Adaptive control and synchronization of a class of chaotic systems in which all parameters are unknown

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Abstract—This paper proposes an adaptive algorithm for the control and synchronization of a class of second order chaotic systems which exact dynamics or parameters are unknown in priori. The proposed control scheme includes a feedback controller and a feedforward compensator. Both the gains of the controller and the compensator are updated by an adaptation algorithm derived from Model Reference Adaptive Control (MRAC) theory. In the proposed approach, the optimal adaptation gains are identified using the Nelder-Mead simplex algorithm. This algorithm does not require the derivatives of the performance index to be optimized, and is therefore particularly applicable to complex systems or problems with undifferentiable elements, discontinuities or uncertainties. The feasibility and effectiveness of the proposed approach are demonstrated by way of numerical simulations using general Duffing’s systems for illustration purposes.

Keywords—chaos, adaptive control, synchronization, Nelder-Mead simplex algorithm.

I. INTRODUCTION

A characteristic property of chaotic dynamic systems is their sensitivity to the initial conditions, external disturbances and parameter variations [1,2]. However, Pecora and Carroll [3] demonstrated that synchronization of chaotic systems is nevertheless achievable by using the output of the drive (or master) system to control the response (or slave) system such that the two systems oscillate in a synchronized manner. The problem of chaos synchronization has received extensive attention in a variety of fields, ranging from physical, chemical and ecological sciences to secure communications, and so forth [4-17].

In the present study, an assumption is made that all the parameters of both the drive system and the response system (including the harmonic external force) are unknown. The approach proposed by Seraji in [18] for the design of adaptive systems for a class of chaotic systems is then applied to drive the output of the response system in such a way that it tracks the output of the drive system. In the proposed approach, the control force is generated in part by an adaptive feedforward compensator which behaves as the inverse of the response system and is driven by the output of the drive system. In addition, an adaptive feedback controller is introduced to enhance the stability of the closed-loop system and to accelerate the adaptation process. The proposed method has a simple structure, is computationally efficient and requires no prior knowledge of the exact dynamic equations of both the drive system and the response system.

Accordingly, this paper proposes a method for determining the optimal adaptation gains of adaptive control system using the Nelder-Mead simplex algorithm [19-26]. This algorithm is a robust, nonlinear multi-dimensional optimization scheme which is easily implemented and has been extensively applied to solve parameter estimation and other optimization problems in a variety of mathematical, engineering and scientific fields. Importantly, the simplex method does not require the derivatives of the function to be optimized, and hence it is particularly applicable to problems with discontinuities or in which the function values include noise. The feasibility of the proposed method is confirmed by simulating the control of a chaotic system, namely Duffing’s system.

II. ADAPTIVE CONTROL AND SYNCHRONIZATION

Let the chaotic drive system be described by the following nonautonomous dynamic equation:

\[ m_D(x,t)\ddot{x} + c_D(x,\dot{x},t)\dot{x} + k_D(x, t)x = d(t), \]  

where \( m_D \) ( \( m_D > 0 \) ), \( c_D \), \( k_D \), and \( d \) are unknown time-variant functions. Additionally, the response system with a control input \( u(t) \) added to its right side is introduced as follows:

\[ m_R(y,t)\ddot{y} + c_R(y,\dot{y},t)\dot{y} + k_R(y, t)y = u(t), \]  

where \( m_R \) ( \( m_R > 0 \) ), \( c_R \), and \( k_R \) are also unknown time-variant functions. The aim of the current adaptive scheme is to drive the output of the response system, \( y(t) \), such that it tracks the output of the drive system, \( x(t) \). Let the control input be

\[ u = u_{FB} + u_{FF}, \]
where $u_{ff}$ and $u_{fb}$ represent the feedforward compensator and the feedback controller, respectively. The feedforward compensator has the form

$$u_{ff} = m(t)\ddot{x} + c(t)\dot{x} + k(t)x,$$  

(4)

where $m$, $c$ and $k$ are the estimated values of $m_\text{R}$, $c_\text{R}$, and $k_\text{R}$, respectively, and $\ddot{x}$, $\dot{x}$ and $x$ are the outputs of the drive system. The feedback control law is formulated as

$$u_{fb} = k_1(t)e + k_2(t)e,$$  

(5)

where $e = x - y$ and $\dot{e} = \ddot{x} - \ddot{y}$. Substituting (3) into (2) gives

$$m(t)\dddot{y} + c(t)\ddot{y} + k(t)y = m(t)\dddot{x} + c(t)\ddot{x} + k(t)x + k_1(t)e + k_2(t)e.$$  

(6)

Rearranging the above equation yields the following error equation:

$$\dddot{e} + m_1^{-1}(c_1 + k_2)e + m_2^{-1}(k_1 + k) = m_1^{-1}(m_\text{R} - m)\dddot{x} + m_1^{-1}(c_\text{R} - c)\ddot{x} + m_1^{-1}(k_\text{R} - k)x.$$  

(7)

