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A New Theory in Stiffness Control for Dextrous Manipulation

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Abstract

A new discovery on the stiffness control in robotics, as well as its applications in grasping and dextrous manipulation, is presented. Extended from the Conservative Congruence Transformation (CCT) theory, the new theory accounts for the change in geometry and the non-commutative rotational property of the Cartesian stiffness in grasping and dextrous manipulation using stiffness control when the external force is applied. The theory, with the consideration of rotational effect in SE(3), is developed by using the geometrical method. The method along with an example presented in this paper provides a systematic way of constructing 6×6 *Cartesian stiffness matrices in robotic grasping/manipulation and stiffness control for dextrous manipulation.*

1 Introduction

Robot dextrous hands with multifingers offer great advantages to change grasping configuration in response to changing task requirements. Like human hands, they manipulate the fingers and exert certain force/moment upon an object surface to ensure stable grasping and manipulation. In general, the planning strategies for multi-fingered hands include position and force controls along with sliding or rolling motion.

The stiffness/compliance control, a compromise between the force control and position control, is equally applicable at free workspace as well as at constrained manipulation. The fingers of a dextrous hand have a stiffness that depends on servoing and on the elasticity in the joint drive and in the fingertip materials if rigid links are assumed. Thus, by controlling the joint stiffness to approach the required Cartesian stiffness which depends on the contact condition of fingers is practical in dextrous manipulation. In addition, the stiffness matrices indicate how the force/moment vary as the arm or fingers move under dextrous manipulation, such as sliding and/or rolling. Stiffness control has been used with sliding manipulation in robotics, such as that in [9, 10, 11].

In this paper, a new theory of stiffness control for applications in dextrous manipulation is presented. We build upon the previous research results of the conservative congruence transformation (CCT)

$$\mathbf{J}_\theta^T \mathbf{K}_p \mathbf{J}_\theta = \mathbf{K}_\theta - \mathbf{K}_g \tag{1}$$

where $\mathbf{K}_p$ and $\mathbf{K}_\theta$ denote the Cartesian and joint stiffness matrices of robot manipulator, respectively. The $\mathbf{K}_g$ matrix defines the changes in geometry through the differential Jacobian matrix, and externally applied loads. The CCT represents the relationship of stiffness control between the linear $\mathcal{R}^{3\times3}$ Cartesian space at the fingertip and the joint space of a robot system at the joint space, is given by [2, 3]. In addition, the CCT corrects the well-known Salisbury's conventional formulation $\mathbf{J}_\theta^T \mathbf{K}_p \mathbf{J}_\theta = \mathbf{K}_\theta$ [16] as the correct and general relationship for congruence mapping of stiffness control in $\mathcal{R}^{3\times3}$.

However, the 6×6 Cartesian stiffness matrix of conservative systems with the consideration of linear and rotational motions in $SE(3)$ is quite different from the 3×3 linear stiffness matrix. The difference can be readily seen due to the well-known fact that the rotational components do not commute. This indicates that the Cartesian coordinates are not on a *coordinate basis* [17]. Unlike the formulation in equation (1) which expresses a direct mapping from the joint space to the Cartesian space, the 6×6 Cartesian stiffness of manipulators needs to be derived with the addition of an intermediate generalized coordinate basis between the joint and Cartesian spaces, using the geometrical methods with the changing basis strategy. Notice that the stiffness matrix associated with a *coordinate basis*, or a commutative basis, is always symmetric. Based on the introduction of the intermediate coordinate basis through certain coordinate transformation matrices under special configuration, the stiffness congruence transformation, the isomorphism of the CCT in equation (1), and the Cartesian stiffness matrix via this stiffness congruence transformation with the modified CCT are derived. Through the stiffness congruence transformation, the 6×6 Cartesian stiffness is shown to be an asymmetric matrix, referenced to both inertial (fixed) and moving (body) frames. In addition, the skew-symmetric part of the 6×6 stiffness matrix is shown to be the negative one-half of the externally applied load expressed in the spatial cross product matrix [4]. These results are consistent with the prior investigations presented in [7, 5, 8].

2 Theoretical Background

The set of $n \times n$ invertible matrices, $GL(n)$, is an algebraic group under the operation of matrix multiplication. A continuous manifold that is also an algebraic group is known as a *Lie group*, $SE(n)$, if the group product is continuous. The continuous motion of a rigid body is the parameterized set of linear transformations, $\mathbf{T}(t) : \mathcal{R} \rightarrow SE(n)$. These transformations form a special Euclidean group $SE(3)$ in three-dimensions. Thus, we can represent the trajectory of a rigid body motion by using rigid body transformation to describe the instantaneous position and orientation of a moving body coordinate frame B relative to an inertial frame O. This can be realized by the following 4×4 homogeneous transformation matrix [6]

$$\mathbf{T}(t) = \begin{bmatrix} \mathbf{R}(t) & \mathbf{p}(t) \\ \mathbf{0} & 1 \end{bmatrix} \tag{2}$$

where $\mathbf{R}(t)$ is a 3×3 rotation matrix with two key properties of $\mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{R}^T = \mathbf{I}$ and $\det(\mathbf{R}) = +1$, $\mathbf{p}(t)$ is a 3×1 displacement vector, and $\mathbf{0}$ is a 1×3 row vector of zeros.

