STABILITY OF THE $k$TH ORDER LYNNESS’ EQUATION WITH A PERIOD-$k$ COEFFICIENT

E. J. JANOWSKI and M. R. S. KULENOVIĆ

Department of Mathematics, University of Rhode Island, Kingston, RI 02881-0816, USA
*kulenn@math.uri.edu

Z. NURKANOVIC

Department of Mathematics, University of Tuzla, 75000 Tuzla, Bosnia and Herzegovina

Received May 10, 2005; Revised January 30, 2006

We first investigate the Lyapunov stability of the period-three solution of Todd’s equation with a period-three coefficient:

$$x_{n+1} = 1 + x_n + x_{n-1} - p_n x_{n-2}, \ n = 0, 1, \ldots$$

where

$$p_n = \begin{cases} 
\alpha, & \text{for } n = 3l \\
\beta, & \text{for } n = 3l + 1 \\
\gamma, & \text{for } n = 3l + 2, \ l = 0, 1, \ldots,
\end{cases}$$

$\alpha, \beta,$ and $\gamma$ positive.

Then for $k = 2, 3, \ldots$ we extend our stability result to the $k$-order equation,

$$x_{n+1} = 1 + x_n + \cdots + x_{n-k+2} - p_n x_{n-k+1}, \ n = 0, 1, \ldots$$

where $p_n$ is a periodic coefficient of period $k$ with positive real values and $x_{-k+1}, \ldots, x_1, x_0 \in (0, \infty)$.

Keywords: Invariant; Lyapunov function; Lyapunov stability; Lyness; periodic coefficient.

1. Introduction

The systematic analysis of difference equations with periodic coefficients was initiated very recently in [Elaydi & Sacker, 2005a, 2005b; Franke & Selgrade, 2003]. There are several recent related papers such as [Henson & Cushing, 1997] and [Kulenović et al., 2004] and a book [Grove & Ladas, 2004] where the global behavior of some specific equations with periodic coefficients was considered. Our goal is to study the effect of the “periodization” of the coefficient on the stability of the equilibrium of an equation with an invariant. In this paper the notion of stability refers to the stability in the sense of Lyapunov, see [Kocic & Ladas, 1993] and [Kulenović & Merino, 2002].

Consider the $k$th order equation

$$y_{n+1} = \frac{p + y_n + \cdots + y_{n-k+2}}{y_{n-k+1}}, \ n = 0, 1, \ldots$$ (1)
where $k = 2, 3, \ldots$. When $k = 2$, Eq. (1) is known as the Lyness’ equation which has been studied intensively by many authors using methods ranging from algebraic and projective geometry [Bastien & Rogalski, 2004a, 2004b; Beukers & Cushman, 1998; Zeeman, 1996] to hard analysis [Barbeau et al., 1995; Kocic et al., 1993; Kulenović, 2000]. Also several authors have studied the case when $k = 3$, known as Todd’s equation, and obtained some results concerning invariants and the stability of the equilibrium [Kocic & Ladas, 1993; Kocic et al., 1993; Kulenović, 2000; Kulenović & Merino, 2002] as well as the periodic solutions of certain periods [Cima et al., 2006]. Cima, Gasull and Manosa used dynamical systems methods to obtain precise results about attainable periods for Todd’s equation [Cima et al., 2006].

By using the substitution $y_n = px_n$, Eq. (1) can be reduced to the form:

$$x_{n+1} = \frac{1 + x_n + x_{n-k+2}}{px_{n-k+1}}, \quad n = 0, 1, \ldots$$  \hspace{1cm} (2)

where $k = 2, 3, \ldots$. Suppose $p$ is periodic with period $k$. Then Eq. (2) becomes

$$x_{n+1} = \frac{1 + x_n + x_{n-k+2}}{p_n x_{n-k+1}}, \quad n = 0, 1, \ldots$$  \hspace{1cm} (3)

where $k = 2, 3, \ldots$ and $p_n$ is a periodic coefficient of period $k$. We will show that Eq. (3) has an invariant and we will use this invariant to investigate the stability of the period-$k$ solution of Eq. (3). In this paper we will assume that all parameters and initial conditions are positive real numbers.

In order to illustrate the method used in the general case, see Sec. 4, we will first extend the result on the stability of the equilibrium for Todd’s equation, which was proven in [Kulenović, 2000], to the case where the coefficient $p_n$ is periodic of period 3. In particular, we consider the following difference equation

$$x_{n+1} = \frac{1 + x_n + x_{n-1}}{p_n x_{n-2}}, \quad n = 0, 1, \ldots$$  \hspace{1cm} (4)

where

$$p_n = \begin{cases} \alpha, & \text{for } n = 3l \\ \beta, & \text{for } n = 3l + 1 \\ \gamma, & \text{for } n = 3l + 2, \quad l = 0, 1, \ldots \end{cases}$$

This equation can be reduced to the following three equations:

$$x_{3l+1} = \frac{1 + x_{3l} + x_{3l-1}}{\alpha x_{3l-2}}$$  \hspace{1cm} (5)

$$x_{3l+2} = \frac{1 + x_{3l+1} + x_{3l}}{\beta x_{3l-1}}$$  \hspace{1cm} (6)

$$x_{3l+3} = \frac{1 + x_{3l+2} + x_{3l+1}}{\gamma x_{3l}}$$  \hspace{1cm} (7)

By setting $u_l = x_{3l}$, $v_l = x_{3l-1}$, $w_l = x_{3l-2}$, we obtain the following system of difference equations:

$$u_{l+1} = 1 + u_l + v_l + \beta w_l$$

$$v_{l+1} = 1 + u_l + v_l + \alpha w_l$$

$$w_{l+1} = 1 + \alpha u_l$$  \hspace{1cm} (8)

We assume that at least two of the parameters $\alpha$, $\beta$, and $\gamma$ are not equal. We will also need the notion of an invariant [Kulenović & Merino, 2002].

