Optimal extended Jacobian inverse kinematics algorithms for robotic manipulators

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Abstract—Extended Jacobian inverse kinematics algorithms for redundant robotic manipulators are defined by combining the manipulator’s kinematics with an augmenting kinematics map in such a way that the combination becomes a local diffeomorphism of the augmented taskspace. A specific choice of the augmentation relies on the optimal approximation by the extended Jacobian of the Jacobian pseudoinverse (the Moore-Penrose inverse of the Jacobian). In this paper we propose a novel formulation of the approximation problem, rooted conceptually in the Riemannian geometry. The resulting optimality conditions assume the form of a Poisson equation involving the Laplace-Beltrami operator. Two computational examples illustrate the theory.

Index terms: Robotic manipulator, inverse kinematics, extended Jacobian, Jacobian pseudoinverse, approximation

I. INTRODUCTION

The inverse kinematics problem for stationary or mobile manipulators consists in computing a manipulator’s configuration corresponding to a prescribed location of the end effector in the taskspace. Usually, this problem is solved numerically by means of Jacobian inverse kinematics algorithms, of which the most widely exploited is the Jacobian pseudoinverse algorithm. An alternative to the pseudoinverse is the extended Jacobian algorithm. The Jacobian pseudoinverse algorithm distinguishes by its speed of convergence, while the extended Jacobian algorithm has the desirable property of repeatability. Repeatable inverse kinematics algorithms transform closed paths in the taskspace into closed paths in the configuration space. The repeatability of stationary manipulators is a traditional subject of robotics [1], [2], [3]. Conditions for repeatability of inverse kinematics algorithms for mobile manipulators were given in [4]. The concept of the extended Jacobian inverse appeared in [5], [6]. Its extension to mobile manipulators can be found in [7], [8].

In this paper we shall concentrate on inverse kinematics algorithms for stationary robotic manipulators. The idea of the optimal approximation of the Jacobian pseudoinverse inverse kinematics algorithm by a repeatable algorithm has been proclaimed and developed in a series of papers by Roberts and Maciejewski [9], [10], [11], [12]. One of the main threads of these papers can be reconstructed in the following way. Suppose that a coordinate representation of the manipulator’s kinematics takes the form of a map

\[ k : \mathbb{R}^n \to \mathbb{R}^m, \quad y = k(x), \]

from the jointspace into the taskspace. We assume that \( n > m \), and let \( J(x) = \frac{\partial k}{\partial x}(x) \) stand for the manipulator’s Jacobian. Then, outside singular configurations of the manipulator, the Jacobian pseudoinverse \( J^p(x) = J^T(x)M^{-1}(x) \), where \( M(x) = J(x)J^T(x) \) denotes the manipulability matrix [13]. It is clear that \( J(x)J^p(x) = I_m \). Now, setting \( s = n - m \), we introduce an augmenting kinematics map

\[ h : \mathbb{R}^s \to \mathbb{R}^s, \quad \tilde{y} = h(x), \]

and the extended kinematics

\[ l = (k, h) : \mathbb{R}^n \to \mathbb{R}^n, \quad \tilde{y} = (k(x), h(x)). \]

The extended Jacobian

\[ J(x) = \begin{bmatrix} \frac{\partial h(x)}{\partial x} \\ \frac{\partial k(x)}{\partial x} \end{bmatrix} \]

is represented by a square \( n \times n \) matrix. At regular points of \( J(x) \) we define an extended Jacobian inverse

\[ J^E(x) = J^{-1}(x)|_{first \ m \ columns}. \]

By definition, \( J^E(x) \) is a right inverse of the Jacobian that is annihilated by the differential of the augmenting map

\[ J(x)J^E(x) = I_m \quad \text{and} \quad \frac{\partial h}{\partial x}(J^E(x) = 0. \]

