Optimal Rate Allocation for Entropy-Coded Uniform Scalar Quantization of Dependent Sources in Nonbinary Hypothesis Testing

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Abstract—We propose a closed-form rate allocation scheme (RAS) for entropy-coded uniform scalar quantization of dependent sources in classification problems. The proposed RAS is applicable to nonbinary classification with piecewise monotonic unquantized Bayes decision boundaries. The RAS is also extended to joint compression and classification.

Index Terms—Rate allocation, entropy-coded uniform scalar quantization, nonbinary classification, joint compression and classification.

I. INTRODUCTION

HYPOTHESIS testing problems involving rate allocation (RA) to multiple information sources are encountered in distributed detection [1], [2] as well as signal compression [3], [4]. Commonly, the sources communicate their quantized observations to a central decision-maker and there is no intersource communication. In general, the RA problem (RAP) with the goal of directly minimizing the quantized Bayes error is intractable and finding the optimal rates under a total rate constraint requires exhaustive search of the rate space. Ali-Silvey distances (ASD’s) [5] measuring the separation between the source distributions have been proposed as a manageable surrogate to the Bayes error to address the RAP. Yu and Varshney [1] proposed a RAS for uniform scalar quantization (USQ) of sources for a subclass of ASD’s having the additive property. Using Poor’s approximation of loss in ASD’s due to USQ [6], Jana and Moulin [3] and Tabesh et al. [4] derived a reverse water-filing solution to the RAP for high-rate entropy-coded USQ (ECUSQ) in the context of transformation and wavelet image coder design, respectively. The ASD’s considered were the Chernoff and Bhattacharya distances (BD). Unfortunately, closed-form RAS’s based on ASD’s are only possible when the observations are independent. For dependent sources, iterative search strategies in the rate space are needed and may lead to local optima.

Hu and Blum [2] considered the general non-uniform scalar quantization problem for dependent sources under nonbinary hypothesis testing. They showed that there is a maximum rate for each source beyond which the overall quantized Bayes error is not improved. They used this result to eliminate some of the candidates in the exhaustive search for optimal integer rates.

This paper presents a RAS for ECUSQ based on a new upper bound on the quantized Bayes error. The proposed scheme has a closed form for dependent sources and nonbinary hypothesis testing, and is applicable to piecewise monotonic unquantized Bayes decision boundaries. The RAS is also extended to joint compression and classification. We consider continuous rates similar to [1], [3], [4].

II. NEW UPPER BOUND ON QUANTIZED BAYES ERROR

Consider the classification problem shown in Fig. 1. The observations are uniformly quantized with the quantizer cell boundaries at \(-3, -2, \ldots, 6\). Let \(\delta^0\) and \(\delta^\Delta\) denote the unquantized and quantized Bayes decision boundaries, i.e., the optimal Bayes boundaries for unquantized and quantized observations, respectively. Note that \(\delta^\Delta\) is one of the boundaries of the cell \(\beta = [1, 2]\) which encompasses \(\delta^0\). Moreover, the observations that fall into quantizer cells other than \(\beta\) will be classified to the same class by both the unquantized and quantized Bayes classifiers. Thus, quantization of such observations does not result in additional misclassifications, whereas quantizing observations falling inside \(\beta\) may incur additional misclassifications. The situation is similar in higher dimensional (vector) quantizers, but \(\delta^0\) may pass through multiple quantizer cells.

Let \(P^\varepsilon_0\) and \(P^\varepsilon_\Delta\) denote the unquantized and quantized Bayes errors, respectively. We can write the above observation as

\[
P^\varepsilon_\Delta = P^\varepsilon_0 + \sum_{i \in B} \varepsilon_i \tag{1}
\]

where \(B\) is the set of quantizer cells that encompass \(\delta^0\), and \(\varepsilon_i\) is the contribution of quantizer cell \(i\) to \(P^\varepsilon_{\Delta}\), where \(i \in B\). We can further write

\[
P^\varepsilon_\Delta \leq P^\varepsilon_0 + \frac{\varepsilon_i}{V_i} \sum_{i \in B} V_i \tag{2}
\]

