϵ about y), then h is a strict contraction in the Carathéodory-Riffen-Finsler metric ρ , and thus has a unique fixed point in \mathcal{D} . Furthermore, one has for all $x, y \in \mathcal{D}$ that $\rho(x, y) \geq m\|x - y\|$ for some constant $m > 0$, and thus $(h^n(x_0))_{n \in \mathbb{N}}$ converges in norm, for any $x_0 \in \mathcal{D}$, to this fixed point. \square

We will now apply this to our situation. The main point will be to check that \mathcal{F}_V is well defined and maps suitably chosen subsets $R_b \subset \mathcal{A}_+$ strictly into itself. (Note that we do not claim that \mathcal{F}_V is a strict contraction on \mathcal{A}_+ itself.)

For $b > 0$, let us define

$$R_b := \{W \in \mathcal{A}_+ \mid \|W\| < b\} \subset \mathcal{A}_+$$

Proposition 3.2.

- (1) Any fixed point W of \mathcal{F}_V satisfies Equation (2.1).
- (2) If $\operatorname{Re} \eta(W) \geq 0$ and $V \in \mathcal{A}_+$, then $V + \eta(W)$ is invertible in \mathcal{A} , thus $\mathcal{F}_V(W) \in \mathcal{A}$ is well-defined, and $\operatorname{Re} \mathcal{F}_V(W) \geq 0$. Furthermore, we have

$$\|\mathcal{F}_V(W)\| \leq \|(\operatorname{Re} V)^{-1}\|.$$

- (3) For $V \in \mathcal{A}_+$ and $b > \|(\operatorname{Re} V)^{-1}\| > 0$, the map \mathcal{F}_V is a bounded holomorphic map on R_b and takes R_b strictly into its interior. We have for $W \in R_b$ that

$$\operatorname{Re} \mathcal{F}_V(W) \geq \frac{1}{m_b^2 \cdot \|(\operatorname{Re} V)^{-1}\|} I$$

with

$$m_b := \|V\| + \sup\{\|\eta(W)\| \mid W \in \mathcal{A}_+, \|W\| \leq b\}.$$

- (4) \mathcal{F}_V maps, for any $V \in \mathcal{A}_+$, the strict upper half plane of \mathcal{A} into itself,

$$\mathcal{F}_V : \mathcal{A}_+ \rightarrow \mathcal{A}_+.$$

It is a bounded holomorphic map there. \square

Proof. (1) This is obvious.

(2) Let A and B be the real and imaginary part of $\eta(W)$, respectively, i.e., $\eta(W) = A + iB$ with $A = A^*$ and $B = B^*$. By our assumption, $A \geq 0$. Put $K := V + \eta(W)$. Then we have:

$$\operatorname{Re} K = \operatorname{Re} V + A \geq \operatorname{Re} V.$$

Since $V \in \mathcal{A}_+$, the real part of K is bounded away from zero by a multiple of identity, and thus K is invertible as a bounded operator. Furthermore, the norm of the inverse $K^{-1} = \mathcal{F}_V(W)$ can be bounded by $\|(\operatorname{Re} V)^{-1}\|$. (For more details about these statements, see Lemma 3.1 in [3].)

To see the positivity of the real part of the inverse, we calculate:

$$\begin{aligned} 2 \cdot \operatorname{Re} \mathcal{F}_V(W) &= [V + A + iB]^{-1} + [V + A + iB]^{*-1} \\ &= [V + A + iB]^{-1} (2A + V + V^*) [V^* + A - iB]^{-1} \\ &= 2\mathcal{F}_V(W) (A + \operatorname{Re} V) \mathcal{F}_V(W)^* \\ &\geq 2\mathcal{F}_V(W) \cdot \operatorname{Re} V \cdot \mathcal{F}_V(W)^* \\ &\geq 0. \end{aligned}$$