Given any right inverse \( J^h(x) \) of the Jacobian, and a desirable taskspace point \( y_d \in \mathbb{R}^s \), a solution of the inverse kinematics problem is obtained by taking a limit at \( t \to +\infty \) of the trajectory of the dynamic system

\[ \dot{x} = -\gamma J^h(x(t))(k(x(t)) - y_d). \]

The positive number \( \gamma \) defines the convergence rate of the algorithm. Given the inverses \( J^p(x) \) and \( J^E(x) \), the approximation problem studied by Roberts and Maciejewski amounts to inventing an augmenting map (2) that minimizes the approximation error

\[ E(h) = \int_{\mathcal{M}} ||J^p(x) - J^E(x)||_F^2 \, dx, \]

where \( ||M||_F = \sqrt{\text{tr}(M^TM)} \) denotes the Frobenius norm, and \( \mathcal{M} \subset \mathbb{R}^s \) is a singularity-free region of the jointspace. Using the fact that both \( J^p(x) \) and \( J^E(x) \) are right inverses of the Jacobian, we deduce that there exists an \( s \times n \) matrix \( W(x) \) such that the difference

\[ J^p(x) - J^E(x) = K(x)W(x), \]
where \( K(x) \) is a matrix with orthonormal columns, spanning the Jacobian kernel, \( J(x)K(x) = 0, K^T(x)K(x) = I_n \). After premultiplying the above expression by \( \frac{\partial h(x)}{\partial x} \), we compute

\[
W(x) = \left( \frac{\partial h(x)}{\partial x}K(x) \right)^{-1} \frac{\partial h(x)}{\partial x} J^{p\#}(x),
\]

and conclude that the error formula (7) becomes

\[
\mathcal{E}(h) = \int_M \text{tr} \left( \left( \frac{\partial h(x)}{\partial x}K(x) \right)^{-1} \frac{\partial h(x)}{\partial x} J^{p\#}(x) J^{p\#T}(x) \times \left( \frac{\partial h(x)}{\partial x}K(x) \right)^{-T} \right) dx.
\]

The optimality conditions for (8) lead to a system of nonlinear partial differential equations, and moreover, a computation of the integrand in (8) becomes ill conditioned close to singularities of the matrix \( \frac{\partial h(x)}{\partial x}K(x) \).

The objective of this paper is to re-define the approximation problem in a way resulting in the optimality conditions that have a sound geometric interpretation and are tractable computationally. Some basic analogies with Riemannian geometry are used as the guidelines. Specifically, the new definition exploits two extended Jacobians: one associated with the augmenting kinematics map, the other based on complementing the range space of the Jacobian with its null space. A measure of the distance between the extended Jacobian inverse and the Jacobian pseudo-inverse is induced by the distance between corresponding extensions. In this way, the error formula standing under the integral (7) gets embedded into the general linear group of matrices, and then integrated using the manipulability as the volume form. The main contribution of this paper consists in providing the optimality conditions in the form of a system of linear, elliptic partial differential equations involving the Laplace-Beltrami operator. The number of these equations equals the redundancy degree of the kinematics. All these equations include the same Laplace-Beltrami operator, and may differ only by the divergence term on the right hand side. This means that, in fact, we need to solve a single equation with different substitutions on the right hand side, and different boundary conditions. The benefits of this result are twofold: first, there exists an advanced theory of the Laplace-Beltrami operator [14], that may provide a theoretic insight into the approximation problem, second, numeric algorithms for solving linear, elliptic partial differential equations are offered by all commercial and scientific software packages dedicated to partial differential equations. Additionally, when the right hand side of the equation is equal to zero, the optimal augmenting kinematics map becomes a harmonic map [15]: the object that has already been recognized in robotics as a potential function in the motion planning problem [16] or as a dexterozous kinematics map minimizing the distortion [17]. If this is the case, all components of the augmenting map are obtained as a solution of the same partial differential equation.

The composition of this paper is the following. In section II we present our main result. Section III is devoted to computational examples. The paper is concluded with section IV.

