

## MULTIPLE QUANTUM WALKERS ON THE LINE USING HYBRID COINS: A POSSIBLE TOOL FOR QUANTUM SEARCH

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**Abstract.** *In this paper discrete quantum walks with different coins used for odd and even time steps are studied. These coins are called hybrid. The calculation results are compared with the most frequently used coin, the Hadamard transform. Furthermore, quantum walks on the line which involve two or more quantum walkers with hybrid coins are studied. Quantum walks with entangled walkers and hybrid coins are also studied. The results of these calculations show that the proposed types of quantum walks can be used for quantum search, because the walker can be directed towards preferred directions and can also be confined in certain segments of the line.*

**Key words:** *Quantum walk, quantum computing, simulation max*

### 1. INTRODUCTION

Quantum walks are quantum versions of classical random walks. They were first introduced in 1993 [1] and since then considerable work has been done on this subject. Quantum walks are useful models for physical processes such as Brownian motion and may serve as a basis for the development of new quantum algorithms [2], [3]. Furthermore, quantum walk is a natural model for quantum search using parallel quantum computer architectures [4], [5]. Quantum walks may also become an effective tool for studying biological systems [6]. Several studies of continuous-time and discrete-time quantum walks on the line [7], [8], and some implementation proposals have been published [9]-[11]. The effect of noise on the discrete-time quantum walk has also been studied [12].

Recently, a study of a quantum walk on the line with one walker and several coins has been published [13]. On the other hand, in [14] a quantum walk on the line with two entangled walkers and one coin, the Hadamard transform, is studied. In this paper a study of discrete quantum walks in which multiple walkers use hybrid coins is presented. "Hybrid coin" means the use of two different coins, one for odd and one for even time steps. The question to be answered by this study is: Is it possible to use multiple quantum

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walkers and hybrid coins to direct the search towards a desired direction and, furthermore, is it possible to confine the search in a desirable segment of the line? Calculation results show that this is possible.

## 2. QUANTUM WALK ON THE LINE WITH HYBRID COINS

In discrete quantum walk a walker (which can be a particle or a state) moves on a one-dimensional periodic lattice. The sites of this lattice are numbered by:

$$i = 0, \pm 1, \pm 2, \dots \pm n \quad (1)$$

The Hilbert space of the discrete quantum walk comprises two subspaces, the location subspace  $H_L$ , which is spanned by the basis:

$$|i\rangle = |-n\rangle, \dots |-2\rangle, |-1\rangle, |0\rangle, |1\rangle, |2\rangle, \dots |n\rangle \quad (2)$$

and the two-dimensional coin subspace,  $H_C$ , which is spanned by the two coin basis states  $|0\rangle$  and  $|1\rangle$ . The Hilbert space,  $H$ , of the quantum walk is:

$$H = H_L \otimes H_C \quad (3)$$

The state of the quantum walker found at location  $j$  with coin in state  $|0\rangle$  is  $|j, 0\rangle$ .

The quantum walk usually starts with the walker in state  $|0, 0\rangle$ . At each step of the walk two operations are applied to the walker state. The coin toss operation,  $C$ , which acts on the coin state, is applied first:

$$C |j, 0\rangle = c_{0,0} |j, 0\rangle + c_{0,1} |j, 1\rangle \quad (4)$$

$$C |j, 1\rangle = c_{1,0} |j, 0\rangle + c_{1,1} |j, 1\rangle$$

Any two-dimensional unitary transformation can be used as a coin toss operation. Usually, the Hadamard transform,  $H$ , is used. In this case:

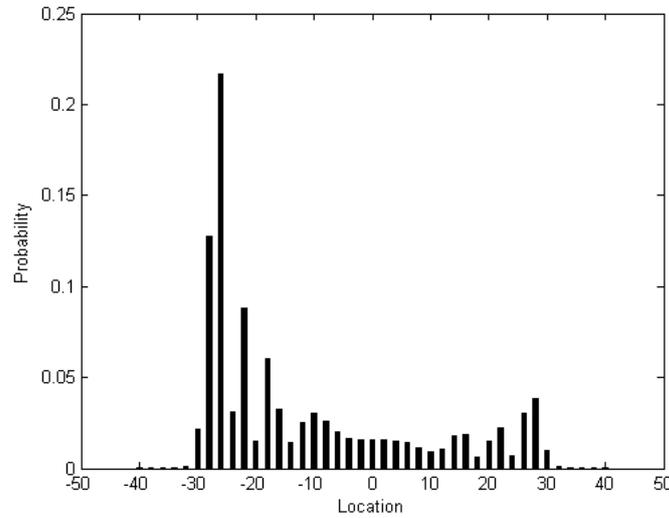
$$H = \begin{bmatrix} c_{0,0} & c_{0,1} \\ c_{1,0} & c_{1,1} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (5)$$