2.1 Rigid body velocity in twist basis

The tangent vector to the trajectory of $\mathbf{T}$ in equation (2), $\frac{d\mathbf{T}}{dt}$, is the velocity of the rigid body. When we consider the left-invariant vector fields on this tangent group, it satisfies the following relation

$$\mathbf{T}^{-1}\dot{\mathbf{T}}(t) = \begin{bmatrix} \mathbf{\Omega}(t) & \mathbf{v}(t) \\ \mathbf{0} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_z & \omega_y & v_x \\ \omega_z & 0 & -\omega_x & v_y \\ -\omega_y & \omega_z & 0 & v_z \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3}$$

where $\mathbf{\Omega}(t) = \mathbf{R}^T\dot{\mathbf{R}}$ is a skew-symmetric matrix representing the 3× 3 angular velocity of the rigid body, and $\mathbf{v}(t) = \mathbf{R}^T\dot{\mathbf{p}}$ is the 3×1 linear velocity of the origin of the body (moving) coordinate frame B relative to the inertial (fixed) frame O, both viewed in the current body frame B [6, 12, 13, 18]. These elements are essentially Ball's screws [1], or rather the instantaneous screws. Therefore, $\mathbf{T}^{-1}\dot{\mathbf{T}}(t)$, a six-dimensional manifold, is the instantaneous twist in the body frame which belongs to *Lie Algebra* $se(3)$ [17]. Moreover, it does not depend on the choice of the inertial frame with the left-invariant translation. Namely, the original vector is translated by left-invariant vector fields to every point on the manifold. Thus, the twist, $\mathbf{T}^{-1}\dot{\mathbf{T}}(t)$, in the new inertial frame remains unchanged [8, 18]. On the other hand, the right-invariant vector fields on the tangent group, *i.e.,* $\dot{\mathbf{T}}(t)\mathbf{T}^{-1}$, indicate the spatial velocity relative to the inertial (fixed) coordinate frame [13]. Thus, the angular velocity $\mathbf{\Omega}(t)$ and the linear velocity $\mathbf{v}(t)$ are all relative to the inertial frame O for this right-invariant vector fields, $\dot{\mathbf{T}}(t)\mathbf{T}^{-1}$.

We can choose an ordered unit twist basis $\mathbf{S} = \{\mathbf{S}_1; \mathbf{S}_2; \mathbf{S}_3; \mathbf{S}_4; \mathbf{S}_5; \mathbf{S}_6\}$ which is identical to the derivative of $\mathbf{T}(t)$ at the origin $t = 0$, *i.e.,* $\mathbf{T}(0) = \mathbf{I}$, for the six-dimensional twist matrix group [17, 8, 18]. Subgroup $\{\mathbf{S}_1; \mathbf{S}_2; \mathbf{S}_3\}$ are related to linear translational motion, which commute with one another; but subgroup $\{\mathbf{S}_4; \mathbf{S}_5; \mathbf{S}_6\}$ are related to the rotational motion of rigid body and do not commute with one another [4]. Based on this twist basis, equation (3) can be rewritten with respect to the unit twist basis $\mathbf{S}$ as

$$\mathbf{T}^{-1}\dot{\mathbf{T}}(t) = s_1\mathbf{S}_1 + s_2\mathbf{S}_2 + s_3\mathbf{S}_3 + s_4\mathbf{S}_4 + s_5\mathbf{S}_5 + s_6\mathbf{S}_6 \tag{4}$$

where the twist coordinates $\mathbf{s} = [s_1, s_2, s_3, s_4, s_5, s_6]^T$ are identical to the rigid body velocity $\dot{\mathbf{x}} = [v_x, v_y, v_z, \omega_x, \omega_y, \omega_z]^T$ in the Cartesian space. That is, $[s_1, s_2, s_3]^T$ is related to the translational motion along the X, Y, and Z axes, respectively, and $[s_4, s_5, s_6]^T$ is related to the consecutive rotational motion about the same axes [4].

2.2 Coordinate and non-coordinate bases

We consider the trajectory of a rigid body, $\mathbf{T}(t)$, which is a set of points (a manifold) having a continuous one-to-one map onto an open set of independent generalized coordinates $(\xi_1, \xi_2, \xi_3, \xi_3, \xi_4, \xi_5, \xi_6)$. This integral curve $\mathbf{T}(t)$ describes the rigid body motion of the frame B attached to the body relative to a fixed or inertial frame O. Therefore, the directional derivatives along the curve, $\frac{d\mathbf{T}}{dt}$, form a vector space which can be differentiated by the chain rule as

$$\dot{\mathbf{T}} = \sum_{i=1}^{6} \dot{\xi}_i \frac{\partial \mathbf{T}}{\partial \xi_i} \tag{5}$$