Defining the error vector as

$$e(t) = \begin{bmatrix} e(t) \\ \dot{e}(t) \end{bmatrix},$$  

(8)

Equation (7) can be rewritten in the following state-space form:

$$\dot{e}(t) = \begin{bmatrix} 0 & 1 \\ -m_1^{-1}(c_1 + k_1) & -m_1^{-1}(k_1 + k) \end{bmatrix}e(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\ddot{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\dot{x}(t) + \begin{bmatrix} 0 \\ -m_1^{-1}(k_\text{R} - k) \end{bmatrix}x(t).$$  

(9)

Defining the following parameters:

$$w_1 = m_1^{-1}(k_1 + k),$$  

(10)

$$w_2 = m_1^{-1}(c_1 + k_2),$$  

(11)

$$w_3 = m_1^{-1}(m_\text{R} - m),$$  

(12)

$$w_4 = m_1^{-1}(c_\text{R} - c),$$  

(13)

$$w_5 = m_1^{-1}(k_\text{R} - k).$$  

(14)

Equation (9) can be simplified to the form

$$\dot{e}(t) = \begin{bmatrix} 0 & 1 \\ -w_1 & -w_2 \end{bmatrix}e(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\ddot{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\dot{x}(t) + \begin{bmatrix} 0 \\ -w_5 \end{bmatrix}x(t).$$  

(15)

We call (15) the “adjustable system”. Define the “reference model” as follows

$$\dot{e}_m(t) = \begin{bmatrix} 0 & 1 \\ -h_1 & -h_2 \end{bmatrix}e_m(t) = He_m(t).$$  

(16)

where $H$ is Hurwitz (i.e., has all its eigenvalues strictly in the left half plane). Meanwhile, defining

$$\epsilon(t) = e_m(t) - e(t),$$  

(17)

We have

$$\dot{\epsilon}(t) = \begin{bmatrix} 0 & 1 \\ -w_1 & -w_2 \end{bmatrix}\epsilon(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\ddot{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\dot{x}(t) + \begin{bmatrix} 0 \\ -w_5 \end{bmatrix}x(t).$$  

(18)

The solution of (16) can be expressed as

$$e_m(t) = e_0 e(t),$$  

(19)

where $e_m(0)$ is the initial state of the reference model. Let $e_m(0) = 0$. Hence, from (19), $e_m(t) = 0$ for all $t > 0$.

Since $H$ is Hurwitz, there exists a symmetric positive definite matrix $P$ which satisfies the Lyapunov algebraic equation, i.e.

$$H^T P + P H = -I.$$  

(20)

Let

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$$  

(21)

$$f(t) = p_2 e(t) + p_3 \dot{e}(t).$$  

(22)
Using the improved MRAC theory described in [18], the following adaptation laws can be obtained:

\[ k_i(t) = k_i(0) + af(t)e(t) + b \int_0^t f(\tau)e(\tau)d\tau, \]  
\[ k_\gamma(t) = k_\gamma(0) + af(t)\dot{e}(t) + b \int_0^t f(\tau)\dot{e}(\tau)d\tau, \]  
\[ m(t) = m(0) + af(t)\ddot{x}(t) + b \int_0^t f(\tau)\ddot{x}(\tau)d\tau, \]  
\[ c(t) = c(0) + af(t)\dot{x}(t) + b \int_0^t f(\tau)\dot{x}(\tau)d\tau, \]  
\[ k(t) = k(0) + af(t)x(t) + b \int_0^t f(\tau)x(\tau)d\tau, \]

where the scalar adaptation gains \( a \) and \( b \) are nonnegative and positive, respectively; and \( k_i(0) \), \( k_\gamma(0) \), \( m(0) \), \( c(0) \) and \( k(0) \) are the initial parameters of the feedback controller and feedforward compensator. With the adaptation laws given in (23)-(27), the point \( (e(t), \dot{e}(t), \ddot{x}(t)) \) converges to that of the drive system, \( (e(t), \dot{e}(t), \ddot{x}(t)) \), as \( t \to \infty \). The schematic illustration of the adaptive chaos synchronization system is shown in Figure 1.

\[ \min_{k \in \mathbb{R}^n} J(k), \]  
\[ J(k) = \int_0^T (e^2(t, k) + \dot{e}^2(t, k))dt, \quad T > 0. \]

The Nelder–Mead simplex algorithm generates a sequence of simplices, in which each simplex is defined by \( n+1 \) distinct vertices \( \mathbf{k}_0, \ldots, \mathbf{k}_n \), with corresponding function values \( J_0, \ldots, J_n \). The points \( \mathbf{k}_0, \ldots, \mathbf{k}_n \) are assumed to be ordered such that \( J_0 \leq \ldots \leq J_n \) and

\[ \mathbf{k} \equiv \frac{1}{n} \sum_{i=0}^n \mathbf{k}_i \]  