The second operation applied is the walker shift operation,  $S$ , which acts on the location state and is given by:

$$S = \sum_{j=-n}^n |1\rangle\langle 1| \otimes |j+1\rangle\langle j| + |0\rangle\langle 0| \otimes |j-1\rangle\langle j| \quad (6)$$

This operation shifts the walker to the right (towards  $+n$ ) if the coin state is  $|1\rangle$  and to the left if the coin state is  $|0\rangle$ . The probability distribution for a discrete quantum walk with initial walker state  $|0, 0\rangle$ , in which the Hadamard transform is used as coin, is shown in

Figure 1. The probability distribution is biased towards left because of quantum interference. The initial walker state  $|0, 1\rangle$  results in probability distribution biased towards right.



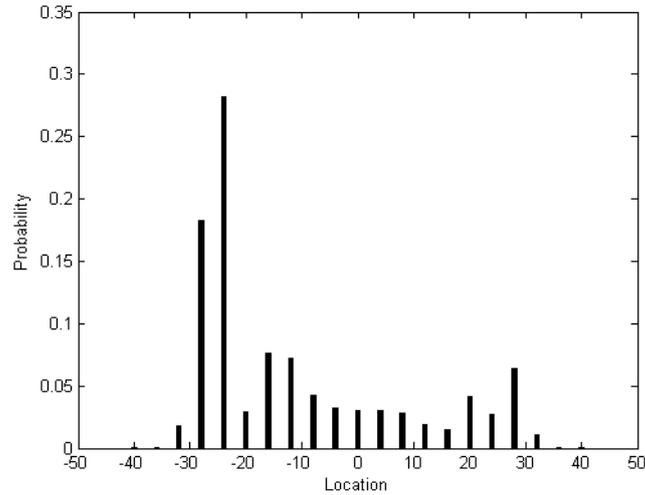
**Fig. 1** Probability distribution for a quantum walk after 40 steps. The initial state is  $|0, 0\rangle$  and the Hadamard transform is used for coin toss

The dependence of the probability distribution on the coin initial state leads naturally to the question: What is the probability distribution in the case where two coins are used alternatively?

The case where the coin used in odd steps is the Hadamard transform and the coin used in even time steps is a phase shift,  $P$ , is considered first. The phase shift is given by:

$$P = \begin{bmatrix} 1 & 1 \\ 1 & e^{i\phi} \end{bmatrix} \tag{7}$$

Figure 2 shows the probability distribution in this case. The initial walker state is  $|0, 0\rangle$  and the phase angle,  $\phi$ , is  $60^\circ$ . The probability distribution is biased towards left as in the case where only the Hadamard transform was used, but the probability to find the walker in certain locations, which are periodically distributed, is larger. On the other hand, there is a zero probability to find the walker in much more locations than the case where only the Hadamard transform was used. This is an expected result for all phase angles, in the case of a single walker. Phase shift is important in the case of multiple walkers, because it affects quantum interference.

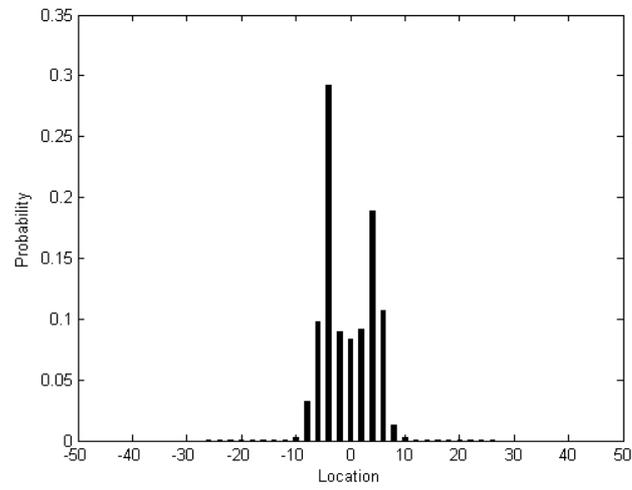


**Fig. 2** Probability distribution for a quantum walk after 40 steps in the case where two coins are used alternatively, namely  $H$  and  $P$ . The initial walker state is  $|0, 0\rangle$