where the derivations are with respect to the generalized coordinates $(\xi_1, \xi_2, \xi_3, \xi_3, \xi_4, \xi_5, \xi_6)$ and true for any function $\mathbf{T}(t)$. Equation (5) shows a linear combination of the particular derivatives $\frac{\partial \mathbf{T}}{\partial \xi_i}$ with the components $\dot{\xi}_i$, which is the rate of the coordinates $(\xi_1, \cdots, \xi_6)$. It follows that $\{\frac{\partial}{\partial \xi_i}\}$ is a differential operator for the coordinate basis for this vector space with the components $\dot{\xi} = [\dot{\xi}_1, \dot{\xi}_2, \dot{\xi}_3, \dot{\xi}_4, \dot{\xi}_5, \dot{\xi}_6]^T$. Thus, the set of basis vector, $\mathbf{E} = \{\frac{\partial \mathbf{T}}{\partial \xi_1}; \ldots; \frac{\partial \mathbf{T}}{\partial \xi_6}\}$, is called the *coordinate basis* corresponding to derivatives with respect to any of the coordinates at any point represented by $\mathbf{T}$. In particular, the operators $\frac{\partial}{\partial \xi_j}$ and $\frac{\partial}{\partial \xi_k}$ commute for all j, k because each is a derivative along a line on which the other is fixed. That is, the coordinates $\xi = [\xi_1, \cdots, \xi_6]^T$ are independent of each other. For example, the components of stiffness matrix in coordinate basis will satisfy

$$\frac{\partial^2 \Phi}{\partial \xi_j \partial \xi_k} = \frac{\partial^2 \Phi}{\partial \xi_k \partial \xi_j} \tag{6}$$

where the potential energy function of the system, Φ, is a scaler. We notice that equation (6) is the standard Hessian formulation which is always symmetric due to the commutative property with respect to the coordinate basis.

On the other hand, a basis that fails to satisfy the commutative property in equation (6) is called a *non-coordinate basis* [17]. The 6×1 Cartesian coordinate basis is a non-coordinate basis because it does not meet the commutative property when involving angular displacement, although the subset of linear displacement does.

3 The Cartesian Stiffness Matrix of A Rigid Body

Cartesian stiffness matrix describes how the components of the Cartesian wrench applied at a rigid body change as the body moves along the twist basis. We will discuss how to find the coordinate transformation matrix through the relationship between bases. Through this coordinate transformation and the principle of virtual work, the stiffness congruence transformation which represents the mapping relationship between Cartesian stiffness and standard Hessian matrices is developed.

3.1 Coordinate transformation

Let us pre-multiply equation (5) by $\mathbf{T}^{-1}$ associated with the same parameterization $(\xi_i, i = 1, \ldots, 6)$. We obtain

$$\mathbf{T}^{-1}\dot{\mathbf{T}} = \dot{\xi}_1 {}^B\mathbf{L}_1 + \cdots + \dot{\xi}_6 {}^B\mathbf{L}_6 \tag{7}$$

where ${}^B\mathbf{L}_i = \mathbf{T}^{-1}\frac{\partial \mathbf{T}}{\partial \xi_i}$ referenced to body frame B belongs to the *Lie Algebra* $se(3)$, because the basis ${}^B\mathbf{L}_i$ are the linear combination of unit twist basis $\mathbf{S}$. However, the ordered parameters $\dot{\xi} = [\dot{\xi}_1, \dot{\xi}_2, \dot{\xi}_3, \dot{\xi}_4, \dot{\xi}_5, \dot{\xi}_6]^T$ are with respect to the new basis, ${}^B\mathbf{L} = \{{}^B\mathbf{L}_1; {}^B\mathbf{L}_2; {}^B\mathbf{L}_3; {}^B\mathbf{L}_4; {}^B\mathbf{L}_5; {}^B\mathbf{L}_6\}$ instead of the coordinate basis, $\mathbf{E} = \{\frac{\partial \mathbf{T}}{\partial \xi_1}; \ldots; \frac{\partial \mathbf{T}}{\partial \xi_6}\}$. The new basis ${}^B\mathbf{L}$ is no longer a coordinate basis because they do not commute with each other (see Appendix A). However, we can easily show that the new basis (non-coordinate basis), ${}^B\mathbf{L}$, will become identical to the coordinate basis, $\mathbf{E}$, at the identity configuration, *i.e.,* when $\mathbf{T}(t) = \mathbf{I}$ (see Figure 1).

We notice that basis $\mathbf{S}$ and basis ${}^B\mathbf{L}$ belong to the same dimensional matrix group space because equations (4) and (7) represent the same vector space, $\mathbf{T}^{-1}\dot{\mathbf{T}}$. Thus, the relationship between bases $\mathbf{S}$ and ${}^B\mathbf{L}$ can be given by

$${}^B\mathbf{L} = \mathbf{M}^T\mathbf{S} = m_{1i}\mathbf{S}_1 + \cdots + m_{6i}\mathbf{S}_6 \tag{8}$$

where the vector $[m_{1i}, \cdots, m_{6i}]^T$ is the i-th $(i = 1 \ldots 6)$ column of the coordinate transformation matrix $\mathbf{M}$. It follows from equation (8) that when $\mathbf{M} = \mathbf{I}$, the two bases ${}^B\mathbf{L}$ and $\mathbf{S}$ are identical, and equation (7) also becomes identical to equation (4).

Based on equation (8), we can derive the relationships, from equations (7) and (4), between the coordinates $\dot{\xi}$ associated with basis ${}^B\mathbf{L}$ and $\mathbf{s}$ associated with the basis $\mathbf{S}$ using the method of change of basis [17, 14]. We can obtain

$$\dot{\mathbf{x}} = \mathbf{s} = \mathbf{M}\dot{\xi} \tag{9}$$

where the Cartesian coordinates is $\dot{\mathbf{x}} = [v_x, v_y, v_z, m_x, m_y, m_z]^T$.