**Definition 1.1.** Consider the difference equation

$$x_{n+1} = f(x_n), \quad n = 0, 1, \ldots$$  \hspace{1cm} (9)

where $x_n$ is in $R^k$ and $f : D \rightarrow D$ is continuous for $D \subset R^k$. We call a nonconstant continuous function $I : R^k \rightarrow R$ an invariant for system (10) if $I(x_{n+1}) = I(f(x_n)) = I(x_n)$, for every $n = 0, 1, \ldots$.

Invariants are a powerful tool for finding solutions of difference equations in exact form and investigating the short-term and long-term behavior of solutions [Kulenović & Merino, 2002, Chap. 4; Sedaghat, 2003]. By using the well-known invariant of the Lyness’ equation Bastien and Rogalski found very precise results about the global behavior of that equation. See also [Gardini et al., 2003] for some related results.

In the case of nonlinear systems of difference equations we present the following result, which establishes a connection between invariants, Lyapunov functions and the stability of the equilibrium points. This result has been used extensively to prove the stability of several difference equations.

**Theorem 1.1 (Discrete Dirichlet Theorem).** Consider the difference equation (10) where $x_n$ is in $R^k$ and $f : D \rightarrow D$ is continuous for $D \subset R^k$. Suppose that $I : R^k \rightarrow R$ is a continuous invariant of Eq. (10). If $I$ attains an isolated local minimum or maximum value at the equilibrium point $\bar{x}$
of this system, then there exists a Lyapunov function equal to
\[ \pm (I(x) - I(\overline{x})) \]
and so the equilibrium \( \overline{x} \) is stable.


2. Nonhyperbolic Equilibrium Solution

The equilibrium solution \((\overline{u}, \overline{v}, \overline{w})\) of system (10) satisfies the system of equations
\[
\begin{align*}
1 + \overline{u} + \overline{v} + \beta \overline{v} + \beta \overline{u} + \beta \overline{u} & = \overline{u} + \overline{v} + \overline{w} + \alpha \overline{w} + \alpha \overline{u} \overline{w} + \alpha \beta \overline{u} \overline{w}, \\
\overline{u} & = \frac{1 + \overline{u} + \overline{v} + \alpha \overline{w} + \alpha \overline{u} \overline{w} + \alpha \beta \overline{u} \overline{w}}{\alpha \beta \overline{u} \overline{v}}, \\
\overline{v} & = \frac{1 + \overline{u} + \overline{v} + \alpha \overline{w} + \alpha \overline{u} \overline{w}}{\alpha \beta \overline{v}}, \\
\overline{w} & = \frac{1 + \overline{u} + \overline{v}}{\alpha \overline{w}},
\end{align*}
\]
which implies
\[
\begin{align*}
\gamma \overline{u}^2 & = 1 + \overline{u} + \overline{v}, \\
\beta \overline{v}^2 & = 1 + \overline{u} + \overline{w}, \\
\alpha \overline{w}^2 & = 1 + \overline{u} + \overline{v}. \\
\end{align*}
\]

Notice
\[
\begin{align*}
\gamma \overline{u}^2 + \overline{v} &= \beta \overline{v}^2 + \overline{v} = \alpha \overline{w}^2 + \overline{w} = 1 + \overline{u} + \overline{v} + \overline{w},
\end{align*}
\]
which implies that \( \overline{u} = \overline{v} \) if and only if \( \gamma = \beta \). Similarly \( \overline{v} = \overline{w} \) if and only if \( \alpha = \beta \) and \( \overline{v} = \overline{w} \) if and only if \( \alpha = \gamma \). This shows that the equilibrium of (10) is a periodic solution of Eq. (4) with minimal period three.

