Fuzzy Entropy on Intuitionistic Fuzzy Sets

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In this article we exploit the concept of probability for defining the fuzzy entropy of intuitionistic fuzzy sets (IFSs). We then propose two families of entropy measures for IFSs and also construct the axiom definition and properties. Two definitions of entropy for IFSs proposed by Burillo and Bustince in 1996 and Szmidt and Kacprzyk in 2001 are used. The first one allows us to measure the degree of intuitionism of an IFS, whereas the second one is a nonprobabilistic-type entropy measure with a geometric interpretation of IFSs used in comparison with our proposed entropy of IFSs in the numerical comparisons. The results show that the proposed entropy measures seem to be more reliable for presenting the degree of fuzziness of an IFS. © 2006 Wiley Periodicals, Inc.

1. INTRODUCTION

Out of several fuzzy set generalizations for various objectives, the notion introduced by Atanassov1 in defining intuitionistic fuzzy sets (IFSs) is interesting and useful. Because these IFSs can present the degrees of both membership and non-membership with a degree of hesitancy, they have been widely studied and applied in a variety of areas (see Ref. 2).

Let a set $X$ be fixed. An IFS $A$ in $X$ is an objective having the form

$$A = \{(x, \mu_A(x), \nu_A(x)) | x \in X\}$$

where

$$\mu_A : X \to [0,1], \quad \nu_A : X \to [0,1]$$

with the condition

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1, \quad \forall x \in X$$

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The numbers $\mu_A(x)$ and $\nu_A(x)$ denote the degree of membership and nonmembership of $x$ to $A$, respectively. Obviously, a fuzzy set $A$ corresponds to the following IFS with

$$A = \{(x, \mu_A(x), 1 - \mu_A(x)) | x \in X\}$$

For each IFS $A$ in $X$, we will call

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$$

the intuitionistic index of $x$ in $A$. It is a hesitancy degree of $x$ to $A$ (see Refs. 1–3). It is obvious that

$$0 \leq \pi_A(x) \leq 1, \quad \forall x \in X$$

Note that for an IFS $A$, if $\mu_A(x) = 0$, then $\nu_A(x) + \pi_A(x) = 1$, and if $\mu_A(x) = 1$, then $\nu_A(x) = 0$ and $\pi_A(x) = 0$.

It is very common in the theory of IFS to measure its fuzziness. A measure of fuzziness often used in the literature is an entropy first mentioned by Zadeh.\(^4\) Two definitions of entropy for IFSs have recently been proposed by Burillo and Bustince\(^5\) and Szmidt and Kacprzyk.\(^6\) These two definitions are completely different frameworks. The first one allows us to measure the degree of intuitionism of an IFS. The second one is a nonprobabilistic-type entropy measure that is the result of a geometric interpretation of IFSs and uses a ratio of distances between two of the IFSs proposed in Ref. 7. In this article, we will exploit the concept of probability for defining the entropy of an IFS. We then propose two families of entropy measures for IFSs.

In Section 2, we give the axiom construction of IFS entropy, which is a generalization of De Luca and Termini.\(^8\) Two families of entropy for IFSs are then presented. The degree of fuzziness with Refs. 5 and 6 is compared in Section 3. Our conclusions are presented in Section 4.

### 2. FUZZY ENTROPY ON INTUITIONISTIC FUZZY SETS

Measures of fuzziness in contrast to fuzzy measures indicate the degree of fuzziness of a fuzzy set. The entropy of a fuzzy set is a measure of the fuzziness of a fuzzy set. De Luca and Termini\(^8\) introduced the axiom construction of fuzzy set entropy and referred to Shannon’s probability entropy. They gave an axiom definition of an entropy of fuzzy sets. Let $R^+ = [0, \infty)$, $X = \{x\}$, $FS(X)$ be the set of all fuzzy sets of $X$ and $IFS(X)$ be the set of all IFSs of $X$.