3.2 Stiffness congruence transformation

We can now evaluate the relationship between: (*i*) the twist coordinate system, $\mathbf{s} = [s_1, s_2, s_3, s_4, s_5, s_6]^T$, with respect to the twist basis $\mathbf{S}$, and (*ii*) the generalized coordinate system, $\dot{\xi} = [\dot{\xi}_1, \dot{\xi}_2, \dot{\xi}_3, \dot{\xi}_4, \dot{\xi}_5, \dot{\xi}_6]^T$, with respect to the coordinate basis $\mathbf{E}$. From equation (7), the rigid body velocity, $\mathbf{T}^{-1}\dot{\mathbf{T}}$, can be expressed with respect to the coordinate basis only at $\mathbf{T} = \mathbf{I}$ [4]. Namely, the stiffness matrix with respect to the coordinate basis $\mathbf{E}$ will be identical to the stiffness matrix with respect to the basis ${}^B\mathbf{L}$ when $\mathbf{T} = \mathbf{I}$, and they are symmetric, *i.e.,* $[\mathbf{K}_\xi]_\mathbf{E} = [\mathbf{K}_\xi]_\mathbf{L}$.

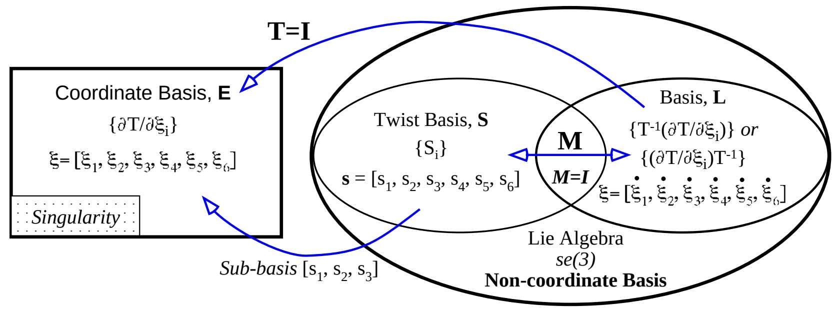

Figure 1: Relationships betw een the coordinate systems in vector space.

In general, we can find a nonsingular initial point, *i.e.,* $\xi_0 = [0,0,0,0,0,0]^T$, to satisfy $\mathbf{T}(0) = \mathbf{I}$. Hence, an expression for a stiffness matrix deriv ed from a coordinate system cannot be valid ev erywhere due to singularities in $SE(3)$ and, therefore, must be a local result [13 , 8]. Thus, w e will represent the Cartesian stiffness matrix based on the twist coordinate system which will be a global result, a non-coordinate system, and valid everywhere in $SE(3)$ (see Figure 1).

F rom the principle of virtual work, we can equate the instantaneous work done in Cartesian coordinate with the work done in generalized coordinate. We consider the condition at $\mathbf{T}(0) = \mathbf{I}$ under which the bases ${}^B\mathbf{L}$ and $\mathbf{E}$ ha ve iden tical coordinate representation. Thus, w e can deriv e the relationship of coordinate transformation betw een the generalized and the twist coordinates, *i.e.,* $d\mathbf{x} = \mathbf{M}d\xi$, by the coordinate transformation matrix $\mathbf{M}$ (see Section 3.1, and Figure 1). Using the definition of stiffness, *i.e.*, $d\mathbf{w} = \mathbf{K}d\mathbf{x}$ in Cartesian coordinate system and $d\mathbf{f} = \mathbf{K}_\xi d\xi$ in generalized coordinate system, the Cartesian stiffness matrix at $\mathbf{T}(0) = \mathbf{I}$ can be obtained

$$\mathbf{K} = \mathbf{M}^{-T} \left(\mathbf{K}_\xi - \mathbf{K}_m\right) \mathbf{M}^{-1} \qquad (10)$$

where $\mathbf{K}_m = \left[\frac{\partial \mathbf{M}^T}{\partial \xi_n}\mathbf{w}\right]$ denotes a 6×6 matrix with the i-th column element being $\left(\frac{\partial \mathbf{M}^T}{\partial \xi_i}\mathbf{w}\right)$ and $i = 1\cdots 6$. Here, the stiffness matrix $\mathbf{K}_\xi$ is with respect to the generalized coordinate basis, ξ, and has no direct geometrical in terpretation. We notice that $\mathbf{K}_\xi$ is the standard Hessian matrix which is always symmetric due to the commutativ e property with respect to the coordinate basis, as defined in equation (6).

4 Application of Stiffness Control Theory to Robot Manipulator

The deriv ations of the Cartesian matrices of manipulator in SE(3) can be applied in dextrous manipulation of robots. The stiffness matrices represent the changes in force and torque that accompany small linear and rotational displacement of the grasped object. In this section, the geometrical method will be presen ted to formulate the 6×6 Cartesian stiffness matrix at the end-effector of robot manipulators, through the change of basis betw een the robot join t and Cartesian spaces, with the introduction of an intermediate coordinate basis.

4.1 The manipulator Jacobian

T raditionally the Jacobian matrix of a manipulator with linear mapping, $\mathcal{R}^n \to \mathcal{R}^6$, is computed by differen tiating the forward kinematics relationship. However, when we consider the forward kinematics mapping as $\mathcal{R}^n \to SE(3)$, the derivation of the Jacobian matrix will be more complex. Although a local coordinates can be chosen for $SE(3)$, it will destro y the natural geometric meaning [13].

