Analysis of Two-Grid Methods for Nonlinear Parabolic Equations by Expanded Mixed Finite Element Methods

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Abstract. In this paper, we present an efficient method of two-grid scheme for the approximation of two-dimensional nonlinear parabolic equations using an expanded mixed finite element method. We use two Newton iterations on the fine grid in our methods. Firstly, we solve an original nonlinear problem on the coarse nonlinear grid, then we use Newton iterations on the fine grid twice. The two-grid idea is from Xu’s work [SIAM J. Numer. Anal., 33 (1996), pp. 1759–1777] on standard finite method. We also obtain the error estimates for the algorithms of the two-grid method. It is shown that the algorithm achieve asymptotically optimal approximation rate with the two-grid methods as long as the mesh sizes satisfy $h = O(H^{4k+1}/(k+1))$.

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Key words: Nonlinear parabolic equations, two-grid scheme, expanded mixed finite element methods, Gronwall’s Lemma.

1 Introduction

In this paper, we consider the following nonlinear parabolic equations

$$\frac{\partial p}{\partial t} - \nabla \cdot (K(p) \nabla p) = f(p, \nabla p), \quad (x, t) \in \Omega \times J, \quad (1.1)$$

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with initial condition
\[ p(x, 0) = p^0(x), \quad x \in \Omega, \quad (1.2) \]
and boundary condition
\[ (K(p) \nabla p) \cdot \nu = 0, \quad (x, t) \in \partial \Omega \times J, \quad (1.3) \]
where \( \Omega \subset \mathbb{R}^2 \) is a bounded and convex domain with \( C^1 \) boundary \( \partial \Omega \), \( \nu \) is the unit exterior normal to \( \partial \Omega \), \( J = (0, T) \), \( K \) is a symmetric positive definite tensor and \( K : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^{2 \times 2} \). Eq. (1.1) can be rewritten as following
\[ \frac{\partial p}{\partial t} + \nabla \cdot \Psi = f(p, \nabla p), \quad (1.4) \]
\[ K(p)^{-1} \Psi + \nabla p = 0. \quad (1.5) \]
Eqs. (1.1)-(1.3) are the simplification of the modeling of groundwater through porous media [9]. In this particular case, \( p \) denotes the fluid pressure; and \( K \) is a symmetric, uniformly positive definite tensor with \( L^\infty(\Omega) \) components representing the permeability divided by the viscosity; \( \Psi \) represents the Darcy velocity of the flow; and \( f(p, \nabla p) \) models the external flow rate.

The two-grid methods was first introduced by Xu [16, 17] as a discretization technique for nonlinear and nonsymmetric indefinite partial differential equations. It based on the fact the nonlinearity, nonsymmetry and indefiniteness behaving like low frequencies are governed by coarse grid and the related high frequencies are governed by some linear or symmetric positive definite operators. The basic idea of the two-grid method is to solve a complicated problem (nonlinear, nonsymmetric indefinite) on a coarse grid (mesh size \( H \)) and then solve an easier problem (linear, symmetric positive) on the fine grid (mesh size \( h \) and \( h \ll H \)) as correction.

In many partial differential equations, the objective functional contains the gradient of the state variables. Thus, the accuracy of the gradient is important in numerical discretization of the coupled state equations. Mixed finite element methods are appropriate for the equations in such cases since both the scalar variable and its flux variable can be approximated to the same accuracy by using such methods. Some specialists have made many important works on some topic of mixed finite element method for linear elliptic or reaction-diffusion equations. In [5, 6], Chen has studied the expanded mixed element methods for some quasilinear second order elliptic equations. However, there doesn't seem to exist much work on theoretical analysis for two-grid methods for mixed finite element approximation of quasilinear or nonlinear parabolic equations in the literature.

Many contributions have been done to the multi-grid schemes for finite element methods, see, for example [11,12]. In [8], the authors have studied a two-grid finite difference scheme for nonlinear parabolic equations. Xu and Zhou have considered some multi-scale schemes for finite element method of elliptic partial differential equations in [18]. Recently, we constructed a new two-grid method of expanded mixed finite
element method for semi-linear reaction-diffusion equation [4]. As a continued work
of Chen [3], in this paper, we shall discuss the case that the coefficient matrix function
K and the reaction term f are nonlinear, namely, \( K = K(p) \) and \( f = f(p, \nabla p) \).