**Lemma 2.1.** System (11) has a unique solution.

**Proof.** Without loss of generality we can assume that \( \gamma = \min\{\alpha, \beta, \gamma\} \). Consider the part of Eq. (12)
\[
\beta \overline{v}^2 + \overline{v} = \gamma \overline{u}^2 + \overline{u}
\]
as a quadratic equation in \( \overline{v} \). By solving this equation for \( \overline{v} \) we obtain
\[
\overline{v} = \frac{1 + \sqrt{1 + 4\beta(\gamma \overline{u}^2 + \overline{u})}}{2\beta}.
\]
Similarly, by solving the part of Eq. (12)
\[
\alpha \overline{w}^2 + \overline{w} = \gamma \overline{u}^2 + \overline{u}
\]
as a quadratic equation in \( \overline{w} \), we obtain
\[
\overline{w} = \frac{-1 + \sqrt{1 + 4\alpha(\gamma \overline{u}^2 + \overline{u})}}{2\alpha}.
\]
By substituting Eqs. (13) and (14) in the first equation of (11) we obtain the following equation:
\[
\begin{align*}
2\alpha \beta \gamma \overline{u}^2 - 2\alpha \beta \gamma \overline{u}^2 + \beta \overline{u}^2 - \alpha \beta \overline{w}^2 + \beta \overline{u}^2 + \alpha \beta \overline{u} \overline{w} & = 0,
\end{align*}
\]
Furthermore, \( \overline{w} \) satisfies the polynomial equation
\[
((2\alpha \beta \gamma x^2 - 2\alpha \beta x + \alpha + \beta)^2 + \alpha \beta \gamma x^2 + \alpha \beta \gamma x^2 + x))
\]
\[
- \beta^2(1 + 4\alpha(\gamma x^2 + x)) = 0.
\]
Set
\[
f(x) = 2\alpha \beta \gamma x^2 - 2\alpha \beta + \alpha + \beta - \sqrt{4\alpha \beta \gamma x^2 + 4\alpha \beta \gamma x^2 + x}) \geq 0.
\]
Clearly \( f(0) = -2\alpha \beta < 0 \) and \( \lim_{x \to \infty} f(x) = \infty \). In addition, in view of \( \gamma = \min\{\alpha, \beta, \gamma\} \) the function \( f \) is concave up for all \( x \geq 0 \) since
\[
\begin{align*}
\frac{f''(x)}{f'(x)} &= 4\alpha \beta \left[ \frac{\beta \gamma x^2 - \alpha \gamma x^2}{(\sqrt{4\alpha \beta \gamma x^2 + 4\alpha \beta \gamma x^2 + x})^3} \right] + \frac{\alpha \gamma x^2}{(\sqrt{4\alpha \beta \gamma x^2 + 4\alpha \beta \gamma x^2 + x})^3} \times \frac{\alpha \gamma x^2}{(\sqrt{4\alpha \beta \gamma x^2 + 4\alpha \beta \gamma x^2 + x})^3} > 0.
\end{align*}
\]
Consequently, \( f \) has a unique positive zero that depends continuously on \( \alpha, \beta, \) and \( \gamma \). In other words there is unique positive \( \overline{v} \) that depends continuously on \( \alpha, \beta, \gamma \). By using this \( \overline{v} \) in Eqs. (13) and (14) we obtain the unique positive \( \overline{v} \) and \( \overline{w} \) that depend continuously on \( \alpha, \beta, \) and \( \gamma \) and this completes the proof of lemma. \( \blacksquare \)

**Observe that**
\[
\begin{align*}
1 + \overline{u} + \overline{v} + \beta \overline{v} + \beta \overline{u} \overline{v} + \beta \overline{v}^2 & = \frac{\alpha \overline{w} + \alpha \overline{u} \overline{w} + \alpha \beta \overline{v} \overline{w}}{\alpha \overline{u} \overline{w}}, \\
1 + \overline{u} + \overline{v} + \beta \overline{v} + \beta \overline{u} \overline{v} + \beta \overline{v}^2 & = \frac{\alpha \overline{w} + \alpha \beta \overline{u} \overline{w}}{\alpha \overline{u} \overline{w}} + 1,
\end{align*}
\]
that is
\[
\beta \gamma \overline{u} \overline{v} > 1.
\]
Similarly
\[
\alpha \beta \overline{u} \overline{w} > 1, \quad \alpha \beta \overline{u} \overline{w} > 1. \tag{17}
\]

\[\tag{16}\]
The equilibrium theorem 2.1. The equilibrium \( E = (\bar{u}, \bar{v}, \bar{w}) \) of system (10) is a nonhyperbolic point with two complex conjugate roots on the unit circle and the third root equal to -1.

Proof. System (10) can be written in the form
\[
\begin{align*}
  u_{l+1} &= f(u_l, v_l, w_l), \\
  v_{l+1} &= g(u_l, v_l, w_l), \\
  w_{l+1} &= h(u_l, v_l, w_l), \quad l = 0, 1, \ldots
\end{align*}
\]
where
\[
\begin{align*}
  f(u, v, w) &= 1 + u + v + \beta v + \beta uv + \beta v^2 \\
  + \alpha w + \alpha u w + \alpha \beta v w \\
  &\quad \div \alpha \beta \gamma u v w, \\
  g(u, v, w) &= 1 + u + v + \alpha w + \alpha u w \\
  &\quad \div \alpha \beta v w, \\
  h(u, v, w) &= 1 + u + v \\
  &\quad \div \alpha w.
\end{align*}
\]