(3) The map $W \mapsto K(W) := V + \eta(W)$ is analytic by the analyticity of η , and $\mathcal{F}_V(W) = K(W)^{-1}$ exists – and is thus also analytic – for $W \in \mathcal{A}_+$, by (2) (note that the

fact that η preserves the positivity of the real part implies that the assumptions of (2) are satisfied). Furthermore, by the norm estimate from (2), we have for all $W \in R_b$ that

$$\|\mathcal{F}_V(W)\| \leq \|(\operatorname{Re} V)^{-1}\| < b. \quad (3.1)$$

Thus \mathcal{F}_V is a bounded holomorphic map on R_b , with image in R_b . To see that the image lies strictly in R_b , we have to see that $\mathcal{F}_V(W)$ stays away from the boundary of R_b by some ε -amount. By (3.1) we stay away from the boundary $\|W\| = b$ by at least $(b - \|(\operatorname{Re} V)^{-1}\|)$. It remains to consider the part of the boundary described by $\operatorname{Re} W = 0$. By refining the last inequality of the calculation from (2) in our present setting we have

$$\begin{aligned} \operatorname{Re} \mathcal{F}_V(W) &\geq \mathcal{F}_V(W) \cdot \operatorname{Re} V \cdot \mathcal{F}_V(W)^* = [K(W)^* \cdot (\operatorname{Re} V)^{-1} \cdot K(W)]^{-1} \\ &\geq \frac{1}{\|K(W)^* \cdot (\operatorname{Re} V)^{-1} \cdot K(W)\|} I \geq \frac{1}{m_b^2 \cdot \|(\operatorname{Re} V)^{-1}\|} I \end{aligned}$$

since $\|K(W)\| \leq m_b$ for all $W \in R_b$. Thus we stay away from the boundary $\operatorname{Re} W = 0$ by at least $1/(m_b^2 \cdot \|(\operatorname{Re} V)^{-1}\|)$.

(4) This follows from the fact that

$$\mathcal{A}_+ = \bigcup_{b>0} R_b;$$

note that the estimate $\|\mathcal{F}_V(W)\| \leq \|(\operatorname{Re} V)^{-1}\|$ does not depend on b . ■

Proof of Theorem 2.1. By the Earle-Hamilton Theorem, each R_b with $b > \|(\operatorname{Re} V)^{-1}\|$ contains exactly one fixed point of (2.1). The estimates for the norm and the real part of W follow from the corresponding estimates in parts (2) and (3) of Proposition 3.2. ■

Remark 3.3. (1) Since the application of Stieltjes inversion formula asks for z very close to the real axis, one might be tempted to try to solve (1.2) directly for real z . Of course, most of the above statements then break down. In particular, one should consider the map \mathcal{F}_V for V with $\operatorname{Re} V \geq 0$ on the domain $\operatorname{Re} W > 0$ instead of \mathcal{A}_+ . In the infinite dimensional case, those two notions are not the same, and using $\operatorname{Re} W > 0$ presents problems. (\mathcal{F}_V is not even well defined there, in general, for $\operatorname{Re} V \geq 0$.) In the finite dimensional case, however, $\operatorname{Re} W > 0$ just says that all eigenvalues of $\operatorname{Re} W$ are positive and different from zero, thus $\operatorname{Re} W$ is positive and invertible, thus $W \in \mathcal{A}_+$ and one can extend some of the above reasoning to the set $\{W \in \mathcal{A} \mid \operatorname{Re} W > 0\}$. This domain is mapped, under \mathcal{F}_V into itself, but now, of course, not necessarily strictly. This implies

(see Proposition 6 in [5]) that, we still get a contraction in the Caratheodory metric, but the contraction constant is not necessarily less than 1. We now state this with formulas. Let \mathcal{A} be finite dimensional (i.e., some matrix algebra). Let ρ denote the Caratheodory metric on the set $\{W \in \mathcal{A} \mid \operatorname{Re} W > 0\}$. Then for $\operatorname{Re} V \geq 0$, there is $c_V \leq 1$ such that

$$\rho(\mathcal{F}_V(W), \mathcal{F}_V(\widetilde{W})) \leq c_V \rho(W, \widetilde{W})$$

if $\operatorname{Re} W > 0$ and $\operatorname{Re} \widetilde{W} > 0$.