In this paper, we adopt the standard notation \( W^{m,p}(\Omega) \) for Sobolev spaces on \( \Omega 
\) with a norm \( \| \cdot \|_{m,p} \) given by
\[
\| v \|_{m,p} = \sum_{|\alpha| \leq m} \| D^\alpha v \|_{L^p(\Omega)},
\]
and a semi-norm \( | \cdot |_{m,p} \) given by
\[
| v |_{m,p} = \sum_{|\alpha| = m} \| D^\alpha v \|_{L^p(\Omega)}.
\]
We set
\[
W^{m,p}_0(\Omega) = \{ v \in W^{m,p}(\Omega) : v|_{\partial \Omega} = 0 \}.
\]
For \( p=2 \), we denote
\[
H^m(\Omega) = W^{m,2}(\Omega), \quad H^m_0(\Omega) = W^{m,2}_0(\Omega),
\]
\( \| \cdot \|_m = \| \cdot \|_{m,2}, \| \cdot \| = \| \cdot \|_{0,2} \).
We denote by \( L^s(J; W^{m,p}(\Omega)) \) the Banach space of all \( L^s \) integrable functions from \( J \)
into \( W^{m,p}(\Omega) \) with norm
\[
\| v \|_{L^s(J; W^{m,p}(\Omega))} = \left( \int_0^T \| v \|_{W^{m,p}(\Omega)}^s dt \right)^{\frac{1}{s}}, \quad \text{for} \ s \in [1, \infty),
\]
and the standard modification for \( s = \infty \). The details can be found in [10]. In addition
\( C \) or \( c \) denotes a general positive constant independent of \( h \).

At first, we make the following assumptions for the nonlinear parabolic equations.

(A1) There exist positive constants \( K_* \) and \( K^* \), such that for \( z \in \mathbb{R}^2 \),
\[
K_* \| z \|^2 \leq z' K(x,s) z \leq K^* \| z \|^2, \quad \text{for} \ x \in \Omega,
\]
and that each element of \( K \) is twice continuously differential with derivatives up to
second order bounded above by \( K^* \).

(A2) For some integer \( k \geq 1 \), we assume that the solution function (1.4)-(1.5) has the following regularity:
\[
p \in L^2(J; W^{k+2,4}(\Omega)), \quad \Psi \in \left( L^2(J; H^{k+1}(\Omega)) \right)^2.
\]
To analyze the discretization on a time interval \((0, T)\), let \( N > 0, \triangle t = T/N, \) and
\( t^n = n \triangle t \), and set

\[
\phi^n = \phi(\cdot, t^n), \quad \partial_t \phi^n = \frac{\phi^n - \phi^{n-1}}{\Delta t},
\]

\[
\|\phi\|_{L^2((0,T);X)} = \left( \sum_{n=1}^{N} \|\phi^n\|^2_X \right)^{1/2},
\]

\[
\|\phi\|_{L^\infty((0,T);X)} = \max_{1 \leq n \leq N} \|\phi^n\|_X,
\]

\[
\|\phi\|_{L^2(\Omega)} = \left( \int_0^T \|\phi^n(\cdot, t)\|_X^2 dt \right)^{1/2}.
\]

The plan of this paper is as follows. In next section, we introduce some notations and projections. The approximation properties of these projections will be recalled. In section 3, we construct a expanded mixed finite element discretization for the nonlinear parabolic equations (1.1)-(1.3). A two-grid scheme and error estimates for two-dimensional nonlinear parabolic equations using an expanded mixed finite element method are discussed in section 4. Finally, we give the conclusion and future work in section 5.

### 2 Projection operators and approximation properties

Let \((\cdot, \cdot)\) denote the \(L^2(\Omega)\) inner product. Let

\[
V = H(\text{div}; \Omega) = \{\mathbf{v} \in (L^2(\Omega))^2, \nabla \cdot \mathbf{v} \in L^2(\Omega)\},
\]

\[
\tilde{V} = V \cap \{\mathbf{v} \cdot \mathbf{n} = 0\}, \quad W = L^2(\Omega).
\]

The Hilbert space \(V\) is equipped with the following norm:

\[
\|\mathbf{v}\|_{H(\text{div}; \Omega)} = \left( \|\mathbf{v}\|^2 + \|\text{div} \mathbf{v}\|^2 \right)^{1/2}.
\]

Let \(\Gamma_h\) denote a quasi-uniform partition of \(\Omega\) into rectangles or triangles with the partition step \(h\). We form \(V_h\) and \(W_h\), discrete subspaces of \(V\) and \(W\), using standard mixed finite element space such as the RT [13] spaces of order \(k\), RT\(_k\), or Brezzi-Douglas-Marini [1] spaces of order \(k\), \(\text{BDM}_k\). In addition, we assume that