where \(V_i\) is the volume of the quantizer cell \(i\), and

\[i^* = \arg \max_{i \in B} \varepsilon_i / V_i\]

denotes the index of the cell with the largest \(\varepsilon_i/V_i\) value. For USQ, we have \(V_i = V, \forall i \in B\). Thus, (2) simplifies to

\[
P^\varepsilon_\Delta \leq P^\varepsilon_0 + N \varepsilon^* \tag{3}
\]

where \(N\) denotes the number of quantizer cells that encompass \(\delta^0\), and \(\varepsilon^* = \varepsilon_{i^*}\). Equation (3) holds for nonbinary problems as well. In that case, \(B\) denotes the set of quantizer cells that encompass all segments of \(\delta^0\) between all classes.
In order for (3) to be beneficial for developing a RAS, we need \( N \) to be the only parameter that is dependent on the allocated rates. However, \( \varepsilon^{*} \) also varies with the particular choice of rates. This dependency problem can be addressed by replacing \( \varepsilon^{*} \) with

\[
\varepsilon^{**} = \sup_{\Delta} \varepsilon^{*}(\Delta),
\]

where \( \varepsilon^{*}(\Delta) \) corresponds to a set of quantizer step sizes \( \Delta = [\Delta_1, ..., \Delta_S] \) satisfying the rate constraint

\[
\sum_{s=1}^{S} \log \Delta_s \geq K
\]

and \( S \) is the number of sources (see (7) below).

The bound in (3) with the constant \( \varepsilon^{**} \) may exceed 1 as \( \varepsilon^{**} \) may become large. This is due to two factors: large \( \varepsilon^{*} \) for a given quantizer, and/or the large disparity between \( \varepsilon^{*} \) across the quantizers to be compared. The impact of the first factor may be controlled as follows. Let \( B' \subset B \) such that \( N(B')\varepsilon^{*}(B') - \sum_{i \in B} \varepsilon_i < \eta \) where \( \eta \) is a parameter. Note that \( N \) and \( \varepsilon^{*} \) are now defined with respect to \( B' \) rather than \( B \). However, it is reasonable to choose \( B' \) such that \( \varepsilon^{*}(B') = \varepsilon^{*}(B) \). Thus, (3) may be modified as

\[
P_{\varepsilon}(B') \leq P_{\varepsilon}^{0}(B') + N(B')\varepsilon^{*}(B'),
\]

where \( P_{\varepsilon}^{0}(B') \) and \( P_{\varepsilon}(B') \) are the unquantized and quantized Bayes errors evaluated over \( B' \). Note that (5) bounds the classification error for a subset of quantizer cells \( B' \) that encompass \( \delta^{0} \) and is ensured to not exceed \( P_{\varepsilon}^{0}(B') + \eta \). The challenge in using (5) for comparing classifiers is that \( B' \) as defined above depends on the particular choice of the quantizer. We propose instead to use \( B' = B \cap \Omega \) where \( \Omega \) is a bounded region of the source space that encompasses a large portion of the class-conditional densities of the classes. While this choice does not guarantee a specific degree of tightness in the bound, it does guarantee a tighter bound than (3) by making \( N(B')\varepsilon^{*}(B') - \sum_{i \in B'} \varepsilon_i \) smaller. In the following sections, \( \Omega \) is assumed to be a rectangle for two sources and a hypercube for multiple sources, but in general it may have any other shape.

The second factor mentioned above may be addressed by noting that \( \varepsilon^{*} \) becomes larger when the quantization step sizes along sources with smaller contributions to \( P_{\varepsilon}^{\Delta} \) are larger. That is, the bound is looser for more suboptimal quantizers. Thus, the bound may be tightened by only considering a more viable subset of possible step sizes satisfying the rate constraint.

### III. THE RATE ALLOCATION SCHEME

Of the terms on the right-hand side of (5), we can control \( N \) via RA. In the following, we first derive the proposed RAS for two sources and then generalize the RAS to an arbitrary number of sources.