A necessary background material from Riemannian geometry is sketched in the Appendix.
Alternatively, we may consider the right error
\[ E_h^R(x) = \text{tr}(B(x)A^{-1}(x) - I_n)^* (B(x)A^{-1}(x) - I_n). \]  
This time the matrix \( C(x) = B(x)A^{-1}(x) - I_n \) corresponds to a linear map \( C(x) : T_xX \rightarrow T_xX \), so the dual map \( C^*(x) = G^{-1}(x)CT(x)G(x) \). Using the identity \( G^{-1}(x) = P(x) \), we compute
\[ E_h^R(x) = \text{tr}(P(x)(J^h(x)P(x)) + K(x) \frac{\partial h(x)}{\partial x} - I_n)^T G(x) \times \\
(J^h(x)P(x)) + K(x) \frac{\partial h(x)}{\partial x} - I_n \),
and, after suitable developments, conclude that
\[ E_h^R(x) = \text{tr} \left( \frac{\partial h(x)}{\partial x} P(x) \left( \frac{\partial h(x)}{\partial x} \right)^T - 2 \frac{\partial h(x)}{\partial x} K(x) + I_n \right), \]
i.e. the left and the right errors coincide. Having established this equivalence, we shall define the problem of optimal approximation of the Jacobian pseudoinverse by an extended Jacobian inverse as the minimization of the integrated left error (13) over a subset \( \mathcal{M} \subset X \), using the volume form \( V = m(x)dx \),
\[ E(h) = \\
\int_{\mathcal{M}} \text{tr} \left( \frac{\partial h(x)}{\partial x} P(x) \left( \frac{\partial h(x)}{\partial x} \right)^T - 2 \frac{\partial h(x)}{\partial x} K(x) + I_n \right) m(x)dx. \]
The approximation error is a functional of the augmenting map. The Lagrangian appearing under the integral (16) is equal to
\[ L(x, h; \frac{\partial h}{\partial x}) = \sum_{i=1}^k \left( \frac{\partial h_i}{\partial x} \right)^T m(x)P(x) \frac{\partial h_i}{\partial x} - 2 \sum_{i=1}^k \frac{\partial h_i}{\partial x} m(x)K_i(x), \]
\( K_i(x) \) denoting the \( i \)th column of \( K(x) \). Consequently, the optimality conditions assume the form of the Euler equations [18], i.e. for every \( i = 1, \ldots, s \)
\[ \text{tr} \frac{\partial}{\partial x} \left( \frac{\partial L}{\partial \frac{\partial h}{\partial x}} \right) = \text{tr} \frac{\partial}{\partial x} \left( m(x)P(x) \frac{\partial h_i}{\partial x} - \frac{\partial}{\partial x} (m(x)K_i(x)) \right) = 0 \]
or, after expansion,
\[ \sum_{i=1}^s \left( \sum_{k=1}^n \left( m(x)P_{ik}(x) \frac{\partial^2 h_i(x)}{\partial x_k \partial x_r} + \sum_{k=1}^n \frac{\partial (m(x)P_{kr}(x))}{\partial x_r} \frac{\partial h_i(x)}{\partial x_k} \\
- \frac{\partial (m(x)K_i(x))}{\partial x_r} \right) \right) = 0. \]
Using these definitions we observe that the optimality conditions (17) require that the result of the action of the Laplace-Beltrami operator, cf. (27), associated with \( G(x) \) on every component of the augmenting map should be equal to the divergence (28) of a corresponding column of the Jacobian kernel, i.e. for every \( i = 1, \ldots, s \),
\[ \Delta h_i = \text{div} K_i. \]
This equation corresponds to the Poisson equation associated with the Laplace-Beltrami operator. When the right hand side of (19) vanishes, the function \( h_i(x) \) is called harmonic. It follows that in order to compute the augmenting map \( h(x) \) we need to solve \( s = n - m \) linear, elliptic partial differential equations (19) with the same operator on the left hand side and suitably chosen boundary conditions.