(A3) The following inclusion hold for the RT spaces or BDM spaces

\[
\nabla \cdot \mathbf{v}_h \in W_h, \quad \forall \mathbf{v}_h \in V_h.
\]

We shall employ some projection operators. Let \(Q_h\) denote the \(L^2\) projection defined by

\[
(\phi, w_h) = (Q_h \phi, w_h), \quad \forall w_h \in W_h,
\]  

(2.1)
for any \( \phi \in L^2(\Omega) \), and
\[
(\phi, v_h) = (Q_h\phi, v_h), \quad \forall v_h \in V_h, \tag{2.2}
\]
for any vector valued function \( \phi \in (L^2(\Omega))^2 \).

We also assume that the following approximation properties hold. For \( 1 < q \leq \infty \), there exists a positive constant \( C \) independent of \( h \) such that [1]: for all \( \phi \in W^{k+1,q}(\Omega) \) (or \( \phi \in (W^{k+1,q}(\Omega))^2 \))
\[
\|Q_h\phi\|_{0,q} \leq C\|\phi\|_{0,q}, \quad 2 \leq q < \infty, \tag{2.3}
\]
\[
\|\phi - Q_h\phi\|_{0,q} \leq C\|\phi\|_{r,q}h^r, \quad 0 \leq r \leq k+1. \tag{2.4}
\]

We use the well known \( \Pi_h \) projection for mixed finite element approximation spaces. We shall assume that there exists a projection operator \( \Pi_h : (H^1(\Omega))^2 \rightarrow V_h \) such that
\[
(\nabla \cdot \Pi_h q, w_h) = (\nabla \cdot q, w_h), \quad \forall w_h \in W_h. \tag{2.5}
\]

The following approximation properties [1] hold for the projection \( \Pi_h \):
\[
\|q - \Pi_h q\|_{0,q} \leq C\|\phi\|_{r,q}h^r, \quad \frac{1}{q} < r \leq k+1, \tag{2.6}
\]
\[
\|\nabla \cdot (q - \Pi_h q)\|_{0,q} \leq C\|\nabla \cdot q\|_{r,q}h^r, \quad 0 \leq r \leq k+1. \tag{2.7}
\]

We also assume that
\[
\|\phi - Q_h\phi\|_\infty \leq C\|\phi\|_{r,q}h^r, \quad 0 \leq r \leq k+1, \tag{2.8}
\]
\[
\|v_h\|_\infty \leq C\|v_h\|_{\infty}h^{-1}, \quad \forall v_h \in V_h. \tag{2.9}
\]

Now, we recall the discrete Gronwall’s Lemma (see, e.g., [14]):

**Lemma 2.1.** Assume that \( k_n \) is a non-negative sequence, and that the sequence \( q_n \) satisfies
\[
\begin{cases}
q_0 \leq g_0, \\
q_n \leq g_0 + \sum_{s=0}^{n-1} p_s + \sum_{s=0}^{n-1} k_s q_s, \quad n \geq 1.
\end{cases} \tag{2.10}
\]

Then \( q_n \) satisfies
\[
\begin{cases}
q_1 \leq g_0(1 + k_0) + p_0, \\
q_n \leq g_0 \prod_{s=0}^{n-1} (1 + k_s) + \sum_{s=0}^{n-2} p_s \prod_{r=s+1}^{n-1} (1 + k_r) + p_{n-1}, \quad n \geq 2.
\end{cases} \tag{2.11}
\]

Moreover, if \( g_0 \geq 0 \) and \( p_n \geq 0 \) for \( n \geq 0 \), it follows
\[
q_n \leq \left(g_0 + \sum_{s=0}^{n-1} p_s\right) \exp\left(\sum_{s=0}^{n-1} k_s\right), \quad n \geq 1. \tag{2.12}
\]
3 Expanded mixed finite element discretization

First, we set three variables: the pressure \( p \), the gradient \( Y = - \nabla p \), and the flux \( \Psi = K(p)Y \).

We suppose that the following assumption is satisfied.