The characteristic equation of the Jacobian matrix
\[
\lambda^3 - r \lambda^2 - s \lambda - t = 0
\]
where
\[
\begin{align*}
  r &= f_u + g_v + h_w, \\
  s &= g_u f_v - f_u h_w + h_u f_w - g_v h_w + g_w h_v - f_u g_v,
\end{align*}
\]
and
\[
\begin{align*}
  t &= f_u g_v h_w - h_u f_w g_v - g_u f_w h_v \\
  &\quad - f_u g_w h_v + h_u f_v g_w + g_u f_v h_w.
\end{align*}
\]
Clearly \( f_u (\bar{u}, \bar{v}, \bar{w}) < 0, g_u (\bar{u}, \bar{v}, \bar{w}) < 0, \) and \( h_u (\bar{u}, \bar{v}, \bar{w}) < 0, \) which implies that \( r < 0. \) Furthermore,
\[
\begin{align*}
  r &= \left( -1 + \frac{1 + \beta \bar{v} + \alpha \bar{w}}{\alpha \beta \gamma \bar{u} \bar{v} \bar{w}} \right) + (-1) + \left( \frac{1}{\alpha \beta \bar{v} \bar{w}} - 1 \right) \\
  &= -3 + \frac{1 + \beta \bar{v} + \alpha \bar{w}}{\alpha \beta \gamma \bar{u} \bar{v} \bar{w}} + \frac{1}{\alpha \beta \bar{v} \bar{w}} \\
  &> -3
\end{align*}
\]
and so
\[
-3 < r < 0.
\]
Now we also have that
\[
\begin{align*}
  s &= -\left( -1 + \frac{1 + \beta \bar{v} + \alpha \bar{w}}{\alpha \beta \gamma \bar{u} \bar{v} \bar{w}} \right) - \left( -1 + \frac{1}{\alpha \beta \bar{v} \bar{w}} \right) \frac{1}{\alpha \bar{v}} \\
  &\quad + \frac{1 + \alpha \bar{w}}{\alpha \beta \bar{v} \bar{w}} \left( \frac{1}{\alpha \bar{v} \bar{w}} - \frac{1}{\bar{u}} + \frac{1}{\alpha \beta \bar{u} \bar{v} \bar{w}} \right) \\
  &\quad - \frac{1 + \beta \bar{v}}{\beta \bar{v}} \frac{1}{\alpha \bar{w}} \\
  &= -2 + 2 \frac{1 + \beta \bar{v} + \alpha \bar{w}}{\alpha \beta \bar{u} \bar{v} \bar{w}} + \frac{1}{\alpha \beta \bar{v} \bar{w}} - \frac{1 + \beta \bar{v} + \alpha \bar{w}}{\alpha \beta \bar{v} \bar{w}} - \frac{1 + \beta \bar{v}}{\alpha \beta \bar{v} \bar{w}} \\
  &\quad - \frac{1 + \alpha \bar{w}}{\alpha \beta \bar{u} \bar{v} \bar{w}} + \frac{1 + \alpha \bar{w}}{\alpha \beta \bar{u} \bar{v} \bar{w}} - \frac{1 + \beta \bar{v}}{\alpha \beta \bar{u} \bar{v} \bar{w}} \\
  &= -3 + \frac{1 + \beta \bar{v} + \alpha \bar{w}}{\alpha \beta \bar{u} \bar{v} \bar{w}} + \frac{1}{\alpha \beta \bar{v} \bar{w}}
\end{align*}
\]
\[
= r
\]
Theorem 3.1. Equation (4) has an invariant of the form:

\[
I(x_n, x_{n-1}, x_{n-2}) = \left(1 + \frac{1}{p_n x_{n-1}}\right) \left(1 + \frac{1}{p_{n-2} x_{n-1}}\right) \\
\times \left(1 + \frac{1}{p_{n-1} x_n}\right) (1 + x_n + x_{n-1} + x_{n-2})
\]

for \(n = 0, 1, \ldots\).

Proof. By taking into account that \(p_n\) is a periodic sequence of period 3, we get that:

\[
I(x_{n+1}, x_n, x_{n-1}) = \left(1 + \frac{1}{p_{n+1} x_{n-1}}\right) \left(1 + \frac{1}{p_n x_n}\right) \times \left(1 + \frac{1}{p_{n-1} x_n}\right) (1 + x_n + x_{n-1} + x_{n-2})
\]

\[
= \left(1 + \frac{1}{p_{n+1} x_{n-1}}\right) \left(1 + \frac{1}{p_n x_n}\right) \left(1 + \frac{1}{p_{n-1} x_n}\right) (1 + x_n + x_{n-1} + x_{n-2})
\]

\[
= I(x_n, x_{n-1}, x_{n-2}).
\]

Similarly, we can show that the corresponding system has an invariant of the form:

\[
I(u_l, v_l, w_l) = \left(1 + \frac{1}{\alpha w_l}\right) \left(1 + \frac{1}{\beta v_l}\right) \left(1 + \frac{1}{\gamma u_l}\right) \\
\times (1 + u_l + v_l + w_l) \quad \text{for } l = 0, 1, \ldots.
\]

By using this invariant we can obtain the following result.