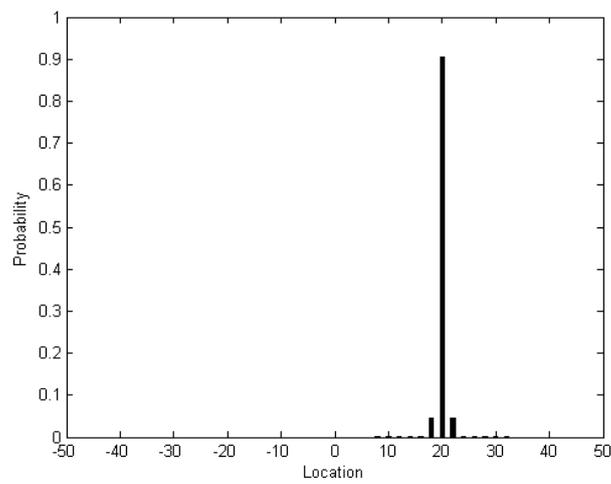
The case where the coin used in odd steps is the Hadamard transform and the coin used in even time steps is a more general transform,  $G$ , is considered next. The transform  $G$  is given by:

$$G = \begin{bmatrix} \cos(\varphi) & \sin(\varphi) \\ \sin(\varphi) & -\cos(\varphi) \end{bmatrix} \quad (8)$$

Figure 3(a) shows the probability distribution after 40 steps in this case where  $\varphi = 30^\circ$ . The initial walker state is  $|0, 0\rangle$ . The walker is localized between in the region  $[-10, +10]$ . The order of magnitude of the probability to find the walker in locations outside this region is  $10^{-3}$ . It is therefore acceptable to say that using the aforementioned hybrid coin we can confine the walker within a certain segment of the line. This walker can be confined in any segment of the line  $[x-10, x+10]$  by setting the initial walker state to  $|x, 0\rangle$ . Different values of  $\varphi$  result in walker confinement in regions with different sizes. For example, Figure 3(b) shows the probability distribution after 40 steps in this case where  $\varphi = 55^\circ$  and initial walker state  $|20, 0\rangle$ . In this case the walker is confined in the region  $[18, 22]$  or  $[20-2, 20+2]$ , that is  $\pm 2$  around its initial location.



(a)



(b)

**Fig. 3** Probability distribution for a quantum walk after 40 steps in the case where the coins H and P are used alternatively. (a) Initial walker state  $|0, 0\rangle$  and  $\varphi=30^\circ$ . (b) Initial walker state  $|20, 0\rangle$  and  $\varphi=55^\circ$ .

## 3. MULTIPLE WALKERS ON THE LINE

More than one walker can be used in order to exploit quantum interference. A number of  $w$  walkers can be used. These walkers can be distinct particles each one with a different initial state. In this case the initial state of the quantum walk,  $|w_{in}\rangle$ , is given by:

$$|w_{in}\rangle = a_1 |w_1, c_1\rangle + a_2 |w_2, c_2\rangle + a_3 |w_3, c_3\rangle + \dots + a_w |w_w, c_w\rangle \quad (9)$$

with

$$|a_1|^2 = |a_2|^2 = |a_3|^2 = \dots = |a_w|^2 = 1 \quad (10)$$

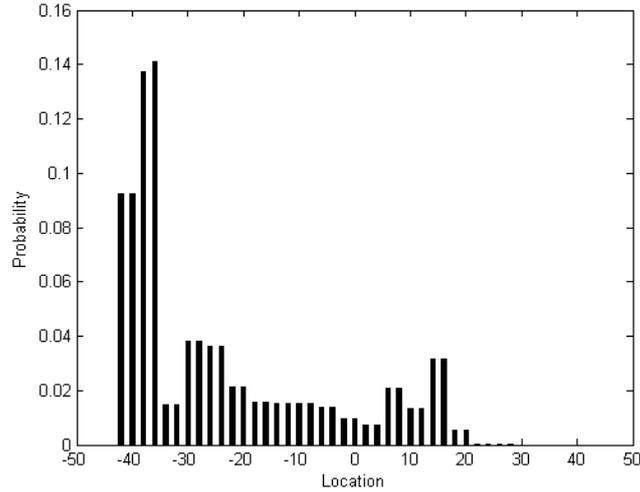
Multiple walkers can also be states of the same particle located initially at different locations. In this case:

$$|a_1|^2 + |a_2|^2 + |a_3|^2 + \dots + |a_w|^2 = 1 \quad (11)$$

Figure 4 shows the probability distribution after 40 steps of quantum walk. A hybrid coin  $H$  and  $P$  with  $\phi = 40^\circ$  is used. The initial state for this walk is:

$$|w_{in}\rangle = \frac{1}{\sqrt{2}} |-14, 0\rangle + \frac{1}{\sqrt{2}} |-12, 0\rangle \quad (12)$$

The calculation results show that the walk is directed towards left.

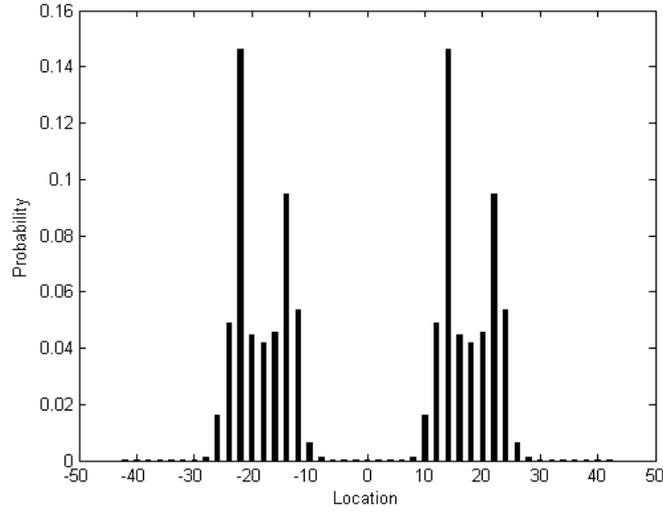


**Fig. 4** Probability distribution after 40 steps of quantum walk with a hybrid coin  $H$  and  $P$  with  $\phi = 40^\circ$ . The initial state for this walk is given by equation (12).

Figure 5 shows the probability distribution after 40 steps of quantum walk in which a hybrid coin  $H$  and  $G$  with  $\varphi = 30^\circ$  is used. The initial state for this walk is:

$$|w_{in}\rangle = \frac{1}{\sqrt{2}} | -18, 0 \rangle + \frac{1}{\sqrt{2}} | 18, 1 \rangle \quad (13)$$

The walkers are confined into two segments of the line. From Figure 5 it is evident that the two probability patterns are located symmetrically to the left and to the right of the origin and are exactly the same. This walk results in two probability patterns which are the same and are displaced by 36 line sites.



**Fig. 5** Probability distribution after 40 steps of quantum walk with a hybrid coin  $H$  and  $G$  with  $\varphi = 30^\circ$ . The initial state for this walk is given by equation (13).

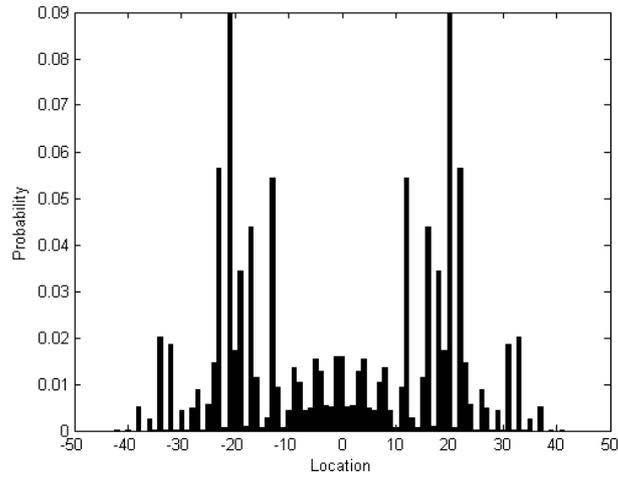
A quantum walk with two entangled walkers will be considered next. The initial state of the walk is:

$$|w_{in}\rangle = \frac{1}{\sqrt{2}} | -6, 0 \rangle + \frac{1}{\sqrt{2}} | 5, 1 \rangle \quad (14)$$

A hybrid coin  $H$  and  $G$  with  $\varphi = 50^\circ$  is used. The calculation results are shown in Figure 6. The probability distribution pattern has mirror symmetry with respect to the origin.