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**A. Two Sources and Linear Decision Boundary**

Consider the linear decision boundary

\[
\Psi(x) = \beta^T x = \beta^T (\mathbf{x}_1, \mathbf{x}_2) = b,
\]

where \( \Psi(x) \) is a decision boundary that separates the two classes. The challenge in using (5) for comparing classiﬁcation errors is that the bound may become large. This is due to two factors: large \( \varepsilon^{*} \) for a given quantizer, and/or the large disparity between \( \varepsilon^{*} \) across the quantizers to be compared. The impact of the first factor may be controlled as follows. Let \( B' \subset B \) such that \( N(B')\varepsilon^{*}(B') - \sum_{i \in B} \varepsilon_i < \eta \) where \( \eta \) is a parameter. Note that \( N \) and \( \varepsilon^{*} \) are now defined with respect to \( B' \) rather than \( B \). However, it is reasonable to choose \( B' \) such that \( \varepsilon^{*}(B') = \varepsilon^{*}(B) \). Thus, (3) may be modified as

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class $c_i$. Assuming $p_i(x) \sim N(\mu_i, I)$, $i = 0, 1$, where $\mu_0 = 0$, $\mu_1 = \mu$, and $I$ is the identity matrix, the method of Lagrange multipliers yields
\[ \frac{\Delta_2}{\Delta_1} = \frac{\mu_1}{\mu_2}, \]
which is identical to (8). Thus, in this case minimization of the BD loss and the new bound result in the same RAS.

B. Two Sources and Piecewise Monotonic $\delta^0$

We now consider nonlinear, monotonic $\delta^0$’s such as the one shown in Fig. 3(a). Let $(x_1, x_2)$ and $(y_1, y_2) \in \delta^0$. A boundary $\delta^0$ is monotonic if for all $y_1 > x_1$, $y_2$ always satisfies either $y_2 \geq x_2$ or $y_2 \leq x_2$. That is, the boundary is either completely non-increasing or non-decreasing. It is straightforward to see from Fig. 3(a) that the approximation of $N$ in (6) applies to any monotonic $\delta^0$ over a finite $\Omega$. Hence, the scheme in (8) is a minimizer of (7) for all such $\delta^0$’s.

Moreover, (6) can be generalized to $\delta^0$ consisting of $L$ monotonic segments. Fig. 3(b) shows an example $\delta^0$ having two segments. For a $\delta^0$ having $L$ monotonic segments,
\[ N \approx \sum_{i=1}^{L} \left[ \frac{U_{1i}}{\Delta_1} \right] + \left[ \frac{U_{2i}}{\Delta_2} \right] - 1 \approx \sum_{i=1}^{L} \frac{1}{\Delta_2} \sum_{i=1}^{L} U_{1i}, \]
where $U_{1i}$ denotes the support of the bounding box of segment $l$ for source $s$. The RAP can again be represented in a similar form to (7) and solved using the method of Lagrange multipliers to get
\[ \frac{\Delta_2}{\Delta_1} = \frac{\sum_i U_{2i}}{\sum_i U_{1i}}. \]

The right-hand side can be thought of as the “average” slope magnitude of the segments of $\delta^0$.

C. Multiple Sources

A key observation in the two-source case helps generalize the above development to an arbitrary number of sources. Note that in Fig. 2 and Fig. 3 all quantizer cells encompassing the monotonic decision boundary/segment have at least one side that directly faces one of the source axes without occlusion by other encompassing cells. That is, each cell has at least one unoccluded projection onto the source axes. Moreover, the unoccluded projections of the encompassing cells are mutually exclusive. In other words, each grid cell along the source axes corresponds to at most one encompassing cell. Thus, counting the number of grid cells along the source axes yields an upper bound on the number of cells within $\Omega$ that encompass $\delta^0$. This observation was the basis for the approximations in (6) and (10) and can be generalized to more than two sources as follows.