Theorem 3.2. The period-three solution of Eq. (4) is stable. The Lyapunov function for Eq. (4) has the form:

\[
V(x, y, z) = I(x, y, z) - I(\bar{u}, \bar{v}, \bar{w}).
\]
Proof. First, we will find the minimizer of $I(x, y, z)$. The necessary conditions for the existence of critical points gives:

$$\frac{\partial I}{\partial x} = \left(1 + \frac{1}{\alpha z}\right) \left(1 + \frac{1}{\beta y}\right) \left(1 - \frac{1 + y + z}{\gamma x^2}\right) = 0,$$

$$\frac{\partial I}{\partial y} = \left(1 + \frac{1}{\alpha z}\right) \left(1 + \frac{1}{\gamma x}\right) \left(1 - \frac{1 + x + z}{\beta y^2}\right) = 0,$$

and

$$\frac{\partial I}{\partial z} = \left(1 + \frac{1}{\beta y}\right) \left(1 + \frac{1}{\gamma x}\right) \left(1 - \frac{1 + x + y}{\alpha z^2}\right) = 0,$$

which is equivalent to

$$\gamma x^2 = 1 + y + z,$$

$$\beta y^2 = 1 + x + z,$$

$$\alpha z^2 = 1 + x + y.$$

Thus we conclude that the only positive critical point of $I(x, y, z)$ is the equilibrium point $E = (\bar{u}, \bar{v}, \bar{w})$.

Next, the Hessian matrix evaluated at the equilibrium point $E = (\bar{u}, \bar{v}, \bar{w})$ has the form:

$$H = \begin{pmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{12} & a_{22} & a_{23} \\
  a_{13} & a_{23} & a_{33}
\end{pmatrix}$$

where

$$a_{11} = \frac{\partial^2 I}{\partial x^2} (\bar{u}, \bar{v}, \bar{w}) = \frac{2}{\bar{w}} \left(1 + \frac{1}{\alpha \bar{w}}\right) \left(1 + \frac{1}{\beta \bar{v}}\right) = \frac{2(1 + \alpha \bar{w})^2}{\alpha \beta \bar{w} \bar{v}},$$

$$a_{12} = \frac{\partial^2 I}{\partial x \partial y} (\bar{u}, \bar{v}, \bar{w}) = \left(1 + \frac{1}{\alpha \bar{w}}\right) \left(1 + \frac{1}{\gamma \bar{u}}\right),$$

$$a_{13} = \frac{\partial^2 I}{\partial x \partial z} (\bar{u}, \bar{v}, \bar{w}) = \left(1 + \frac{1}{\beta \bar{v}}\right) \left(1 + \frac{1}{\gamma \bar{u}}\right),$$

$$a_{22} = \frac{\partial^2 I}{\partial y^2} (\bar{u}, \bar{v}, \bar{w}) = \frac{2}{\bar{v}} \left(1 + \frac{1}{\alpha \bar{w}}\right) \left(1 + \frac{1}{\gamma \bar{u}}\right) = \frac{2(1 + \alpha \bar{w})^2}{\alpha \beta \bar{w} \bar{v}},$$

$$a_{23} = \frac{\partial^2 I}{\partial y \partial z} (\bar{u}, \bar{v}, \bar{w}) = \frac{2}{\bar{w}} \left(1 + \frac{1}{\beta \bar{v}}\right) \left(1 + \frac{1}{\gamma \bar{u}}\right) = \frac{2(1 + \alpha \bar{w})^2}{\beta \gamma \bar{w} \bar{v}},$$

$$a_{33} = \frac{\partial^2 I}{\partial z^2} (\bar{u}, \bar{v}, \bar{w}) = \frac{2}{\bar{w}} \left(1 + \frac{1}{\beta \bar{v}}\right) \left(1 + \frac{1}{\gamma \bar{u}}\right) = \frac{2(1 + \alpha \bar{w})^2}{\beta \gamma \bar{w} \bar{v}},$$

Next we will show that the principal minors of the Hessian matrix are positive. Clearly

$$A_{11} = \det(a_{11}) = \frac{2(1 + \alpha \bar{w})^2}{\alpha \beta \bar{w} \bar{v}} > 0$$

and in view of (16)

$$A_{22} = \frac{4(1 + \alpha \bar{w})^4}{\alpha^2 \beta \bar{w}^3 \bar{v}} - \frac{(1 + \alpha \bar{w})^4}{\alpha^2 \beta \bar{w}^2 \bar{v}^2} = \frac{(1 + \alpha \bar{w})^4}{\alpha^2 \beta \bar{w}^2 \bar{v}^2} \left[4 - \frac{1}{\beta \gamma \bar{u}}\right] > 0.$$

Finally we have

$$\det H = A_{33} = a_{11}a_{22}a_{33} + 2a_{12}a_{13}a_{23} - a_{11}a_{23}^2 - a_{12}a_{23}^2 - a_{13}a_{23}^2.$$