**Definition 1** (cf. Ref. 8). A real function $e : FS(X) \rightarrow R^+$ is called an entropy on $FS(X)$ if $e$ has the following properties:

1. $e(\hat{A}) = 0$ if $\hat{A}$ is a crisp set.
2. $e(\hat{A})$ assumes a unique maximum if $\mu_{\hat{A}} = \frac{1}{2}$.
3. $e(\hat{A}) \leq e(\hat{B})$ if $\hat{A}$ is crispier than $\hat{B}$, that is, if $\mu_{\hat{A}} \leq \mu_{\hat{B}}$ for $\mu_{\hat{B}} = \frac{1}{2}$ and $\mu_{\hat{A}} = \mu_{\hat{B}}$ for $\mu_{\hat{B}} \geq \frac{1}{2}$.
4. $e(\hat{A}) = e(\hat{A}^c)$ where $\hat{A}^c$ is the complement of $\hat{A}$.
Because the IFS concept is a generalization of the fuzzy set concept, the following natural question arises: How can we appraise the fuzziness associated with an IFS in the spirit of Ref. 8? It is known that a full description of an IFS A in X has three coordinates \((\mu_A, \nu_A, \pi_A)\) (see Ref. 7) with \(\mu_A + \nu_A + \pi_A = 1\), \(0 \leq \mu_A, \nu_A, \pi_A \leq 1\). By taking the three-parameter characterization of IFS, we may regard \((\mu_A, \nu_A, \pi_A)\) as a probability concept of IFSs. The following axiom definition of IFS entropy is mainly done on the basis of extensions of Definition 1 with a probability concept of IFSs.

**Definition 2.** A real function \(E: IFS(X) \rightarrow R^+\) is called an entropy on IFS\( (X)\) if \(E\) has the following properties:

\begin{align}
(IE1) \quad & E(A) = 0 \text{ if } A \text{ is a crisp set.} \\
(IE2) \quad & E(A) \text{ assumes a unique maximum if } \mu_A = \nu_A = \pi_A = \frac{1}{3}. \\
(IE3) \quad & E(A) = E(B) \text{ if } A \text{ is crisper than } B, \text{ that is, if } \mu_A \leq \mu_B \text{ and } \nu_A \leq \nu_B \text{ for max}\{\mu_B, \nu_B\} \leq \frac{1}{3} \text{ and } \mu_A \geq \mu_B \text{ and } \nu_A \geq \nu_B \text{ for min}\{\mu_B, \nu_B\} \geq \frac{1}{3}. \\
(IE4) \quad & E(A) = E(A^c) \text{ where } A^c \text{ is the complement of } A.
\end{align}

**Property 1.** IE3 implies that

\[
\left| \mu_A - \frac{1}{3} \right| + \left| \nu_A - \frac{1}{3} \right| + \left| \pi_A - \frac{1}{3} \right| \leq \left| \mu_B - \frac{1}{3} \right| + \left| \nu_B - \frac{1}{3} \right| + \left| \pi_B - \frac{1}{3} \right|
\]

(1)

and

\[
\left( \mu_A - \frac{1}{3} \right)^2 + \left( \nu_A - \frac{1}{3} \right)^2 + \left( \pi_A - \frac{1}{3} \right)^2 \leq \left( \mu_B - \frac{1}{3} \right)^2 + \left( \nu_B - \frac{1}{3} \right)^2 + \left( \pi_B - \frac{1}{3} \right)^2
\]

(2)

**Proof.** If \(\mu_A \leq \mu_B\) and \(\nu_A \leq \nu_B\) for \(\max\{\mu_B, \nu_B\} \leq \frac{1}{3}\), then

\[
\mu_A \leq \mu_B \leq \frac{1}{3} \quad \nu_A \leq \nu_B \leq \frac{1}{3} \quad \text{and} \quad \pi_A \geq \pi_B \geq \frac{1}{3}
\]

It implies that Equations 1 and 2 hold. Similarly, if \(\mu_A \geq \mu_B\) and \(\nu_A \geq \nu_B\) for \(\min\{\mu_B, \nu_B\} \geq \frac{1}{3}\) then Equations 1 and 2 also hold.