### III. Examples

In this section we shall compute the optimal augmenting functions for two kinematics having the degree of redundancy 1.

#### A. Example 1

As the first example, we shall study the kinematics of a 3DOF planar manipulator shown in figure 1. The manipulator has 3 joint variables \((x_1, x_2, x_3)\) and 2 tasks coordinates \((y_1, y_2)\) describing the Cartesian position of the car \(W2\) with respect to the inertial coordinate frame fixed to the base. The joint variable \(x_1\) is moved directly by the motor \(M1\), similarly \(x_3\) is driven by \(M3\). The position of \(W2\) along the runner \(P2\) depends on \(x_3\), and also on the revolution angle of the toothed wheel \(z2\). This angle is coupled with the revolution angle of the toothed wheel \(z1\) through a transmission gear whose gear ratio is adjusted by the joint variable \(x_2\) moved by a motor \(M2\) (not shown in the figure). A computation yields the kinematics \(y_1 = c_1x_1\) and \(y_2 = f_2(x_2)x_1 + c_3x_3\) for constants \(c_1\), \(c_3\) and a nonlinear function \(f_2(x_2)\). After a change of coordinates \(q_1 = c_1x_1\), \(q_2 = c_3x_3\) and \(q_3 = \frac{1}{3}f_2(x_2)\), and setting \(x = q\), these kinematics may be given the normal form
\[ k(x) = (x_1, x_2 + x_3). \]

A simple computation provides the Jacobian and the manipulability matrix
\[ J(x) = \begin{bmatrix} 1 & 0 & 0 \\
-x_3 & 1 & x_1 \end{bmatrix}, \quad M(x) = J(x)J^T(x) = \begin{bmatrix} 1 & x_3 \\
x_3 & 1 + x_3^2 + x_3^2 \end{bmatrix}. \]
as well as the manipulability \( m(x) = \sqrt{1 + x_1^2} \) and the Jacobian kernel

\[ K(x) = \frac{1}{m(x)} (0, -x_1, 1)^T. \]

It is easily seen that the kinematics (20) is regular everywhere. The Riemannian metric \( G(x) = P^{-1}(x) \), where

\[ P(x) = \begin{bmatrix}
-\frac{x_3}{1 + x_1^2} & \frac{-x_1 x_3}{1 + x_1^2} & \frac{-x_1 x_3}{1 + x_1^2} \\
\frac{-x_1 x_3}{1 + x_1^2} & \frac{1 - 2 x_1^2 + 2 x_3^2}{(1 + x_1^2)^{3/2}} & \frac{-x_1 x_3}{1 + x_1^2} \\
\frac{-x_1 x_3}{1 + x_1^2} & \frac{-x_1 x_3}{1 + x_1^2} & \frac{1 - 2 x_1^2 + 2 x_3^2}{(1 + x_1^2)^{3/2}}
\end{bmatrix}. \]

Our objective consists in finding an augmenting function \( h(x) \) that minimizes the error (16). Because \( \text{div} K(x) = 0 \), the corresponding Euler equation (18) takes the form \( \Delta h = 0 \) and can be written as

\[
\sqrt{1 + x_1^2} \frac{\partial^2 h(x)}{\partial x_1^2} + \frac{1 + x_1^2 + x_3^2}{(1 + x_1^2)^{3/2}} \frac{\partial^2 h(x)}{\partial x_2^2} + \frac{2 x_3}{(1 + x_1^2)^{3/2}} \frac{\partial^2 h(x)}{\partial x_1 \partial x_2} + \frac{2 x_1 x_3}{(1 + x_1^2)^{3/2}} \frac{\partial^2 h(x)}{\partial x_1 \partial x_3} + \frac{3 x_1 x_3}{(1 + x_1^2)^{3/2}} \frac{\partial h(x)}{\partial x_2} + \frac{2 x_3^2 x_3 - x_3 \Delta h(x)}{(1 + x_1^2)^{3/2}} = 0. \quad (21)
\]