When we consider the motions of a robot parameterized by the joint variables $\theta = (\theta_1, \ldots, \theta_n)$, the homogeneous transformation matrix is denoted by $\mathbf{T}(\theta)$. Thus, w e obtain the motion of end-effector with respect to the (instantaneous) body or moving frame, such as [12, 13]

$$\mathbf{T}^{-1}\dot{\mathbf{T}} = \left(\mathbf{T}^{-1}\frac{\partial \mathbf{T}}{\partial \theta_1}\right)\dot{\theta}_1 + \cdots + \left(\mathbf{T}^{-1}\frac{\partial \mathbf{T}}{\partial \theta_n}\right)\dot{\theta}_n \quad (11)$$

Thus, the manipulator Jacobian matrix, $\mathbf{J}$, which defines the mapping relationship betw een the velocities in the joint and Cartesian spaces can be derived from equation (11) via the method of change of basis in

Section 3.1. Namely,

$$\mathbf{T}^{-1}\dot{\mathbf{T}} = (J_{11}\mathbf{S}_1 + \cdots + J_{61}\mathbf{S}_6)\,\dot{\theta}_1 + \cdots + (J_{1n}\mathbf{S}_1 + \cdots + J_{6n}\mathbf{S}_6)\,\dot{\theta}_n \qquad (12)$$

where J_{ij} denotes the components of the manipulator Jacobian matrix, $\mathbf{J}_\theta$, which relates the change of joint coordinates to the twist (Cartesian) coordinates of the end effector. The term $\mathbf{T}^{-1}\frac{\partial \mathbf{T}}{\partial \theta_i}$ determines the i-th column of the manipulator Jacobian matrix. Similarly, the manipulator Jacobian matrix referenced to the inertial frame can be derived from the relationship between the bases $\{\frac{\partial \mathbf{T}}{\partial \theta_i}\mathbf{T}^{-1}\}$ and $\{\mathbf{S}\}$.

4.2 The stiffness modeling of a robot manipulator

First, we know the homogeneous transformation matrix, $\mathbf{T}$, of a robot manipulator is not equal to identity in general configuration of robots. Therefore, it is impracticable to directly derive the 6×6 Cartesian stiffness matrix via the stiffness congruence transformation in equation (10). The conservative congruence transformation (CCT) which describes the stiffness relationship between two generalized coordinate systems in equation (1) is considered. Thus, equation (1) can be re-written as

$$\mathbf{K}_\xi = \mathbf{J}_\xi^{-T}\left(\mathbf{K}_\theta - \mathbf{K}_g\right)\mathbf{J}_\xi^{-1} \qquad (13)$$

where the matrix $\mathbf{K}_g = \left[\frac{\partial \mathbf{J}_\theta^T}{\partial \theta_n}\mathbf{g}\right]$ denotes a $n \times n$ matrix with the i-th column element being $\frac{\partial \mathbf{J}_\theta^T}{\partial \theta_i}\mathbf{g}$, $\mathbf{g}$ is the generalized force, the $6 \times n$ matrix $\mathbf{J}_\xi$ describes the mapping relation between the joint coordinates $d\theta = [d\theta_1, \cdots, d\theta_n]^T$ and the generalized coordinates $d\xi = [d\xi_1, \cdots, d\xi_6]^T$, *i.e.,* $d\xi = \mathbf{J}_\xi d\theta$, and $\mathbf{K}_\theta$ denotes the $n \times n$ symmetric joint stiffness matrix.

Next, the relationship between 6×6 Cartesian and joint stiffness matrices is determined by substituting the stiffness matrix $\mathbf{K}_\xi$ in equation (13) into the stiffness congruence transformation in equation (10). Therefore, the 6×6 Cartesian stiffness matrix can be obtained at the configuration of $\mathbf{T} = \mathbf{I}$

$$\mathbf{K} = \mathbf{M}^{-T}\left[\mathbf{J}_\xi^{-T}\left(\mathbf{K}_\theta - \mathbf{K}_g\right)\mathbf{J}_\xi^{-1} - \mathbf{K}_m\right]\mathbf{M}^{-1} \qquad (14)$$

Equation (14) can be re-written as

$$\mathbf{K} = \underbrace{\mathbf{M}^{-T}\mathbf{J}_\xi^{-T}\mathbf{K}_\theta\mathbf{J}_\xi^{-1}\mathbf{M}^{-1} - \overbrace{\mathbf{M}^{-T}\mathbf{J}_\xi^{-T}\mathbf{K}_g\mathbf{J}_\xi^{-1}\mathbf{M}^{-1}}^{changes\ in\ geometry}}_{symmetric} - \underbrace{\mathbf{M}^{-T}\mathbf{K}_m\mathbf{M}^{-1}}_{asymmetric} \qquad (15)$$

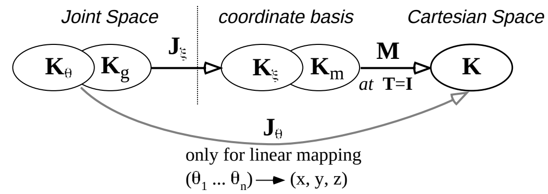

Figure 2: Stiffness matrices of the robot manipulator.