As was shown in Ref. 7, the distances between IFSs should be calculated by taking into account the three parameters \(\mu_A, \nu_A, \text{ and } \pi_A\) of an IFS \(A\). The most popular distances between IFSs are the Hamming and Euclidean distances. Therefore, Equation 1 (or 2) means that the Hamming (or Euclidean) distance (see Ref. 7) between \((\mu_A, \nu_A, \pi_A)\) and \((\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\) is larger than the Hamming (or Euclidean) distance.
between \((\mu_B, \nu_B, \pi_B)\) and \((\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\). This establishes that \((\mu_B, \nu_B, \pi_B)\) is more around \((\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\) than \((\mu_A, \nu_A, \pi_A)\). Thus, IE3 is reasonable from a geometrical viewpoint.

In the following, we will borrow Havrda and Charvát’s\(^9\) entropy \(H_{hc}^\alpha(p)\) and Rényi’s\(^10\) entropy \(H_r^\beta(p)\) to a probability measure \(p = \{p_1, \ldots, p_k\}\) with \(\sum_{i=1}^k p_i = 1\) for which

\[
H_{hc}^\alpha(p) = \begin{cases} 
\frac{1}{\alpha - 1} \left(1 - \sum_{i=1}^k p_i^\alpha\right), & \alpha \neq 1 (\alpha > 0) \\
- \sum_{i=1}^k p_i \log p_i, & \alpha = 1
\end{cases}
\]

and

\[
H_r^\beta(p) = \frac{1}{1 - \beta} \log \left(\sum_{i=1}^k p_i^\beta\right), \quad 0 < \beta < 1
\]

We then propose two families of fuzzy entropy of an IFS \(A\) with

\[
E_{hc}^\alpha(A) = \begin{cases} 
\frac{1}{\alpha - 1} \left[1 - (\mu_A^\alpha + \nu_A^\alpha + \pi_A^\alpha)\right], & \alpha \neq 1 (\alpha > 0) \\
-(\mu_A \log \mu_A + \nu_A \log \nu_A + \pi_A \log \pi_A), & \alpha = 1
\end{cases}
\]

and

\[
E_r^\beta(A) = \frac{1}{1 - \beta} \log (\mu_A^\beta + \nu_A^\beta + \pi_A^\beta), \quad 0 < \beta < 1
\]

Thus, the Shannon-type entropy denoted by \(E_s\) is a special case of \(E_{hc}^\alpha\) with \(\alpha = 1\).

It is natural to pose the question: “Are the proposed entropy measures reasonable?” We answer this question by expressing the axioms IE1–IE4 for these entropy measures of IFSs. It is easy to see that \(E_{hc}^\alpha\) and \(E_r^\beta\) satisfy conditions IE1 and IE4. Using the Lagrange multiplier method, we can also obtain that \(E_{hc}^\alpha\) and \(E_r^\beta\) satisfy condition IE2. To prove condition IE3 we need the following lemma.

**Lemma 1.** Let \(\phi_\alpha(x)\), \(0 < x < 1\) be defined as

\[
\phi_\alpha(x) = \begin{cases} 
\frac{1}{\alpha - 1} (x^\alpha - x), & \alpha \neq 1 (\alpha > 0) \\
x \log x, & \alpha = 1
\end{cases}
\]

Then \(\phi_\alpha(x)\) is a strictly convex function of \(x\).

**Proof.** Because \(\phi_\alpha''(x) = \alpha x^{\alpha - 2} > 0\), for \(\alpha \neq 1 (\alpha > 0)\), then \(\phi_\alpha(x)\) is a strictly convex function of \(x\). Similarly, \(\phi_1(x) = x \log x\) is also a strictly convex function of \(x\).
Lemma 2. Let \( \psi_\beta(x) = (1/(\beta - 1)) \log x^\beta, \) \( 0 < x < 1, \) \( 0 < \beta < 1. \) Then \( \psi_\beta(x) \) is a strictly convex function of \( x. \)

Proof. Because \( \psi_\beta''(x) = -1/(\beta - 1)x^2 > 0, \) for \( 0 < \beta < 1, \) then \( \psi_\beta(x) \) is a strictly convex function of \( x. \)

Property 2. Fuzzy entropies \( E_{hc}^\alpha \) and \( E_r^\beta \) satisfy condition IE3.