In each iteration, the algorithm examines one or more of four different \( \zeta \) values along the line \( \mathbf{k} + \zeta(\mathbf{k} - \mathbf{k}_i) \), \( \zeta \in \mathbb{R} \). These four values, denoted as \( \alpha \), \( \beta \), \( -\beta \), and \( \gamma \), respectively, have to satisfy the following conditions:

\[ 0 < \beta < 1, \quad 0 < \alpha < \gamma \]

where \( \gamma > 1 \). In this paper, these parameters are assigned values of \( \alpha = 1 \), \( \beta = 0.5 \), and \( \gamma = 2 \). The values of \( \alpha \), \( \gamma \), \( \beta \) and \( -\beta \) yield the reflection point \( \mathbf{k}_r \), the expansion point \( \mathbf{k}_e \), the outer contraction point \( \mathbf{k}_o \), and the inner contraction point \( \mathbf{k}_i \), respectively. The values of the performance index at these four points are denoted as \( J_r, J_e, J_o \), and \( J_i \), respectively. If none of the four points represents an improvement on the current worst point \( \mathbf{k}_o \), the algorithm shrinks the points \( \mathbf{k}_i, \ldots, \mathbf{k}_o \) toward the lowest \( \mathbf{k}_o \), thereby producing a new simplex. In the shrinking process, each \( \mathbf{k}_i \) is replaced by \( \mathbf{k}_i + 0.5(\mathbf{k}_i - \mathbf{k}_o) \) for \( i = 1, \ldots, n \). Upon completion of the shrinking process, a new iteration is automatically triggered. This iterative process is continued until the specified termination criteria are satisfied. Figure 2 shows the schematic...
diagram that utilizing the Nelder–Mead simplex algorithm to search the optimal adaptation gains, \(a\) and \(b\), which are applied in (23)-(27).

![Diagram of using the simplex algorithm to search the optimal adaptation gains](image)

**Figure 2.** Schematic diagram of using the simplex algorithm to search the optimal adaptation gains where \(k = [a, b]\).

**IV. NUMERICAL SIMULATION**

In demonstrating the feasibility of the proposed control scheme, it is assumed that the drive system is described by the following Duffing’s oscillator:

\[
\ddot{x} + 0.25\dot{x} - (1 - x^3)x = 0.3 \sin(t), \tag{32}
\]

where \(m_g = 1\), \(c_g = 0.25\), \(k_g = x^2 - 1\), and \(d(t) = 0.3 \sin(t)\) are assumed to be unknown constants or functions. Furthermore, the response system has the form

\[
\ddot{y} + 0.2\dot{y} - (0.8 - 1.1y^2)y = u(t), \tag{33}
\]

where \(m_g = 1\), \(c_g = 0.2\) and \(k_g = -(0.8 - 1.1y^2)\) are also assumed to be unknown. The aim of the proposed adaptive scheme is to control the solution of the response system, \(y(t)\), such that it converges to that of the drive system, \(x(t)\).

The coefficient matrix of the reference model described in (16) is chosen as

\[
H = \begin{bmatrix}
0 & 1 \\
-1 & -2
\end{bmatrix}. \tag{34}
\]

Solving the Lyapunov equation given in (20) yields

\[
P = \begin{bmatrix}
1.5 & 0.5 \\
0.5 & 0.5
\end{bmatrix}. \tag{35}
\]

Thus,

\[
f(t) = 0.5e(t) + 0.5\dot{e}(t). \tag{36}
\]

The adaptation laws are designed in accordance with (23)-(27) where all the initial values of the controller gains, i.e., \(k_1(0), k_2(0), m(0), c(0)\) and \(k(0)\) are set to zero. In performing the numerical simulations, the time step size is specified as 0.01 sec and the initial states are defined as \((x, \dot{x}) = (y, \dot{y}) = (0, 0)\). Next, we utilize the simplex method to search the optimal adaptation gains. Here, the performance
index is defined as the form of (29) where $T = 100$. The initial simplex is defined by the following 3 distinct vertices:

$$a = 1, \ b = 1$$
$$a = 1, \ b = 2$$
$$a = 2, \ b = 1$$

The termination criteria are that the iteration number is greater than 50 or $(J_e - J_0) / J_e < 0.001$. From inspection, the optimal adaptation gains of the system are determined to be

$$a = 153.1, \ b = 864.3$$

Figures 3 and 4 illustrate the time responses of the Duffing’s oscillator under adaptive synchronization with different adaptation gains.

V. CONCLUSION

This paper has presented an adaptive control scheme for the control and synchronization of a class of chaotic systems. The adaptation laws in the control scheme are derived from MRAC theory. In the proposed approach, the control force is generated by an adaptive feedforward compensator and an adaptive feedback controller. The former behaves as the inverse of the response system and is driven by the output of the drive system, while the latter enhances the stability of the closed-loop system and accelerates the adaptation process. The proposed method requires neither an exact dynamic model of the chaotic systems nor any prior knowledge of the system parameters or the harmonic external force. In addition, we apply the Nelder-Mead simplex method to determine the optimal adaptation gains of adaptive control system. Finally, the feasibility and effectiveness of the proposed method are demonstrated by way of numerical simulations using general Duffing’s oscillators for illustration purposes.

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REFERENCES