Thus

$$A_{33} = 8 \frac{(1 + \alpha \bar{w})^6}{\alpha^2 \beta^2 \gamma \bar{w}^2 \bar{v}^2} - 2 \frac{(1 + \alpha \bar{w})^6}{\alpha^2 \beta \bar{w} \bar{v}^3} - 2 \frac{(1 + \alpha \bar{w})^6}{\alpha^3 \beta^2 \gamma \bar{w}^2 \bar{v}},$$

$$-2 \frac{(1 + \alpha \bar{w})^6}{\alpha^2 \beta \bar{w} \bar{v}^3} - 2 \frac{(1 + \alpha \bar{w})^6}{\alpha^3 \beta^2 \gamma \bar{w}^2 \bar{v}},$$

$$= 2 \frac{(1 + \alpha \bar{w})^6}{\alpha^2 \beta \bar{w} \bar{v}^3} \left(4 - \frac{1}{\alpha \beta \bar{w}}\right) - \frac{1}{\alpha \beta \bar{w}} \left(\bar{w} \bar{v} - \frac{1}{\alpha \gamma \bar{w}}\right).$$
\[ \begin{align*}
&= 2 \left( 1 + \frac{\alpha \nu}{\omega} \right)^6 \left[ \left( \frac{\nu}{\alpha \beta^2 \gamma^2 \nu^6} - \frac{1}{\alpha \beta \gamma} \right) + \left( \frac{\nu}{\alpha \beta \gamma^2} - \frac{1}{\alpha \beta} \right) \right] \\
&\quad + \left[ \left( \frac{\nu}{\alpha \beta^2 \gamma^2 \nu^6} - \frac{1}{\alpha \beta \gamma} \right) + \left( \frac{\nu}{\alpha \beta \gamma^2} - \frac{1}{\alpha \beta} \right) \right] \\
&\quad + \left( \frac{\nu (\alpha \beta^2 \gamma^2 \nu - 1)}{\beta \gamma \nu} + \frac{\alpha \beta \gamma \nu - 1}{\alpha \gamma \nu} \right) \\
&\quad + \left( \frac{\nu (\alpha \beta^2 \gamma^2 \nu - 1)}{\beta \gamma \nu} + \frac{\alpha \beta \gamma \nu - 1}{\alpha \gamma \nu} \right) \\
\end{align*} \]

and in view of (16)–(19) is positive.

Therefore the Hessian matrix \( H \) is positive definite at \( E \), which implies that the invariant \( I(x, y, z) \) attains an isolated minimum at \( E = (\overline{w}, \overline{v}, \overline{w}) \) that is

\[ \min \{ I(x, y, z) : (x, y, z) \in (0, \infty)^3 \} = I(\overline{w}, \overline{v}, \overline{w}). \]

By Theorem 1.1 there exists a Lyapunov function

\[ V(x, y, z) = I(x, y, z) - I(\overline{w}, \overline{v}, \overline{w}) \]

which shows that the equilibrium \( E = (\overline{w}, \overline{v}, \overline{w}) \) of system (10) is stable. In other words, the period-three solution of Eq. (4) is stable. \( \blacksquare \)

4. Higher Order Generalization

We now establish the stability of the period-\( k \) solution of the \( k \)th order Lyness’ equation (3), where \( k = 2, 3, \ldots \) and \( p_n \) is a periodic coefficient of period \( k \), that is:

\[ p_n = \begin{cases} 
  p_{k-1} & \text{for } n = kl \\
  p_{k-2} & \text{for } n = kl + 1 \\
  \vdots \\
  p_1 & \text{for } n = kl + k - 2 \\
  p_0 & \text{for } n = kl + k - 1, \quad l = 0, 1, \ldots
\end{cases} \]

Equation (3) can be reduced to the following system of equations:

\[ \begin{align*}
x_{kn+1} &= 1 + x_{kn} + x_{kn-1} + \cdots + x_{kn-k+2}, \\
x_{kn+2} &= 1 + x_{kn+1} + x_{kn} + \cdots + x_{kn+k-3}, \\
&\vdots \\
x_{kn+k-1} &= 1 + x_{kn+k-2} + x_{kn+k-3} + \cdots + x_{kn}, \\
x_{kn+k} &= 1 + x_{kn+k-1} + x_{kn+k-2} + \cdots + x_{kn+1},
\end{align*} \]

By setting

\[ w_n^0 = x_{kn}, \quad w_n^1 = x_{kn-1}, \ldots, w_n^{k-1} = x_{kn-k+1} \]

we obtain the following system of difference equations

\[ \begin{align*}
  w_{n+1}^{k-1} &= 1 + w_n^0 + w_n^1 + \cdots + w_n^{k-2}, \\
  w_{n+1}^{k-2} &= 1 + w_n^{k-1} + w_n^0 + \cdots + w_n^{k-3}, \\
  &\vdots \\
  w_{n+1}^0 &= 1 + w_n^1 + w_n^2 + \cdots + w_n^{k-1}.
\end{align*} \]

We assume that at least two of the parameters \( p_i, i = 0, 1, \ldots, k - 1 \) are not equal.

The equilibrium equations of system (25) have the form:

\[ \begin{align*}
p_{k-1}(\overline{w}^{k-1})^2 &= 1 + \overline{w}^0 + \overline{w}^1 + \cdots + \overline{w}^{k-2}, \\
p_{k-2}(\overline{w}^{k-2})^2 &= 1 + \overline{w}^{k-1} + \overline{w}^0 + \cdots + \overline{w}^{k-3}, \\
&\vdots \\
p_1(\overline{w})^2 &= 1 + \overline{w}^0 + \overline{w}^1 + \cdots + \overline{w}^k, \\
p_0(\overline{w})^2 &= 1 + \overline{w}^1 + \overline{w}^2 + \cdots + \overline{w}^k.
\end{align*} \]

Equations (26) imply

\[ \overline{w}^i(1 + p_i \overline{w}^i) = 1 + \overline{w}^0 + \overline{w}^1 + \cdots + \overline{w}^{k-1}, \quad i = 0, 1, \ldots, k - 1. \]

By using a similar technique as in Lemma 2.1, we can prove that system (26) has a unique positive solution defined for all \( p_0, p_1, \ldots, p_{k-1} \).