Proof. Because

\[
E_{hc}^\alpha(A) = -(\phi_\alpha(\mu_A) + \phi_\alpha(\nu_A) + \phi_\alpha(\pi_A))
\]

Using Lemma 1, \( \phi_\alpha \) is strictly convex. So, \( E_{hc}^\alpha \) is strictly concave on the set \( \{ (\mu_A, \nu_A, \pi_A) | 0 < \mu_A, \nu_A, \pi_A < 1, \mu_A + \nu_A + \pi_A = 1 \}. \) According the result of Property 1, we have \( E_{hc}^\alpha \) satisfies condition IE3. On the other hand,

\[
E_r^\beta(A) = -(\psi_\beta(\mu_A) + \psi_\beta(\nu_A) + \psi_\beta(\pi_A))
\]

Using Lemma 2 and Property 1, we also have \( E_r^\beta \) satisfies condition IE3.

The above formulas describe the entropy for a single point belonging to an IFS. For an IFS \( A \) in \( X = \{ x_1, \ldots, x_n \}, \) the entropy of \( A \) is given by

\[
E(A) = \frac{1}{n} \sum_{i=1}^{n} E(A_i)
\]

where \( A_i = \{ (x_i, \mu_{A_i}(x_i), \nu_{A_i}(x_i)) | x_i \in X_i \} \) are IFSs in \( X_i = \{ x_i \}, i = 1, \ldots, n. \)

3. COMPARATIVE EXAMPLES

Let \( A \) be an IFS in \( X = \{ x_1, \ldots, x_n \}. \) Szmidt and Kacprzyk\(^6\) defined the entropy of \( A \) as follows:

\[
E_{sk}(A) = \frac{1}{n} \sum_{i=1}^{n} \frac{\min(\mu_A(x_i), \nu_A(x_i)) + \pi_A(x_i)}{\max(\mu_A(x_i), \nu_A(x_i)) + \pi_A(x_i)}
\]

and Burillo and Bustince\(^5\) used the following entropy of \( A: \)

\[
E_{bb}(A) = \sum_{i=1}^{n} \pi_A(x_i)
\]

The following examples are used for comparing the proposed entropy measures with \( E_{sk} \) and \( E_{bb}. \)

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Example 1 (cf. Ref. 6). Let us consider the example from Szmidt and Kacprzyk\(^6\) with \(F_1\), \(F_2\), and \(F_3\) being IFSs in \(X\) and defined by

\[
F_1 = \left\{ \left( x, \frac{3}{4}, \frac{1}{6} \right) \right\}, \quad F_2 = \left\{ \left( x, \frac{1}{2}, 0 \right) \right\}, \quad F_3 = \left\{ \left( x, \frac{1}{2}, \frac{1}{4} \right) \right\}
\]

These IFSs are used for comparing the entropy measures \(E_s\), \(E_{hc}^1\), \(E_{hc}^2\), \(E_{hc}^3\), and \(E_{bb}\). The comparison results are summarized in Table I. Intuitively, \(F_3\) describes three characters more homogeneous than \(F_1\) and \(F_2\). Thus, the entropy of \(F_3\) is the largest. The entropy measures of \(E_{hc}\) and ours reflect this phenomenon. But the entropy measure of \(E_{bb}\) fails. Based on Table I, we know that (1) the behaviors of \(E_{hc}\) and ours are the same; (2) the behaviors of \(E_{hc}^{1/3}\), \(E_{hc}^{1/2}\), \(E_s\), \(E_{hc}^{1/3}\), and \(E_{hc}^{1/2}\) are the same. However, it is difficult for us to choose the best entropy measure in this example. Next, we shall use a linguistic example to compare them.

Example 2. Let \(A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}\) be an IFS in \(X\). For any positive real number \(n\), De et al.\(^{11}\) defined the IFS \(A^n\) as follows:

\[
A^n = \{(x, [\mu_A(x)]^n, 1 - [1 - \nu_A(x)]^n) \mid x \in X\}
\]

Using the above operation, they also defined the concentration and dilation of \(A\) as follows:

- concentration: \(CON(A) = A^2\)
- dilation: \(DIL(A) = A^{1/2}\)

Like fuzzy sets, \(CON(A)\) and \(DIL(A)\) may be treated as “very \(A\)” and “more or less \(A\),” respectively.