where the first two terms of equation (15) is symmetric due to the symmetric joint stiffness $\mathbf{K}_\theta$ and the symmetric matrix $\mathbf{K}_g$ [2, 3], and the third term of equation (15) results in the skew-symmetric part of the Cartesian stiffness matrix. Moreover, the second term accounts for the configuration changes in the presence of external load of a robot manipulator. Figure 2 illustrates the relationship between the joint and Cartesian stiffness matrices for both the linear subset and full Cartesian space. In particular, the linear 3×3 Cartesian stiffness is always symmetric for conservative system because the 3×3 matrix $\mathbf{M}$ is equal to identity (basis $\mathbf{E}_i = {}^B\mathbf{L}_i = {}^O\mathbf{L}_i = \mathbf{S}_i$, with $i = 1 \ldots 3$), which results in a zero $\mathbf{K}_m$ matrix. Thus, equation (15) becomes the conservative congruence transformation in equation (13) for the linear mapping, and is equivalent to equation (1) for the purpose of our consideration. This can also be proven easily by only including the homogeneous transformation matrix with the translation motions, such as $\mathbf{T}(\xi) = Trans(x, \xi_1)Trans(y, \xi_2)Trans(z, \xi_3)$.

In addition, the relationship between the Cartesian wrench, $\mathbf{w}$, and the generalized force, $\mathbf{g}$, can be obtained by the principle of virtual work

$$\tau^T d\theta = \mathbf{g}^T d\xi = \mathbf{w}^T d\mathbf{x} \qquad (16)$$

and the definition of $d\xi = \mathbf{J}_\xi d\theta$ and $d\mathbf{x} = \mathbf{M}d\xi$. Namely,

$$\tau = \mathbf{J}_\xi^T\mathbf{g} \quad and \quad \mathbf{g} = \mathbf{M}^T\mathbf{w} \qquad (17)$$

Thus, we obtain the following relationship

$$\mathbf{K}_g = \left[\frac{\partial \mathbf{J}_\theta^T}{\partial \theta_n}\mathbf{M}^T\mathbf{w}\right] \qquad (18)$$

4.3 Computation algorithm of stiffness modeling

The following algorithm presents the procedures to compute the Cartesian stiffness of robot manipulator by geometrical method and the conservative congruence transformation.

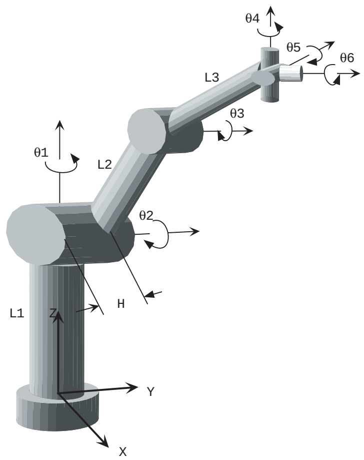

Figure 3: A 6R manipulator.

1. Define a local coordinate basis system for the end-effector of a manipulator, such as $\xi = (\xi_1, \ldots, \xi_6)$ which is chosen to make the homogeneous transformation matrix identity at the origin configuration, *i.e.*, $\mathbf{T} = \mathbf{I}$.

2. Compute the Jacobian matrix, $\mathbf{J}_\xi$, by differentiating the forward kinematic relationship.

3. Determine the coordinate transformation matrix, $\mathbf{M}$, by using the method of change of basis between the coordinate basis and the twist basis.

4. Calculate matrices $\mathbf{K}_m$ and $\mathbf{K}_g$.

5. Substitute matrices $\mathbf{M}$, $\mathbf{J}_\xi$, $\mathbf{K}_\theta$, $\mathbf{K}_g$, and $\mathbf{K}_m$ into the conservative congruence transformation (CCT) equation (14) to obtain the Cartesian stiffness matrix, $\mathbf{K}$.

4.3.1 An example

First of all, we must define local coordinates for the end-effector configuration including translation and rotation in order to find the Cartesian stiffness with respect to the twist basis. The configuration of the end-effector will be of interests at $\mathbf{T} = \mathbf{I}$.

For a 6R manipulator shown in Figure 3, we can choose the local coordinates $(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6)$ to parameterize the motion of the end-effector including the translational and rotational movements. Thus, these six independent parameters are determined by the forward kinematics as follows

$$\begin{aligned}
\xi_1 &= L_2 \cos\theta_1 \cos\theta_2 + L_3 \cos\theta_1 \cos(\theta_2 + \theta_3) - H \sin\theta_1 \\
\xi_2 &= L_2 \sin\theta_1 \cos\theta_2 + L_3 \sin\theta_1 \cos(\theta_2 + \theta_3) + H \cos\theta_1 \\
\xi_3 &= L_1 + L_2 \sin\theta_2 + L_3 \sin(\theta_2 + \theta_3) \\
\xi_4 &= \theta_4 \\
\xi_5 &= \theta_5 \\
\xi_6 &= \theta_6 \qquad (19)
\end{aligned}$$

where L_1, L_2, L_3, and H are the link lengths shown in Figure 3. The Jacobian of the manipulator, $\mathbf{J}_\xi$, $\mathbf{J}_\xi : (\theta_1, \ldots, \theta_6) \to \mathcal{R}^6$, is given by

$$\mathbf{J}_\xi = \begin{bmatrix} J_{11} & J_{12} & J_{13} & 0 & 0 & 0 \\ J_{21} & J_{22} & J_{23} & 0 & 0 & 0 \\ 0 & J_{32} & J_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (20)$$