For $S > 2$ sources, let $X = [x_1, ..., x_i, ..., x_j, ..., x_S]^T$ and $Y = [y_1, ..., y_i, ..., y_j, ..., y_S]^T \in \delta^0$, where $(x_i, x_j)$ and $(y_i, y_j)$, $i \neq j$, are pairs of elements of $X$ and $Y$. Fixing all elements of $X$ and $Y$ except for $(x_i, x_j)$ and $(y_i, y_j)$, respectively, boundary $\delta^0$ is monotonic if for all $y_i \geq x_i$, $y_j$ always satisfies either $y_j \geq x_j$ or $y_j \leq x_j$, $\forall i, \forall j \neq i$. For a monotonic $\delta^0$, the encompassing cells have at least one unoccluded projection onto the sides of $\Omega$, which is an $S$-dimensional hypercube, and all the projections are mutually exclusive.

Applying the above observation to a monotonic $\delta^0$, we have
\[ N \approx \sum_{i=1}^{S} \frac{\Delta_i}{\sum_{s=1}^{S} \frac{\Delta_s}{U_s}}. \]

The above expression yields the total number of grid cells on the $S - 1$-dimensional sides of $\Omega$. Expressing the RAP as in (7), the solution is given by
\[ \frac{\Delta_i}{\Delta_j} = \frac{U_i}{U_j}, \quad \forall i \neq j. \]

This result is a generalization of (8). For a $\delta^0$ consisting of $L$ monotonic segments, we have
\[ N \approx \sum_{i=1}^{L} \left( \frac{U_{1i}}{\Delta_i} \right) \sum_{s=1}^{S} \frac{\Delta_s}{U_{1s}}. \]

Minimizing (14) subject to a rate constraint yields a generalization of (11) given by
\[ \frac{\Delta_i}{\Delta_j} = \frac{\sum_i \frac{U_{1i}}{\Delta_i}}{\sum_i \frac{U_{1i}}{\Delta_i}}, \quad \forall i \neq j. \]

D. Joint Compression and Classification

The objective in joint compression and classification is to achieve a trade-off between minimizing the (possibly conflicting) signal quality criteria for human observers and statistical decision-making systems. Here, we will use a joint distortion criterion consisting of a weighted combination of the weighted mean-square error (WMSE) as the fidelity criterion for human observation and the bound in (5) to characterize classification performance. The joint distortion criterion is given by
\[ J = (1 - \lambda) \sum_s w_s \text{MSE}_s + \lambda N, \]
where $\text{MSE}_s$ and $w_s$ are the mean-square error (MSE) and associated weight for source $s$. Parameter $\lambda \in [0, 1]$ controls the trade-off between the WMSE and the classification criterion. When $\lambda = 0$, $J$ reduces to the WMSE criterion, and when $\lambda = 1$, $J$ reduces to the classification criterion.
Assuming $\text{MSE}_n \approx \Delta_2^2/12$ [8], minimizing (16) subject to a rate constraint yields
\[
\frac{1 - \lambda}{6} w_i - \frac{1}{6} \Delta_2^2 + \left( \frac{\lambda}{\exp(K)} \sum_{i=1}^{L} \left( \prod_{k=1}^{L} U_{ik} \right) \frac{1}{U_{ij}} \right) \Delta_i = \frac{1 - \lambda}{6} w_j - \frac{1}{6} \Delta_2^2 + \left( \frac{\lambda}{\exp(K)} \sum_{i=1}^{L} \left( \prod_{k=1}^{L} U_{ik} \right) \frac{1}{U_{ij}} \right) \Delta_j \quad (17)
\]
for all $i \neq j$. The above quadratic equations must be solved iteratively to obtain the optimal rates.

IV. EXAMPLES

In this section, we show the efficacy of the proposed RAS through numerical examples. Note that while the proposed scheme is applicable to any number of sources, the examples demonstrate its utility for two sources. Moreover, while the joint compression and classification formulation allows for arbitrary weights in the WMS criterion, the corresponding example shows the scheme for equal weights.