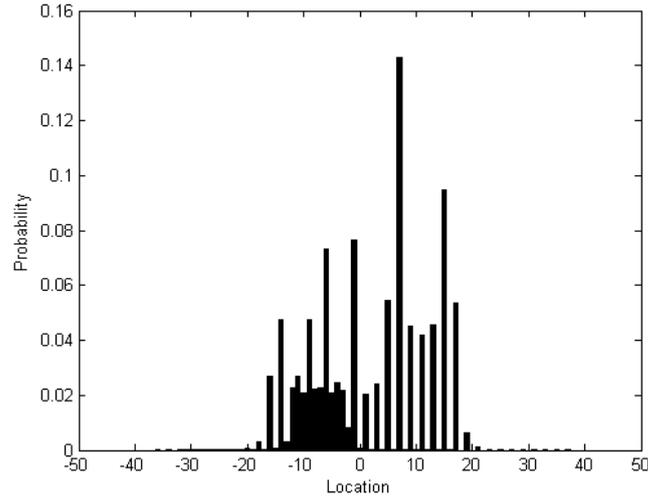
More than two walkers can be used. Let us examine the case of a quantum walk with three walkers in which a hybrid coin  $H$  and  $G$  with  $\varphi = 30^\circ$  is used. The initial state is:

$$|w_{in}\rangle = \frac{1}{2} | -10, 1 \rangle + \frac{1}{2} | -5, 1 \rangle + \frac{1}{\sqrt{2}} | 11, 0 \rangle \quad (15)$$



**Fig. 6** Probability distribution after 40 steps of quantum walk with a hybrid coin  $H$  and  $G$  with  $\varphi = 50^\circ$ . The initial state for this walk is given by equation (14)

The calculation results shown in Figure 7 indicate that the walkers are confined within the region  $[-20, 20]$ . There is a non-zero probability to locate a walker in every location within the region  $[-15, 1]$ .

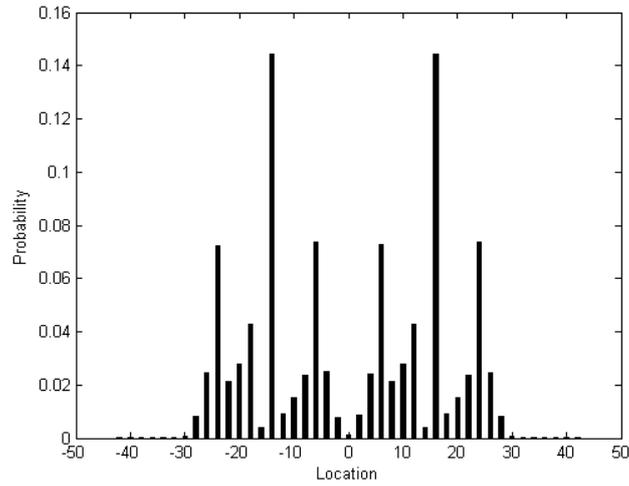


**Fig. 7** Probability distribution after 40 steps of quantum walk with a hybrid coin  $H$  and  $G$  with  $\varphi = 30^\circ$ . The initial state for this walk is given by equation (15)

The periodic structure of probability distribution shown in Figure 8 was obtained using four walkers with hybrid coin  $H$  and  $G$  ( $\varphi = 30^\circ$ ). The initial state was:

$$|w_{in}\rangle = \frac{1}{2} |-20, 0\rangle + \frac{1}{2} |-10, 1\rangle + \frac{1}{2} |10, 0\rangle + \frac{1}{2} |20, 1\rangle \quad (16)$$

A periodic probability distribution with more periods can be obtained using more walkers. Figures 5 and 6 indicate that two periods correspond to two walkers.



**Fig. 8** Probability distribution after 40 steps of quantum walk with a hybrid coin  $H$  and  $G$  with  $\varphi = 30^\circ$ . The initial state for this walk is given by equation (16)

#### 4. CONCLUSIONS

In this paper the study of discrete quantum walks involving multiple walkers and hybrid coins was presented. An analytical study of these quantum walks is probably impossible because the probability distribution patterns depend on the choice of hybrid coins, the number of the walkers and the initial states. A large variety of patterns can be achieved including walker confinement, walker direction and periodic patterns. The results presented here indicate that quantum walk on the line with multiple walkers using hybrid coins is an effective tool for quantum search.

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