In the next, we consider an IFS \(A\) in \(X = \{6, 7, 8, 9, 10\}\) defined in Ref. 11:

\[
A = \{ (6, 0.1, 0.8), (7, 0.3, 0.5), (8, 0.6, 0.2), (9, 0.9, 0.0), (10, 1.0, 0.0) \}\]

Using the operations defined in Ref. 11, we can generate the following IFSs:

\[ A^{1/2}, A^2, A^3, A^4 \]
By taking into account the characterization of linguistic variables, De et al.\textsuperscript{11} regarded $A$ as “LARGE” in $X$. Using the above operations,

\[ A^{1/2} \text{ may be treated as “More or less LARGE”} \]
\[ A^2 \text{ may be treated as “Very LARGE”} \]
\[ A^3 \text{ may be treated as “Quite very LARGE”} \]
\[ A^4 \text{ may be treated as “Very very LARGE”} \]

We used these IFSs to compare the entropy measures $E_s$, $E^\alpha_{hc}$, $E^\beta_r$, $E_{sk}$, and $E_{bb}$, respectively. The comparison results are shown in Table II.

From the viewpoint of mathematical operations, the entropies of these IFSs have the following requirement:

\[ E(A^{1/2}) > E(A) > E(A^2) > E(A^3) > E(A^4) \quad (3) \]

The entropy measure $E_{sk}$ satisfies this requirement, but

\[ E_{bb}(A^{1/2}) < E_{bb}(A) < E_{bb}(A^2) < E_{bb}(A^3) < E_{bb}(A^4) \]

Our proposed entropy measures indicate

\[ E^\alpha_{hc}(A) > E^\alpha_{hc}(A^{1/2}) > E^\alpha_{hc}(A^2) > E^\alpha_{hc}(A^3) > E^\alpha_{hc}(A^4), \quad \alpha = \frac{1}{3}, \frac{1}{2}, 1, 2, 3 \quad (4) \]
\[ E^\beta_r(A) > E^\beta_r(A^{1/2}) > E^\beta_r(A^2) > E^\beta_r(A^3) > E^\beta_r(A^4), \quad \beta = \frac{1}{3}, \frac{1}{2} \quad (5) \]

Therefore, the performance of $E_{sk}$ is good, but $E_{bb}$ is poor.

To see how different IFSs “LARGE” in $X$ affect the above entropy measures, we reduce the hesitancy degree of “8,” which is the middle point of $X$. First, suppose that “LARGE” = $A_1 = \{(6, 0.1, 0.8), (7, 0.3, 0.5), (8, 0.5, 0.4), (9, 0.9, 0.0), (10, 1.0, 0.0)\}

We also use IFSs $A^{1/2}$, $A_1$, $A^2$, $A^3$, and $A^4$ to compare the above entropy measures. The comparison results are presented in Table III. Based on Table III, we see that the Shannon entropy $E_s$ satisfies requirement (3), but

\[ E_{bb}(A_1) > E_{bb}(A^2) > E_{bb}(A^3) = E_{bb}(A^4) > E_{bb}(A^{1/2}) \]

\[ \text{Table II. Comparison of the degree of fuzziness with different entropy measures.} \]