We assume the boundary condition \( h(x_1, x_2, 0) = 0 \), and set \( h(x) = x_3 f(x_1) \). Then the partial differential equation (21) reduces to a 2nd order linear ordinary differential equation

\[
\frac{d^2 f(x_1)}{dx_1^2} = \frac{2 x_3}{1 + x_1^2} \frac{df(x_1)}{dx_1} + \frac{2 x_3^2 - 1}{(1 + x_1^2)^2} f(x_1) = 0. \quad (22)
\]

By inspection we discover a specific solution \( f(x_1) = \sqrt{1 + x_1^2} \) of (22). Since (22) is linear, a standard procedure provides the general solution \( f(x_1) = (ax_1 + b) \sqrt{1 + x_1^2} \), for some constants \( a \) and \( b \). Thus, we have obtained a family of harmonic optimal augmenting functions \( h_{a,b}(x) = x_3 (ax_1 + b) \sqrt{1 + x_1^2} \). Let us choose the constants \( a = 0, b = 1 \). The corresponding augmenting function \( h_{0,1}(x) = x_3 \sqrt{1 + x_1^2} \). The same solution obtained numerically with the help of the MATLAB PDE toolbox is shown in figure 2. It has been demonstrated that for the normal form kinematics (20) the optimal augmenting function can be found analytically. Since the normal form and the original kinematics of the manipulator shown in figure 1 are diffeomorphic, the extended Jacobian inverse for (20) will produce an extended Jacobian inverse (in general not optimal) for the original kinematics.

B. Example 2

The second example involves a 3DOF planar manipulator presented schematically in figure 3, whose kinematics

\[ k(q) = (q_1 + l_2 \cos q_2 + l_3 \cos q_3, l_2 \sin q_2 + l_3 \sin q_3). \quad (23) \]

The manipulator’s Jacobian, the manipulability matrix, the manipulability, and the Jacobian kernel have been computed as follows

\[
J(q) = \begin{bmatrix}
1 & -l_2 \sin q_2 & -l_3 \sin q_3 \\
0 & l_2 \cos q_2 & l_3 \cos q_3
\end{bmatrix},
\]

\[
M(q) = \begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix},
\]

\[
m(q) = \|K(q)\| = l_2^2 \cos^2 q_2 + l_3^2 \cos^2 q_3 + l_2 l_3^2 \sin^2(q_2 - q_3),
\]

\[
K(q) = \frac{1}{m(q)} (l_2 \sin(q_2 - q_3), l_3 \cos q_3, -l_2 \cos q_2)^T,
\]

where \( M_{11} = 1 + l_2^2 \sin^2 q_2 + l_3^2 \sin^2 q_3, M_{12} = M_{21} = -l_2^2 \sin q_2 \cos q_2 - l_3^2 \sin q_3 \cos q_3, M_{22} = l_2^2 \cos^2 q_2 + l_3^2 \cos^2 q_3 \). It is easily checked that the kinematics (23) become singular at \( q_2, q_3 = \pm \frac{\pi}{2} \). We are looking for an augmenting function \( h(q) \) satisfying the optimality conditions (17). Since, by (28),
again div $K(q) = 0$, we obtain
$$\text{tr} \frac{\partial}{\partial q} \left( R(q) \frac{\partial h(q)}{\partial q} \right) = 0,$$
where $R(q) = m(q) P(q)$. The entries of this matrix, $R_{ij}(q)$, computed for the unit arm lengths $l_2 = l_3 = 1$ of the manipulator, are the following:

\begin{align*}
R_{11}(q) & = \frac{2}{m(q)} (\cos^2 q_2 + \cos^2 q_3), \\
R_{12}(q) & = \frac{1}{m(q)} \cos q_2 (\sin q_2 \cos q_2 + \sin q_3 \cos q_3 - \\
& \quad \sin(q_2 - q_3) \cos(q_2 - q_3)), \\
R_{13}(q) & = \frac{1}{m(q)} \cos q_3 (\sin q_2 \cos q_2 + \sin q_3 \cos q_3 + \\
& \quad \sin(q_2 - q_3) \cos(q_2 - q_3)), \\
R_{22}(q) & = \frac{1}{m(q)} \cos^2 q_2 + \cos^2 q_3 + \sin^2(q_2 - q_3) - \\
& \quad 2 \cos q_2 \sin q_3 \sin(q_2 - q_3)), \\
R_{23}(q) & = \frac{1}{m(q)} \cos^2 q_2 + \cos^2 q_3 + \sin^2(q_2 - q_3) - \\
& \quad 2 \sin q_2 \cos q_3 \sin(q_2 - q_3)).
\end{align*}

It follows that the optimal augmenting function will be harmonic. The computation has been accomplished under an additional assumption that $h(q) = h(q_2, q_3)$. The result provided by the MATLAB PDE toolbox is shown in figure 4.

![Fig. 4. Optimal augmenting function $h(q_2, q_3)$](image)

IV. CONCLUSION

Using some instruments of Riemannian geometry we have addressed the optimal synthesis problem of extended Jacobian inverse kinematics algorithms for stationary robotic manipulators. The proposed problem formulation leads to explicit optimality conditions involving the Laplace-Beltrami operator, tantamount with solving a linear, elliptic partial differential equation with functional coefficients. The theoretical concepts have been illustrated with elementary computations. An application of the proposed approach to inverse kinematics algorithms for mobile manipulators will be a subject of future research.

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VI. APPENDIX

The definitions of the approximation error introduced in section II are built on an analogy with some basic concepts of Riemannian geometry. Below we recall them briefly referring the reader for more details e.g. to [19]. Let $(X, G)$ and $(Y, H)$ denote a pair of Riemannian manifolds. Assume that $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_m)$ are local coordinate systems in $X$, and $Y$. In these coordinates the Riemannian metrics can be identified with symmetric, positive definite matrices of suitable size. Choose a pair of vectors $v, w$ tangent to $X$ at the point $x$. Then the Riemannian metric $G$ defines the inner product $g_{v,w} = g^X(x)$, similarly for $H$. A local diffeomorphism $\varphi : X \rightarrow Y$ transfers the metric $H$ from $Y$ to $X$ by the pullback

$$g^H(\varphi(x)) = \left( \frac{\partial \varphi(x)}{\partial x} \right)^T H(\varphi(x)) \frac{\partial \varphi(x)}{\partial x}. \quad (24)$$

Given the Riemannian metric $G(x)$, the volume form on $X$ is represented by

$$V = \sqrt{\det G(x)} dx. \quad (25)$$

Now, let for some points $x, y$ the map $C(x, y) : T_xX \rightarrow T_yY$ be a linear transformation of tangent spaces, represented in coordinates by an $n \times m$ matrix. It can be shown that the dual map $C^*(x, y) : T^*_xY \rightarrow T^*_yX$, transforming suitable dual spaces, can be defined as

$$C^*(x, y) = G^{-1}(x) C^T(x, y) H(y). \quad (26)$$

On the Riemannian manifold $(X, G)$ the operator

$$\Delta f = \frac{1}{\sqrt{\det G}} \frac{\partial}{\partial x} \left( \sqrt{\det G}^{-1} \frac{\partial f}{\partial x} \right) \quad (27)$$

acting on a function $f$ is called the Laplace-Beltrami operator [19], [15], whereas the divergence of a vector field $Z$ on $X$ is equal to

$$\text{div} Z = \frac{1}{\sqrt{\det G}} \frac{\partial}{\partial x} \left( \sqrt{\det G} \frac{\partial Z}{\partial x} \right). \quad (28)$$

REFERENCES