(2) One can generalize our considerations from (1.2) to the equation

$$G(z) = G_1[z - \eta(G(z))]. \quad (3.2)$$

The latter describes [11, 12] the operator-valued Cauchy transform of the sum $x + s$ where $s \in \mathcal{B}$ is an operator-valued semicircular element as before (with covariance function η) and $x = x^* \in \mathcal{B}$ is an element which is free from s with respect to the conditional expectation $E : \mathcal{B} \rightarrow \mathcal{A}$;

$$G_1(z) := E \left[\frac{1}{z - x} \right]$$

is the given operator-valued Cauchy transform of x . Note that for $x = 0$ we have $G_1(z) = 1/z$ and (3.2) reduces to the fixed point version of (1.2). In the scalar-valued case $\mathcal{A} = \mathbb{C}$, equation (3.2) was derived by Pastur [7], describing a 'deformed semicircle'.

The same arguments as before show that for any fixed z with positive imaginary part there exists exactly one solution of (3.2) whose imaginary part is strictly negative. This unique solution can be obtained as the limit of iterates of the mapping $G \mapsto G_1[z - \eta(G)]$, for any initial G with strictly negative imaginary part.

4 Numerical considerations

4.1 Iteration and averaging

Let us come back to our motivating Equation (1.2) for the operator-valued Cauchy transform $G(z)$ of an operator-valued semicircular element. According to Theorem 2.1, we can get the wanted solution $G = -iW$ of Eqn (1.2) by iterating the mapping $W \mapsto \mathcal{F}_z(W) := [-izI + \eta(W)]^{-1}$, starting with any $W_0 \in \mathcal{A}_+$. Even though this seems to completely solve our problem, it turns out that numerically this might not work

too well. Namely, since we want to invoke the Stieltjes inversion formula to recover the wanted eigenvalue distribution from $G(z)$, we need z very close to the real axis. Then \mathcal{F}_z is still a contraction (in the Caratheodory metric), the contraction constant, however, might be very close to 1, and the convergence could be extremely slow. In typical numerical examples, there was, for z very close to the real axis, no numerically observable convergence to a fixed point at all. (In some cases it looked as if \mathcal{F}_z would have a limiting 2-cycle.) To improve on this, we invoked also an averaging procedure along with our iteration. In the simplest case, we replaced the iteration algorithm

$$W \mapsto \mathcal{F}_z(W)$$

by

$$W \mapsto \mathcal{G}_z(W) := \frac{1}{2}W + \frac{1}{2}\mathcal{F}_z(W).$$

Note that \mathcal{G}_z has essentially the same properties as \mathcal{F}_z ; in particular, the fixed points are the same, and \mathcal{G}_z is a bounded holomorphic map on R_b and takes R_b strictly into its interior, as follows from the following lemma.

Proposition 4.1. Let \mathcal{D} be a convex and bounded domain in a Banach space X and $\mathcal{F} : \mathcal{D} \rightarrow \mathcal{D}$ a bounded holomorphic function. Assume that $\mathcal{F}(\mathcal{D})$ lies strictly inside \mathcal{D} . Put

$$\mathcal{G}(W) := \frac{1}{2}W + \frac{1}{2}\mathcal{F}(W).$$

Then $\mathcal{G} : \mathcal{D} \rightarrow \mathcal{D}$ is bounded and holomorphic and maps \mathcal{D} strictly into itself. Thus \mathcal{G} iterations applied to any W^0 inside \mathcal{D} converge to a fixed point for both \mathcal{G} and \mathcal{F} . \square

Proof. Only the fact that $\mathcal{G}(\mathcal{D})$ lies strictly inside \mathcal{D} is not directly clear. Let $\varepsilon > 0$ be such that $B_\varepsilon(\mathcal{F}(W)) \subset \mathcal{D}$ for all $W \in \mathcal{D}$. We claim that $B_{\varepsilon/2}(\mathcal{G}(W)) \subset \mathcal{D}$ for all $W \in \mathcal{D}$. To see this assume there is a $W \in \mathcal{D}$ and an $r \in X$ with $\|r\| \leq \varepsilon/2$ such that $\mathcal{G}(W) + r \notin \mathcal{D}$. But we know that $\mathcal{F}(W) + 2r \in \mathcal{D}$ and thus, since \mathcal{D} is convex, also

$$\mathcal{G}(W) + r = \frac{1}{2}(\mathcal{F}(W) + 2r) + \frac{1}{2}W \in \mathcal{D}.$$

The convergence of \mathcal{G} iterates follows immediately from the Earle-Hamilton Theorem. \blacksquare

As stated, iterations of \mathcal{G}_z must converge to the same fixed point as iterations of \mathcal{F}_z . Numerically, in all our considered reasonable examples, the speed of convergence for the averaged iteration was substantially faster than for the plain iterations and allowed a fast numerical determination of the desired fixed point.