<table>
<thead>
<tr>
<th>IFSs</th>
<th>$E^\alpha_{hc}$</th>
<th>$E^\alpha_{hc}$</th>
<th>$E_s$</th>
<th>$E^\beta_{hc}$</th>
<th>$E^\beta_{hc}$</th>
<th>$E^\alpha_{hc}$</th>
<th>$E^\alpha_{hc}$</th>
<th>$E^\beta_{hc}$</th>
<th>$E^\beta_{hc}$</th>
<th>$E_{sk}$</th>
<th>$E_{bb}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A^{1/2}$</td>
<td>0.982</td>
<td>0.840</td>
<td>0.404</td>
<td>0.328</td>
<td>0.228</td>
<td>0.702</td>
<td>0.661</td>
<td>0.319</td>
<td>0.462</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_1$</td>
<td>1.004</td>
<td>0.865</td>
<td>0.414</td>
<td>0.340</td>
<td>0.236</td>
<td>0.718</td>
<td>0.681</td>
<td>0.307</td>
<td>0.600</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A^2$</td>
<td>0.899</td>
<td>0.754</td>
<td>0.348</td>
<td>0.290</td>
<td>0.203</td>
<td>0.659</td>
<td>0.605</td>
<td>0.301</td>
<td>0.660</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A^3$</td>
<td>0.782</td>
<td>0.642</td>
<td>0.299</td>
<td>0.253</td>
<td>0.179</td>
<td>0.585</td>
<td>0.522</td>
<td>0.212</td>
<td>0.672</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A^4$</td>
<td>0.686</td>
<td>0.557</td>
<td>0.264</td>
<td>0.226</td>
<td>0.163</td>
<td>0.521</td>
<td>0.457</td>
<td>0.176</td>
<td>0.680</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Thus, the performance of measures $E_{hc}$ from the viewpoint of structured linguistic variables. The other entropy measures indicate $E_{hc}$ entropy measures $E_{bb}$

<table>
<thead>
<tr>
<th>IFSs</th>
<th>$E_{1/3}^{hc}$</th>
<th>$E_{1/2}^{hc}$</th>
<th>$E_{r}$</th>
<th>$E_{bb}$</th>
<th>$E_{1/3}^{sk}$</th>
<th>$E_{1/2}^{sk}$</th>
<th>$E_{sk}$</th>
<th>$E_{bb}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{1/2}$</td>
<td>0.989</td>
<td>0.850</td>
<td>0.433</td>
<td>0.342</td>
<td>0.238</td>
<td>0.706</td>
<td>0.667</td>
<td>0.345</td>
</tr>
<tr>
<td>$A_1$</td>
<td>0.999</td>
<td>0.860</td>
<td>0.431</td>
<td>0.344</td>
<td>0.241</td>
<td>0.715</td>
<td>0.678</td>
<td>0.374</td>
</tr>
<tr>
<td>$A_{1/3}$</td>
<td>0.867</td>
<td>0.715</td>
<td>0.327</td>
<td>0.261</td>
<td>0.187</td>
<td>0.643</td>
<td>0.581</td>
<td>0.197</td>
</tr>
<tr>
<td>$A_{1/4}$</td>
<td>0.728</td>
<td>0.576</td>
<td>0.253</td>
<td>0.199</td>
<td>0.146</td>
<td>0.557</td>
<td>0.481</td>
<td>0.131</td>
</tr>
<tr>
<td>$A_{1/5}$</td>
<td>0.614</td>
<td>0.471</td>
<td>0.208</td>
<td>0.161</td>
<td>0.120</td>
<td>0.482</td>
<td>0.401</td>
<td>0.109</td>
</tr>
</tbody>
</table>

The other entropy measures indicate

$$E_{hc}^{\alpha}(A_1) > E_{hc}^{\alpha}(A_{1/2}) > E_{hc}^{\alpha}(A_{2}) > E_{hc}^{\alpha}(A_{3}) > E_{hc}^{\alpha}(A_{4}) , \quad \alpha = \frac{1}{3}, \frac{1}{2}, 1, 2, 3$$

(6)

$$E_{bb}^{\beta}(A_1) > E_{bb}^{\beta}(A_{1/2}) > E_{bb}^{\beta}(A_{2}) > E_{bb}^{\beta}(A_{3}) > E_{bb}^{\beta}(A_{4}) , \quad \beta = \frac{1}{3}, \frac{1}{2}$$

(7)

$$E_{sk}^{\delta}(A_1) > E_{sk}^{\delta}(A_{1/2}) > E_{sk}^{\delta}(A_{2}) > E_{sk}^{\delta}(A_{3}) > E_{sk}^{\delta}(A_{4}) $$

(8)

Thus, the performance of $E_{hc}$ is good, but $E_{bb}$ is poor.