(A4) The reaction term \( f \) is a sufficiently smooth function with bounded derivatives through the second order on \( \Omega \). For all \((\hat{p}, \hat{Y}) \in W \times \mathbf{V}\) there exists \( F > 0\) such that

\[
f_p(\hat{p}, \hat{Y}) \leq F, \quad f_Y(\hat{p}, \hat{Y}) \leq F.
\]

Now, we define the weak form of the nonlinear parabolic equations (1.1)-(1.3) as follows: Find \((p, Y, \Psi) \in W \times \mathbf{V} \times \mathbf{V}\) such that

\[
\left( \frac{\partial p}{\partial t}, w \right) + (\nabla \cdot \Psi, w) = (f(p, -Y), w), \quad w \in W, \tag{3.1}
\]

\[
(Y, \mathbf{v}) = (p, \nabla \cdot \mathbf{v}), \quad \mathbf{v} \in \mathbf{V}, \tag{3.2}
\]

\[
(\Psi, \mathbf{v}) = (K(p)Y, \mathbf{v}), \quad \mathbf{v} \in \mathbf{V}. \tag{3.3}
\]

The discrete time mixed finite element approximation to (3.1)-(3.3) can be defined as follows: Find \((p^n_h, Y^n_h, \Psi^n_h) \in W_h \times \mathbf{V}_h \times \mathbf{V}_h\) such that

\[
\left( \frac{p^n_h - p^{n-1}_h}{\Delta t}, w_h \right) + (\nabla \cdot \Psi^n_h, w_h) = (f(p^n, -Y^n_h), w_h), \quad w_h \in W_h, \tag{3.4}
\]

\[
(Y^n_h, \mathbf{v}_h) = (p^n_h, \nabla \cdot \mathbf{v}_h), \quad \mathbf{v}_h \in \mathbf{V}_h, \tag{3.5}
\]

\[
(\Psi^n_h, \mathbf{v}_h) = (K(p^n)Y^n_h, \mathbf{v}_h), \quad \mathbf{v}_h \in \mathbf{V}_h. \tag{3.6}
\]

At the time of \( t = t^n \), we rewrite (3.1)-(3.3) in the following form

\[
\left( \frac{\partial p^n}{\partial t}, w_h \right) + (\nabla \cdot \Psi^n, w_h) = (f(p^n, -Y^n), w_h), \quad w_h \in W_h, \tag{3.7}
\]

\[
(Y^n, \mathbf{v}_h) = (p^n, \nabla \cdot \mathbf{v}_h), \quad \mathbf{v}_h \in \mathbf{V}_h, \tag{3.8}
\]

\[
(\Psi^n, \mathbf{v}_h) = (K(p^n)Y^n, \mathbf{v}_h), \quad \mathbf{v}_h \in \mathbf{V}_h. \tag{3.9}
\]

Using the definition of \( \Pi_h \) and \( Q_h \), and the assumption (A3) that \( \nabla \cdot \mathbf{V}_h \subset W_h \), we can obtain

\[
\left( \frac{Q_h p^n - Q_h p^{n-1}}{\Delta t}, w_h \right) + (\nabla \cdot \Pi_h \Psi^n, w_h)
\]

\[
= (f(p^n, -Y^n), w_h) + (\partial_t p^n - p^n_i, w_h), \tag{3.10}
\]

\[
(Q_h Y^n, \mathbf{v}_h) = (Q_h p^n, \nabla \cdot \mathbf{v}_h), \tag{3.11}
\]

\[
(\Pi_h \Psi^n, \mathbf{v}_h) = (K(p^n)Y^n, \mathbf{v}_h) + (\Pi_h \Psi^n - \Psi^n, \mathbf{v}_h). \tag{3.12}
\]

Set

\[
\mu^n = Q_h p^n - p^n_h, \quad \xi^n = Q_h Y^n - Y^n_h, \quad \lambda^n = \Pi_h \Psi^n - \Psi^n_h.
\]
Subtracting (3.10)-(3.12) from (3.4)-(3.6), we derive the error equations:

\[
\left( \frac{\mu^n - \mu^{n-1}}{\Delta t}, w_h \right) + (\nabla \cdot \lambda^n, w_h) = (f(p^n, -Y^n) - f(p_h^n, -Y_h^n), w_h) + (\partial_t p^n - p^n_t, w_h),
\]

(3.13)

\[
(\xi^n, \varphi_h) = (\mu^n, \nabla \cdot \varphi_h),
\]

(3.14)

\[
(\lambda^n, \varphi_h) = (K(p^n)Y^n - K(p_h^n)Y_h^n, \varphi_h) + (\Pi_h Y^n - \Psi^n, \varphi_h).
\]

(3.15)