where $J_{11} = -L_2 s_1 c_2 - L_3 s_1 c_{23} - H c_1$, $J_{12} = -L_2 c_1 s_2 - L_3 c_1 s_{23}$, $J_{13} = -L_3 c_1 s_{23}$, $J_{21} = L_2 c_1 c_2 + L_3 c_1 c_{23} - H s_1$, $J_{22} = -L_2 s_1 s_2 - L_3 s_1 s_{23}$, $J_{23} = -L_3 s_1 s_{23}$, $J_{32} = L_2 c_2 + L_3 c_{23}$, and $J_{33} = L_3 c_{23}$. Note that the Jacobian matrix in equation (59) happens to be the same as the conventional Jacobian matrix defined by $\mathbf{J}_\theta = \frac{\partial x}{\partial \theta}$ due to the choice of local coordinates. Next, we adopt the end-effector with ZYX Euler angles to describe the orientation of the body coordinate frame B relative to the inertial coordinate frame O with local coordinates $\xi = [\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6]^T$. The displacement including the translations and rotations of the moving body is given by [15, 6]

$$\mathbf{T}(\xi) = \underbrace{Rot(z, \xi_6) Rot(y, \xi_5) Rot(x, \xi_4)}_{rotation} \underbrace{Trans(z, \xi_3) Trans(y, \xi_2) Trans(x, \xi_1)}_{translation} \qquad (21)$$

where the rate of the coordinates, $[\dot{\xi}_1, \dot{\xi}_2, \dot{\xi}_3]$, are related to the translational motion, and $[\dot{\xi}_3, \dot{\xi}_4, \dot{\xi}_5]$, are related to the translational motion. Through the geometrical method described in Section 3.1, the coordinate transformation matrix $\mathbf{M} = \frac{d\mathbf{x}}{d\xi}$ corresponding to the basis ${}^O\mathbf{L}_i = \frac{\partial \mathbf{T}}{\partial \xi_i} \mathbf{T}^{-1}$ referenced to the inertial frame is obtained by

$$\mathbf{M} = \begin{bmatrix} c_5 c_6 & s_4 s_5 c_6 - c_4 s_6 & c_4 s_5 c_6 + s_4 s_6 & 0 & 0 & 0 \\ c_5 s_6 & s_4 s_5 s_6 + c_4 c_6 & c_4 s_5 s_6 - s_4 c_6 & 0 & 0 & 0 \\ -s_5 & s_4 c_5 & c_4 c_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_5 c_6 & -s_6 & 0 \\ 0 & 0 & 0 & c_5 s_6 & c_6 & 0 \\ 0 & 0 & 0 & -s_5 & 0 & 1 \end{bmatrix} \qquad (22)$$

For simplicity, we choose the configuration, $(\theta_1, \ldots, \theta_6) = (0, -30°, 30°, 0, 0, 0)$ degree, with the specification of manipulator, $L_1 = 1$ m, $L_2 = 0.6$ m, $L_3 = 0.5$ m, and $H = 0.2$ m. Thus, the matrices $\mathbf{K}_g$ and $\mathbf{K}_m$ associated with the inertial frame are given by

$$\mathbf{K}_g = \begin{bmatrix} -1.02 f_x - 0.2 f_y & 0.3 f_y & 0 & 0 & 0 & 0 \\ 0.3 f_y & -1.02 f_x + 0.3 f_z & -0.5 f_x & 0 & 0 & 0 \\ 0 & -0.5 f_x & -0.5 f_x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (23)$$

$$
\mathbf{K}_m = \begin{bmatrix} 0 & 0 & 0 & 0 & -f_z & f_y \\ 0 & 0 & 0 & f_z & 0 & -f_x \\ 0 & 0 & 0 & -f_y & f_x & 0 \\ 0 & 0 & 0 & 0 & -m_z & m_y \\ 0 & 0 & 0 & 0 & 0 & -m_x \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{24}
$$

where $\mathbf{f} = [f_x, f_y, f_z, m_x, m_y, m_z]^T$ is the wrench in Cartesian twist basis. Substituting the matrix $\mathbf{K}_g$ in equation (23) and the matrix $\mathbf{K}_m$ in equation (24) into the conservative congruence transformation in equation (14), we obtain the Cartesian stiffness of the 6R manipulator at this specified configuration as

$$
\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & 0 & f_z & -f_y \\ k_{12} & k_{22} & k_{23} & -f_z & 0 & f_x \\ k_{13} & k_{23} & k_{33} & f_y & -f_x & 0 \\ 0 & 0 & 0 & k_{\theta 44} & m_z & -m_y \\ 0 & 0 & 0 & 0 & k_{\theta 55} & m_x \\ 0 & 0 & 0 & 0 & 0 & k_{\theta 66} \end{bmatrix} \tag{25}
$$

where the components of upper-left 3×3 stiffness matrix are given by

$$
\begin{aligned}
k_{11} &= 11.773 f_x - 3.333 f_z + 11.111 k_{\theta 22} + 46.205 k_{\theta 33} \\
k_{12} &= 2.309 f_x - 0.981 f_y - 0.654 f_z + 2.18 k_{\theta 22} + 9.063 k_{\theta 33} \\
k_{13} &= -3.464 f_x - 13.595 k_{\theta 33} \\
k_{22} &= 1.434 f_x - 0.192 f_y - 0.128 f_z + 0.962 k_{\theta 11} + 0.428 k_{\theta 22} + 1.759 k_{\theta 33} \\
k_{23} &= -0.68 f_x - 2.667 k_{\theta 33} \\
k_{31} &= 2 f_x + 4 k_{\theta 33}
\end{aligned}
$$