Finally, we consider an IFS “LARGE” in $X$ defined as

$$A_2 = \{ (6, 0.1, 0.8), (7, 0.3, 0.5), (8, 0.5, 0.5), (9, 0.9, 0.0), (10, 1.0, 0.0) \}$$

In $A_2$, the hesitancy degree of “8” is equal to 0. Based on Table IV, we see that the entropy measures $E_{hc}^{\alpha}, \alpha = \frac{1}{3}, \frac{1}{2}, 1, 2, 3$ satisfy requirement (3), but

$$E_{bb}(A_2) = E_{bb}(A_{2}) > E_{bb}(A_{3}) > E_{bb}(A_{4}) > E_{bb}(A_{2}) $$

The other entropy measures indicate

$$E_{1/3}^{r}(A_2) > E_{1/3}^{r}(A_{1/2}) > E_{1/3}^{r}(A_{2}) > E_{1/3}^{r}(A_{3}) > E_{1/3}^{r}(A_{4}) $$

(9)

$$E_{1/2}^{r}(A_{1/2}) > E_{1/2}^{r}(A_{2}) > E_{1/2}^{r}(A_{3}) > E_{1/2}^{r}(A_{4}) $$

(10)

$$E_{sk}^{\delta}(A_1) > E_{sk}^{\delta}(A_{1/2}) > E_{sk}^{\delta}(A_{2}) > E_{sk}^{\delta}(A_{3}) > E_{sk}^{\delta}(A_{4}) $$

(11)

Thus, the performance of $E_{hc}^{\alpha}, \alpha = \frac{1}{3}, \frac{1}{2}, 1, 2, 3$ is good, but $E_{bb}$ is poor.

According to the results in Tables II–IV, we see that the proposed entropy measures $E_{hc}$ seem to be reliable. Furthermore, the behavior of $E_{hc}$ is reasonable from the viewpoint of structured linguistic variables.

<table>
<thead>
<tr>
<th>IFSs</th>
<th>$E_{1/3}^{hc}$</th>
<th>$E_{1/2}^{hc}$</th>
<th>$E_{r}$</th>
<th>$E_{bb}$</th>
<th>$E_{1/3}^{sk}$</th>
<th>$E_{1/2}^{sk}$</th>
<th>$E_{sk}$</th>
<th>$E_{bb}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{1/2}$</td>
<td>0.883</td>
<td>0.772</td>
<td>0.438</td>
<td>0.336</td>
<td>0.237</td>
<td>0.644</td>
<td>0.615</td>
<td>0.352</td>
</tr>
<tr>
<td>$A_1$</td>
<td>0.876</td>
<td>0.763</td>
<td>0.427</td>
<td>0.328</td>
<td>0.235</td>
<td>0.647</td>
<td>0.615</td>
<td>0.407</td>
</tr>
<tr>
<td>$A_{1/3}$</td>
<td>0.737</td>
<td>0.608</td>
<td>0.313</td>
<td>0.233</td>
<td>0.171</td>
<td>0.569</td>
<td>0.510</td>
<td>0.168</td>
</tr>
<tr>
<td>$A_{1/4}$</td>
<td>0.603</td>
<td>0.475</td>
<td>0.238</td>
<td>0.171</td>
<td>0.127</td>
<td>0.482</td>
<td>0.409</td>
<td>0.110</td>
</tr>
<tr>
<td>$A_{1/5}$</td>
<td>0.499</td>
<td>0.382</td>
<td>0.196</td>
<td>0.138</td>
<td>0.103</td>
<td>0.409</td>
<td>0.334</td>
<td>0.095</td>
</tr>
</tbody>
</table>

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4. CONCLUSIONS

We extended the De Luca and Termini\textsuperscript{8} axiom definition to IFSs and proposed two families of entropy measures on IFSs. These entropy measures satisfy the extended entropy measure axiom definition of IFSs. Furthermore, we used two examples to make comparisons with Refs. 5 and 6. Tables II–IV show that the proposed measures are more reliable for presenting the degree of fuzziness of an IFS.

References