A. Binary Hypothesis Testing in Dependent Gaussian Noise

Let $p_i(x) \sim N(\mu_i, C)$, $i = 0, 1$, where $\mu_0 = 0$, $\mu_1 = \mu = [2 \ 0.2]^T$, and
\[
C = \begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix}.
\]
In this case, $\delta^0$ is given by $\mu^T C^{-1} x + a = 0$, where $a$ is a constant. From (8), we have
\[
\frac{\Delta_2}{\Delta_1} = \frac{2 + 0.2r}{2r + 0.2}.
\]
Note that as $\delta^0$ is linear, any finite $\Omega$ will yield the RAS in (18).

Table I shows $P^\Delta_a$ for $K = 2$ and four choices of quantization step sizes: (a) $\Delta_1 = \Delta_2$, which minimizes the MSE in expressing $x$; (b) $\Delta_1 = (\mu_2/\mu_1) \Delta_2 = \Delta_2/10$, which minimizes the loss in SD assuming independent observations (cf. (9)); and (c) step sizes from (18). Two values $r = 0$, $-0.7$ were considered in the simulations, for which (18) yields $\Delta_1 = \Delta_2/10$ (identical to BD loss minimizer) and $\Delta_1 = (20/31) \Delta_2$, respectively. Noting that the choice of quantization grid placement affects $P^\Delta_a$, we considered shifting the grid locations in increments of $\Delta_2/100$ for source $s$. The table shows the average $P^\Delta_a$ taken over all shifted grids, i.e., when the optimal grid placement is not known, as well as the minimum $P^\Delta_a$. For $r = 0$, the proposed scheme offers the largest gain in the average $P^\Delta_a$. Moreover, the variations in $P^\Delta_a$ due to grid shifts are much larger for the MSE-minimizing scheme than for the proposed scheme. For $r = -0.7$, the proposed scheme offers a small improvement in average $P^\Delta_a$ over the BD loss-minimizing scheme. Additionally, the high-rate MSE due to the latter scheme (3.367) is much larger than that of the proposed scheme (0.7317), even though MSE is not explicitly considered in this example.

In this case, $\Delta_0$ is given by $\mu^T C^{-1} x + a = 0$, where $a$ is a constant. From (8), we have
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B. Three-Class Hypothesis Testing in Independent Gaussian Noise

Let $p_i(x) \sim N(\mu_i, I)$, $i = 0, 1, 2$, where $\mu_0 = -\mu_1 = [3 \ 0]^T$, $\mu_2 = [0 \ 1]^T$. In this case, $\delta^0$ consists of three line segments with slopes $3$, $-3$, and $-\infty$ with respect to the $x_1$-axis. Equation (11) yields $\Delta_1 = (2/9) \Delta_2$. This result holds for any bounded $\Omega$ centered at $[0 \ -4]^T$, the intersection of the three segments. Table II shows $P^\Delta_a$ for $\Delta_1 = \Delta_2$ and $\Delta_1 = (2/9) \Delta_2$, where the rate constraint is $K = 2$. As expected, the proposed scheme offers substantial improvements in classification accuracy.

![Fig. 3. (a) A monotonic and (b) a two-piece, piecewise monotonic $\delta^0$ for two sources and their encompassing quantizer cells marked in bold.](image-url)

| TABLE I QUANTIZED BAYES ERROR $P^\Delta_a$ FOR EXAMPLE A; (A) $r = 0$, $P^\Delta_a$ = 0.1574; (B) $r = -0.7$, $P^\Delta_a$ = 0.0666. |
|----------------------------------|-------------------|
| (A)                             | (B)               |
| average $P^\Delta_a$            | 0.1940            | 0.1447            |
| minimum $P^\Delta_a$            | 0.1583            | 0.1274            |
| average $P^\Delta_a$            | 0.1574            | 0.1396            |
| minimum $P^\Delta_a$            | 0.1350            | 0.1038            |

Note that as $\delta^0$ is linear, any finite $\Omega$ will yield the RAS in (18).