Let us, however, point out that there is no theoretical reason that the averaged iteration has a better contraction constant than the plain iteration (actually it might even be worse), and that one can produce artificial examples where the averaging does not improve the plain iteration. For example, in the case of 2×2 matrices where

$$\eta(W) = AWA, \quad \text{with} \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and for initial condition

$$W_0 = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix},$$

there is no clear improvement of the averaged iteration over the plain iteration for z close to 0.

Of course, the usual algorithm of choice for solving equations like (1.2) or (2.1) would be a kind of Newton's method applied at later iterations. This has much faster convergence properties (second-order local convergence compared to first-order convergence rate of our iteration). However, Newton's method does in general not respect the positivity requirement for our solution. So even if we start in \mathcal{A}_+ , it might happen that the Newton's method leads out of \mathcal{A}_+ and converges to another root which is not in \mathcal{A}_+ . One might expect that iterating (or iterating and averaging) for a while will get in the basin of attraction of the correct point and then switching to the Newton's method should be fairly safe and fast.

Whereas we do not have any theoretical results to justify averaging or a modification of the Newton's method in all cases, we want to point out that the uniqueness part in our theorem allows us to check in a concrete example whether our algorithm of choice works or not. We only have to check whether the solution produced by our algorithm is in \mathcal{A}_+ .

4.2 Numerical examples

We consider the block Toeplitz random matrix

$$X = \frac{1}{\sqrt{3N}} \begin{bmatrix} A & B & C \\ B & A & B \\ C & B & A \end{bmatrix}, \quad (4.1)$$

where A, B, C are independent selfadjoint $N \times N$ matrices with i.i.d. entries of unit variance. In [9] it was shown that, in the limit $N \rightarrow \infty$, the Cauchy transform H of the eigenvalue distribution of X is given by $H(z) = \text{tr}_3(G(z))$ where tr_3 denotes the normalized trace on 3×3 matrices and where the operator-valued Cauchy transform $G(z)$ is a 3×3 matrix of the form

$$G = \begin{bmatrix} f & 0 & h \\ 0 & g & 0 \\ h & 0 & f \end{bmatrix}. \quad (4.2)$$

G satisfies Equation (1.2), where η acts as follows:

$$\eta(G) = \frac{1}{3} \begin{bmatrix} 2f+g & 0 & g+2h \\ 0 & 2f+g+2h & 0 \\ g+2h & 0 & 2f+g \end{bmatrix}. \quad (4.3)$$

In order to get the eigenvalue distribution of X we have to solve Equation (1.2) for z running along the real axis. We solve Equation (1.2) either by our modified iteration procedure or by Newton's algorithm. In both cases we start from the center where z_0 is an imaginary number close to the origin ($z_0 = 10^{-9} \cdot i$ for the following results) and we use the solution at each $z_n = z_0 + d \cdot \Delta \cdot n$ as the initial point for the next $z_{n+1} = z_0 + d \cdot \Delta \cdot (n+1)$ where $\Delta \in \mathbb{R}$ is the resolution step size in calculating the spectrum and $d = \mp 1$ depending on negative or positive side of the spectrum. In other words, we calculate the spectrum in two phases both starting from the center, one for the positive side of the spectrum ($d = 1$), and then for the negative side ($d = -1$).

4.2.1 Solution by iteration. With the convention $W(z) = iG(z)$, we use

$$W_0(z) = \begin{bmatrix} 1 - 0.1 \cdot i & 0 & 1 \\ 0 & 1 - 0.1 \cdot i & 0 \\ 1 & 0 & 1 - 0.1 \cdot i \end{bmatrix} \cdot i$$

as the initializing matrix at center ($z = 10^{-9} \cdot i$) for the iteration method. Figure 1 depicts the calculated spectrum, which matches completely with the true spectrum [9].