Our next result gives the existence of an invariant for Eq. (3).

**Theorem 4.1.** Equation (3) has an invariant of the form:

\[ I(x_n, x_{n-1}, \ldots, x_{n-k+1}) = \left( 1 + \frac{1}{p_n x_{n-k+1}} \right) \left( 1 + \frac{1}{p_{n-k+1} x_{n-k+2}} \right) \cdots \times \left( 1 + \frac{1}{p_{n-1} x_n} \right) (1 + x_n + \cdots + x_{n-k+1}) \]

for \( n = 0, 1, \ldots \) and \( k = 2, 3, \ldots \).
Proof. Observe that \( p_n \) is a periodic sequence of period \( k \):
\[
I(x_{n+1}, x_n, \ldots, x_{n+k-2})
= \left(1 + \frac{1}{p_{n+1}x_{n+k-2}}\right) \cdots \left(1 + \frac{1}{p_n x_{n+1}}\right)
\times \left(1 + x_n + \cdots + x_{n-k+2}\right)
\times \left(1 + x_n + \cdots + x_{n-k+2}\right)
= \left(1 + \frac{1}{p_{n-k+1}x_{n+k-2}}\right) \cdots \left(1 + \frac{1}{p_{n-1} x_n}\right)
\times \left(1 + x_n + \cdots + x_{n-k+2}\right)
\times (1 + x_n + \cdots + x_{n-k+1})
\left(1 + \frac{1}{p_{n-k+1}x_{n-k+1}}\right)
= I(x_n, x_{n-1}, \ldots, x_{n-k+1}).
\]

Now by using this invariant we can present the following result.

**Theorem 4.2.** For \( k = 2, 3, \ldots \) the period-\( k \) solution of Eq. (3) is stable. The Lyapunov function for Eq. (3) has the form:
\[
V(z_0, z_1, \ldots, z_{k-1}) = I(z_0, z_1, \ldots, z_{k-1})
- I(\bar{z}_0, \bar{z}_1, \ldots, \bar{z}_{k-1}).
\]

**Proof.** First, we will find the minimizer of \( I(z_0, z_1, \ldots, z_{k-1}) \). The necessary conditions for the existence of critical points gives:
\[
\frac{\partial I}{\partial z_i} = \frac{1}{p_i z_i} \prod_{j=0}^{k-1} \left(1 + \frac{1}{p_j z_j}\right) [z_i(1 + p_i z_i)
- (1 + z_0 + z_1 + \cdots + z_{k-1})]
= 0
\]
for \( i = 0, 1, \ldots, k - 1 \) and so
\[
z_i(1 + p_i z_i) - (1 + z_0 + z_1 + \cdots + z_{k-1}) = 0,
\]
\( i = 0, 1, \ldots, k - 1 \). (29)

Thus the critical point of \( I \) satisfies the equilibrium equation (27), and this implies that the critical point is exactly the equilibrium point \((\bar{w}^0, \ldots, \bar{w}^{k-1})\).

Now we check the Hessian matrix at the positive critical point \((\bar{w}^0, \ldots, \bar{w}^{k-1})\). The second order derivatives of \( I \) with respect to \( z_i \) are
\[
\frac{\partial^2 I}{\partial z_i^2} = \frac{2 [z_0(1 + p_0 z_0)]^{k-1}}{z_i} \prod_{j=0}^{k-1} \left(p_j z_j^2\right)
\]
and
\[
\frac{\partial^2 I}{\partial z_i \partial z_l} = -\frac{[z_0(1 + p_0 \bar{z}_0)]^{k-1}}{z_i} \prod_{j=0}^{k-1} \left(p_j z_j^2\right)
\]
for \( i, l = 0, 1, \ldots, k - 1, i \neq l \). Let \( \bar{w}^i = \bar{z}_i \) for \( i = 0, \ldots, k - 1 \). Then the second order derivatives evaluated at the equilibrium point \((\bar{w}^0, \ldots, \bar{w}^{k-1})\) are
\[
\frac{\partial^2 I}{\partial z_i^2}(\bar{w}^0, \ldots, \bar{w}^{k-1}) = 2p_i \bar{z}_i B
\]
and
\[
\frac{\partial^2 I}{\partial z_i \partial z_l}(\bar{w}^0, \ldots, \bar{w}^{k-1}) = -B
\]
where
\[
B = \frac{[\bar{w}^0(1 + p_0 \bar{w}^0)]^{k-1}}{\prod_{j=0}^{k-1} p_j \bar{z}_j^2}.
\]

Thus the Hessian matrix takes the form
\[
H_k = \begin{pmatrix}
2p_0 \bar{w}_0 B & -B & \cdots & -B
-2p_0 \bar{z}_0 B & 2p_1 \bar{z}_1 B & \cdots & -B
\vdots & \vdots & \ddots & \vdots
-2p_{k-1} \bar{w}_{k-1} B & \cdots & -B & 2p_{k-1} \bar{z}_{k-1} B
\end{pmatrix}.
\]