The above results are obtained assuming a diagonal joint stiffness matrix, $\mathbf{K}_\theta$, with the diagonals $(k_{\theta 11}, k_{\theta 22}, k_{\theta 33}, k_{\theta 44}, k_{\theta 55}, k_{\theta 66})$. The 6×6 Cartesian stiffness matrix in equation (25) is asymmetric. However, the upper-left 3×3 stiffness remains symmetric as expected. In addition, the skew-symmetric part of the Cartesian stiffness is

$$
\mathbf{K}_{skew} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.5 f_z & -0.5 f_y \\ 0 & 0 & 0 & -0.5 f_z & 0 & 0.5 f_x \\ 0 & 0 & 0 & 0.5 f_y & -0.5 f_x & 0 \\ 0 & 0.5 f_z & -0.5 f_y & 0 & 0.5 m_z & -0.5 m_y \\ -0.5 f_z & 0 & 0.5 f_x & -0.5 m_z & 0 & 0.5 m_x \\ 0.5 f_y & -0.5 f_x & 0 & 0.5 m_y & -0.5 m_x & 0 \end{bmatrix} \tag{26}
$$

Equation (26) indicates that the skew-symmetric part of the Cartesian stiffness referenced to the inertial frame is equal to negative one-half of the external load, *i.e.*, $-\frac{1}{2}\mathbf{w}\times$. In addition, we notice that the Cartesian stiffness referenced to the inertial and moving frames at coincident points are transposes of each other [4].

5 Conclusions

We propose a new theory of stiffness control for modeling dextrous manipulation. The fingers of a dextrous hand manipulate an object by exerting both forces and moments at the contact. The stiffness control indicates how the fingertip forces and moments change when the fingertips are moved by small amounts. In this paper, the geometrical method provides a systematic way of constructing 6×6 Cartesian stiffness matrices in robotic grasping/manipulation and stiffness control. Equation (14) represents the mapping relationship between the Cartesian and joint stiffness matrices. Namely, the manipulation using Cartesian-based stiffness control can be achieved by controlling the joint stiffnesses. On the other hand, the joint torques corresponding to the joint stiffness matrices can be computed with the known contact stiffnesses. The conservative properties of a 6×6 Cartesian stiffness matrix are represented by a symmetric part which satisfies the conservative criteria, and by an anti-symmetric part which equals to the negative one-half of the cross-product matrix formed by the externally applied load referenced to the inertial frame. These results are illustrated by the example of a spatial 6R manipulator.

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Appendix:

A Non-coordinate Basis ${}^B\mathbf{L}$

Given two arbitrary operators with respect to the basis ${}^B\mathbf{L}$, $L_i = \mathbf{T}^{-1}\frac{\partial}{\partial \xi_i}$ and $L_j = \mathbf{T}^{-1}\frac{\partial}{\partial \xi_j}$, then

$$
\begin{aligned}
L_i L_j - L_j L_i &= \mathbf{T}^{-1}\frac{\partial}{\partial \xi_i}\left(\mathbf{T}^{-1}\frac{\partial}{\partial \xi_j}\right) - \mathbf{T}^{-1}\frac{\partial}{\partial \xi_j}\left(\mathbf{T}^{-1}\frac{\partial}{\partial \xi_i}\right) \\
&= \mathbf{T}^{-1}\frac{\partial \mathbf{T}^{-1}}{\partial \xi_i}\frac{\partial}{\partial \xi_j} - \mathbf{T}^{-1}\frac{\partial \mathbf{T}^{-1}}{\partial \xi_j}\frac{\partial}{\partial \xi_i} \\
&\quad + \mathbf{T}^{-1}\mathbf{T}^{-1}\underbrace{\left(\frac{\partial}{\partial \xi_i}\frac{\partial}{\partial \xi_j} - \frac{\partial}{\partial \xi_j}\frac{\partial}{\partial \xi_i}\right)}_{0} \\
&= \mathbf{T}^{-1}\frac{\partial \mathbf{T}^{-1}}{\partial \xi_i}\left(\frac{\partial}{\partial \xi_j}\right) - \mathbf{T}^{-1}\frac{\partial \mathbf{T}^{-1}}{\partial \xi_j}\left(\frac{\partial}{\partial \xi_i}\right)
\end{aligned}
$$

In general, the Lie bracket $[L_i, L_j] = L_i L_j - L_j L_i \neq 0$ which means that the operators L_i and L_j do not commute for all i, j, except $\mathbf{T} = constant$ or $\mathbf{T} = \mathbf{I}$. Namely, the basis $\mathbf{T}^{-1}\frac{\partial}{\partial \xi_i}$ is not a derivative holding ξ_j fixed, so $\mathbf{T}^{-1}\frac{\partial}{\partial \xi_i}$ and $\mathbf{T}^{-1}\frac{\partial}{\partial \xi_j}$ do not commute. Nevertheless, the basis ${}^B\mathbf{L}$ becomes identical to the coordinate basis $\mathbf{E}$ when $\mathbf{T} = \mathbf{I}$, the identity matrix, though the basis ${}^{B}\mathbf{L}$ is generally a non-coordinate basis. Similarly, ${}^{O}\mathbf{L}$ is also a non-coordinate basis. The detailed geometrical interpretation can be found in [17].

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