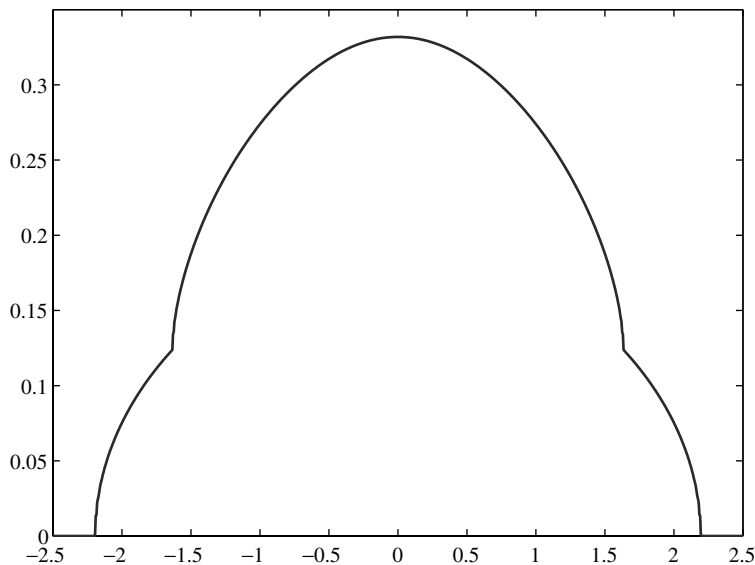


Figure 1 Iteration method calculation result for the asymptotic eigenvalue distribution of the 3×3 block Toeplitz matrix as in (4.1).

4.2.2 Failure of Newton's method. Replacing Equations (4.2) and (4.3) in Equation (1.2), one reaches to the following system of equations:

$$zf = 1 + \frac{g(f+h) + 2(f^2 + h^2)}{3}, \quad (4.4)$$

$$zg = 1 + \frac{g(g + 2(f+h))}{3}, \quad (4.5)$$

$$zh = \frac{4fh + g(f+h)}{3}. \quad (4.6)$$

We used Newton's method to solve this system numerically, starting with the initial values

$$\begin{bmatrix} f_0 \\ g_0 \\ h_0 \end{bmatrix} = \begin{bmatrix} 1 - 0.1 \cdot i \\ 1 - 0.1 \cdot i \\ 1 \end{bmatrix},$$

for $z = 10^{-9}i$ on the imaginary axis and then using the fixed point for one z_n as the initial point for the next z_{n+1} . The result is shown in Figure 2. Clearly this is not the desired result and the algorithm has converged to some different roots, which in this case do not yield a distribution function (though it is positive everywhere). In other words, using

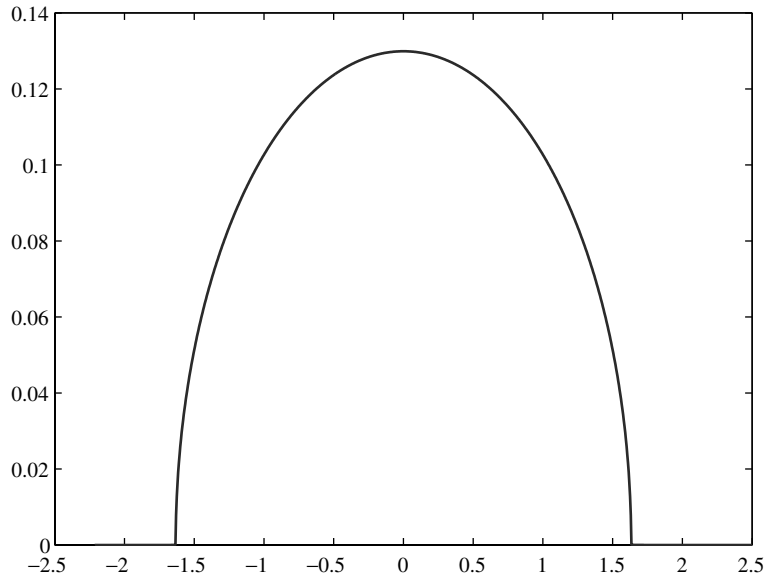


Figure 2 Newton's method calculation result for the eigenvalue distribution of the 3×3 block Toeplitz matrix as in (4.1).

Newton's method, one has to check the positivity of the final result to make sure that it has converged to the desired solution. Of course, in our example the 3×3 matrix G produced by Newton's algorithm is not positive.

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