Letting \( w_h = \mu^n, \varphi_h = \lambda^n, \varphi_h = \xi^n \) in Eqs. (3.13)-(3.15) respectively, we get

\[
\left( \frac{\mu^n - \mu^{n-1}}{\Delta t}, \mu^n \right) + (\nabla \cdot \lambda^n, \mu^n) = (f(p^n, -Y^n) - f(p_h^n, -Y_h^n), \mu^n) + (\partial_t p^n - p^n_t, \mu^n),
\]

(3.16)

\[
(\xi^n, \lambda^n) = (\mu^n, \nabla \cdot \lambda^n),
\]

(3.17)

\[
(\lambda^n, \xi^n) = (K(p^n)Y^n - Q_h Y^n, \xi^n) - ((K(p^n) - K(p^n_h))Y^n, \xi^n)
+ (K(p_h^n)\xi^n, \xi^n) + (\Pi_h Y^n - \Psi^n, \xi^n).
\]

(3.18)

Noticing that

\[
- \frac{1}{2} \left( \|\mu^{n-1}\|^2 + \|\mu^n\|^2 \right) \leq - (\mu^{n-1}, \mu^n),
\]

(3.19)

\[
(K(p_h^n)\xi^n, \xi^n) = \|K(p_h^n)\|^2 \xi^n.\]

(3.20)

We have

\[
(\partial_t \mu^n, \mu^n) + (K(p_h^n)\xi^n, \xi^n) \geq \frac{1}{2\Delta t} \left( \|\mu^n\|^2 - \|\mu^{n-1}\|^2 \right) + \|K(p_h^n)\|^2 \xi^n.\]

(3.21)

Applying the Taylor expansions, we have

\[
f(p^n, -Y^n) - f(p_h^n, -Y_h^n) = f_p(\bar{\rho}, \bar{Y})(p^n - p_h^n) - f_Y(\bar{\rho}, \bar{Y})(Y^n - Y_h^n).
\]

Then,

\[
(f(p^n, -Y^n) - f(p_h^n, -Y_h^n), \mu^n) \leq (F(p^n - Q_h p^n), \mu^n) + (F\mu^n, \mu^n) - (F(Y^n - Q_h Y^n), \mu^n) - (F\xi^n, \mu^n).
\]

(3.22)

Moreover,

\[
|\partial_t p^n - p^n_t, \mu^n| = \left| \left( \int (t - t^{n-1}) p_{ht}(\cdot, s) ds, \mu^n \right) \right| \leq \Delta t \int_{t^{n-1}}^{t^n} \|p_{ht}(\cdot, s)\| ds \|\mu^n\| \leq (\Delta t)^2 \left( \int_{t^{n-1}}^{t^n} \left\| \frac{\partial^2 p}{\partial t^2}(\cdot, s) \right\| \right) + \|\mu^n\|^2,
\]

(3.23)
and
\[
((K(p^n_h) - K(p^n))Y^n, \xi^n) \leq K^* \| Y^n \|_{\infty} \| p^n_h - p^n \| \cdot \| \xi^n \|
\]
\[
\leq CK^* (\| p^n_h - Q_h p^n \| + \| Q_h p^n - p^n \|) \| \xi^n \|
\]
\[
= CK^* (\| \mu^n \| + \| Q_h p^n - p^n \|) \| \xi^n \|
\]
\[
\leq C (\| \mu^n \|^2 + \| Q_h p^n - p^n \|^2) + \delta \| \xi^n \|^2,
\]
(3.24)

where we have used the assumptions (A1)-(A2). From (3.16)-(3.24), applying the Hölder's inequality gives
\[
\frac{1}{2\Delta t} (\| \mu^n \|^2 - \| \mu^{n-1} \|^2) + \| K(p^n_h)^{\frac{1}{2}} \xi^n \|^2
\]
\[
\leq \left( \frac{\mu^n - \mu^{n-1}}{\Delta t}, \mu^n \right) + \| K(p^n_h)^{\frac{1}{2}} \xi^n \|^2
\]
\[
\leq (\partial_t p^n - p^n, \mu^n) + (\Pi_h \Psi^n - \Psi^n, \xi^n) + (F(p^n - Q_h p^n), \mu^n)
\]
\[
+ (F\mu^n, \mu^n) - (F(Y^n - Q_h Y^n), \mu^n) - (F\xi^n, \mu^n)
\]
\[
+ (K(p^n_h)(Y^n - Q_h Y^n), \xi^n) + ((K(p^n_h) - K(p^n))Y^n, \xi^n)
\]
\[
\leq C\| \mu^n \|^2 + C\Delta t^2 + \delta\| \xi^n \|^2 + C\| \Pi_h \Psi^n - \Psi^n \|^2
\]
\[
+ \| Y^n - Q_h Y^n \|^2 + \| p^n - Q_h p^n \|^2.
\]
(3.25)