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C. Joint Compression and Classification in Independent Gaussian Noise

Consider the problem described in Example A with \( r = 0 \). The goal in this example is to minimize (16) with \( w_1 = w_2 = 1, U_1 = 0.2, \lambda = 0.05, 1 \), and \( K = 2 \). From the previous example, we have \( \Delta_1 = \Delta_2 \) and \( \Delta_1 = \Delta_2/10 \), for \( \lambda = 0.1 \), respectively. For \( \lambda = 0.5 \), the solution of (17) is found at \( \Delta_1 = 0.5 \Delta_2 \). Table III shows the MSE and \( P^\Delta_\varepsilon \) for the above choices of \( \lambda \). As expected, both the MSE and \( P^\Delta_\varepsilon \) for \( \lambda = 0.5 \) fall between the corresponding values for \( \lambda = 0.1 \). Note that achieving the lowest average \( P^\Delta_\varepsilon \) (\( \lambda = 1 \)) comes at a very high MSE cost.

D. Binary Hypothesis Testing with Unequal Class-Conditional Gaussian Distributions

Let \( p_i(x) \sim N(\mu_i, C_i), i = 0,1 \), where \( \mu_0 = -\mu_1 = [-1 \ 0]^T \), and

\[
C_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1.5 \end{bmatrix}.
\]

In this case, \( \delta^0 \) is a parabola consisting of two monotonic segments. For \( \Omega = \{(x, y) : x \in [-5, 5], y \in [-5, 5]\} \), we have \( \Pi = 0.9999 \). Using (11), we obtain \( \Delta_1 = 0.3964 \Delta_2 \). The \( P^\Delta_\varepsilon \) for \( \Delta_1 = \Delta_2 \) and \( \Delta_1 = 0.3964 \Delta_2 \), where \( K = 2 \), are shown in Table IV. As the table indicates, the proposed scheme improves the classification accuracy.

E. Binary Hypothesis Testing with Two-Component Gaussian Mixtures

Let \( p_i(x) \sim (1/2) N(\mu_{i1}, C_{i1}) + (1/2) N(\mu_{i2}, C_{i2}), i = 0,1 \), where \( \mu_{01} = -\mu_{11} = [-1 \ 0]^T, \mu_{02} = -\mu_{12} = [-1 \ 2]^T \),

\[
C_{i1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad C_{i2} = \begin{bmatrix} 2 & -1.4 \\ -1.4 & 2 \end{bmatrix}, i = 0,1,
\]

and let \( \Omega = \{(x, y) : x \in [-4, 4], y \in [-7, 7]\} \), corresponding to \( \Pi = 0.991 \). Within \( \Omega \), \( \delta^0 \) consists of one monotonic segment; thus (8) can be invoked to obtain \( \Delta_1 = 0.2253 \Delta_2 \). Table V shows \( P^\Delta_\varepsilon \) for \( \Delta_1 = \Delta_2 \) and \( \Delta_1 = 0.2253 \Delta_2 \), where \( K = 2 \), confirming the improved classification accuracy achievable using the proposed scheme.

V. Conclusions

In this paper, we presented a new upper bound on the quantized Bayes error and used it to devise a closed-form RAS for piecewise monotonic unquantized Bayes decision boundaries. The new bound was derived based on the observation that the difference between unquantized and quantized Bayes errors is only due to the misclassifications occurring in quantizer cells encompassing the unquantized Bayes decision boundary, and not other cells. Thus, the difference between the unquantized and quantized Bayes errors can be upper-bounded by the total volume of the quantizer cells that encompass the Bayes decision boundary times a constant.

The proposed RAS can handle problems involving non-binary hypothesis testing and/or dependent source observations and piecewise monotonic unquantized Bayes boundaries. For binary hypothesis testing with Gaussian class-conditional distributions having a common, diagonal covariance matrix, we showed that the proposed scheme is equivalent to that obtained from minimizing the BD loss [4]. We also showed that if the independence assumption does not hold, the new scheme offers a small improvement over the scheme based on minimizing the BD loss. Moreover, we showed that the proposed scheme allows for controlling the trade-off between minimizing the WMSE and classification performance. Finally, the new scheme was shown to improve the classification performance over the MSE-minimizing scheme in a nonbinary problem involving Gaussian class-conditional distributions as well as in binary problems involving Gaussian distributions with unequal covariance matrices and Gaussian mixtures.

References