Now for \( m = 1, 2, \ldots, k \), the \( m \)th order principal minor \( H_m \) of the Hessian matrix is given by:
\[
\det H_1 = 2p_0 \bar{w}_0 B > 0,
\]
\[
\det H_2 = B^2(4p_0 p_1 \bar{w}_0 \bar{z}_1 - 1) > 0,
\]
and
\[
\det H_m = B^m \cdot Q_m
\]
where
\[
Q_m = \begin{vmatrix}
2p_0 \bar{w}_0 & -1 & \cdots & -1 & -1
-1 & 2p_1 \bar{z}_1 & \cdots & -1 & -1
\vdots & \vdots & \ddots & \vdots & \vdots
-1 & -1 & \cdots & 2p_{m-2} \bar{w}_{m-2} & -1
-1 & -1 & \cdots & -1 & 2p_{m-1} \bar{z}_{m-1}
\end{vmatrix}.
\]

By manipulating the rows of \( Q_m \) we get that
\[
Q_m = (1 + 2p_{m-1} \bar{w}_{m-1})Q_{m-1}
+ (1 + 2p_{m-2} \bar{w}_{m-2})T_{m-1}
\]
where
\[ T_{m-1} = \begin{vmatrix} 2p_0z_0 & -1 & \ldots & -1 & -1 \\ -1 & 2p_1z_1 & \ldots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \ldots & 2p_{m-3}z_{m-3} & -1 \\ -1 & -1 & \ldots & -1 & -1 \end{vmatrix} = -\prod_{i=0}^{m-3} (1 + 2p_i z_i). \]

Thus for \( m = 2, 3, \ldots \), \( Q_m \) satisfies the following first order difference equation:
\[ Q_m = (1 + 2p_{m-1}z_{m-1})Q_{m-1} - \prod_{i=0}^{m-2} (1 + 2p_i z_i), \]

where \( Q_1 = 2p_0z_0 \) and whose solution is
\[ Q_m = \prod_{i=0}^{m-1} (1 + 2p_i z_i) \left[ 1 - \sum_{j=0}^{m-1} \frac{1}{1 + 2p_j z_j} \right]. \]

Hence
\[ \det H_m = B^m \prod_{i=0}^{m-1} (1 + 2p_i z_i) = \prod_{i=0}^{m-1} \frac{\bar{z}_i}{z_i + 2p_i \bar{z}_i} > 0. \]

Indeed by using (27) we obtain:
\[
\begin{align*}
\sum_{i=0}^{m-1} \frac{1}{1 + 2p_i z_i} &= \sum_{i=0}^{m-1} \frac{\bar{z}_i}{z_i + 2p_i \bar{z}_i} \\
&= \sum_{i=0}^{m-1} \frac{\bar{z}_i}{z_i + 2p_i \bar{z}_i + \bar{z}_i - \bar{z}_i} \\
&= \sum_{i=0}^{m-1} \frac{\bar{z}_i}{2(\bar{z}_i + p_i z_i)} - \frac{\bar{z}_i}{z_i} \\
&= \sum_{i=0}^{m-1} \frac{\bar{z}_i}{2(1 + \bar{z}_0 + \cdots + \bar{z}_{m-1}) - \bar{z}_i} - (1 + \bar{z}_0 + \cdots + \bar{z}_{m-1}) \\
&< \sum_{i=0}^{m-1} \frac{\bar{z}_i}{1 + \bar{z}_0 + \cdots + \bar{z}_{m-1}}.
\end{align*}
\]

Thus, the Hessian matrix \( H_k \) is positive definite, and so the invariant \( I \) attains an isolated minimum at the equilibrium point \((z_0, \ldots, z_{k-1})\).

Thus,
\[
\begin{align*}
\min \{ I(z_0, z_1, \ldots, z_{k-1}) : (z_0, z_1, \ldots, z_{k-1}) \in (0, \infty)^k \} &= I(z_0, \ldots, z_{k-1}) \\
&= \frac{\bar{z}_0 + \cdots + \bar{z}_{m-1}}{1 + \bar{z}_0 + \cdots + \bar{z}_{m-1}} < 1.
\end{align*}
\]

Fig. 1. The bifurcation diagram of the solution of Eq. (4) for \( \beta = 1.5, \gamma = 2 \) and \( \alpha \in (0.01, 6) \) with the initial point \((2, 1, 4)\).

Fig. 2. The bifurcation diagram of the solution of Eq. (4) for \( \beta = 1.5, \gamma = 2 \) and the initial point \((2, 1, 4)\) focussed around the value \( \alpha = 3.8 \).
Note that, for \( k = 2 \) and \( k = 3 \) we get Lyness’ and Todd’s equations, respectively.

The dynamics of Todd’s equation are very rich [Cima et al., 2006] and we believe that the dynamics of Eq. (4) are even more complex. As an illustration we provide the following bifurcation diagrams in the region of parameter \( \alpha \) while \( \beta \) and \( \gamma \) are kept fixed (see Figs. 1 and 2). The plots have been obtained by using the software package Dynamica [Kulenović & Merino, 2002].

References