We next multiply both sides of (3.25) by $2\Delta t$ and sum over $n$, $n = 1, \cdots, N$, applying the discrete Gronwall's Lemma, to obtain
\[
\| \mu^N \|^2 + \sum_{n=1}^N \Delta t \| K(p^n_h)^{\frac{1}{2}} \xi^n \|^2
\]
\[
\leq C \sum_{n=1}^N \Delta t \left[ \| \Pi_h \Psi^n - \Psi^n \|^2 + \| Y^n - Q_h Y^n \|^2
\]
\[
+ \| p^n - Q_h p^n \|^2 \right] + \| \mu^0 \|^2 + C(\Delta t)^2.
\]

With a proper choice of the initial function $p^0_h = Q_h p^0$, we have $\mu^0 = 0$, combining (2.4) and (2.6), we get
\[
\| \mu^N \|^2 + \sum_{n=1}^N \Delta t \| K(p^n_h)^{\frac{1}{2}} \xi^n \|^2 \leq C \left( (\Delta t)^2 + h^{2k+2} \right).
\]

Then, we have
\[
\| \mu^N \| + \left( \sum_{n=1}^N \Delta t \| K(p^n_h)^{\frac{1}{2}} \xi^n \|^2 \right)^{\frac{1}{2}} \leq C \left( \Delta t + h^{k+1} \right),
\]
(3.26)

Applying the triangle inequality, we have the following result.
Theorem 3.1. Suppose that assumptions (A1)-(A4) are valid. Let \((p^n_H, \Psi^n_H, \varPsi^n_H) \in W_H \times V_H \times \tilde{V}_H\) be the solution of the mixed finite element equations (3.4)-(3.6) for \(n \geq 1\). If we choose the initial function

\[ p^0_H = Q_hp_0, \]

for \(1 \leq M \leq N\) and \(k \geq 1\), then there exists a positive constant \(C\), independent of \(h\) such that

\[
\|p^M - p^M_H\| + \left( \sum_{n=1}^{M} \Delta t \| K(p^n_H) \|^1/2 \right)^{1/2} + \left( \sum_{n=1}^{M} \Delta t \| \Psi^n - \Psi^n_H \|^2 \right)^{1/2} \leq C \left( \Delta t + h^{k+1} \right).
\]

\[ (3.27) \]

4 Two-grid method and error estimates

Note that the two-grid algorithms used in Wu and Allen [15] involve only one Newton iteration on the fine grid. Nevertheless, some more interesting results can be derived if one more Newton iteration is carried out on the fine grid. The idea is from Xu’s work [16]. Thus, based on the works of Dawson [7], we can construct an algorithm of the two-grid method for Eqs. (1.1)-(1.3) discretized by expanded mixed finite element methods.

To iteratively solve the nonlinear system (3.4)-(3.6), we find \((\tilde{p}^n_H, \tilde{\Psi}^n_H, \tilde{\varPsi}^n_H) \in W_H \times V_H \times \tilde{V}_H\) in three steps as follows:

**Step 1**: On the coarse grid \(\Gamma_H\), compute \((p^n_H, \Psi^n_H, \varPsi^n_H) \in W_H \times V_H \times \tilde{V}_H\) to satisfy the following original nonlinear system:

\[
\begin{align*}
\left( \frac{p^n_H - p^{n-1}_H}{\Delta t}, w_H \right) + (\nabla \cdot \Psi^n_H, w_H) &= (f^n, w_H), \quad w_H \in W_H, \\
(\Psi^n_H, \mathbf{v}_H) - (p^n_H, \nabla \cdot \mathbf{v}_H) &= 0, \quad \mathbf{v}_H \in \tilde{V}_H, \\
(\varPsi^n_H, \mathbf{v}_H) &= (K(p^n_H) \Psi^n_H, \mathbf{v}_H), \quad \mathbf{v}_H \in V_H.
\end{align*}
\]

\[ (4.1)-(4.3) \]

**Step 2**: On the fine grid \(\Gamma_h\), compute \((\tilde{p}^n_H, \tilde{\Psi}^n_H, \tilde{\varPsi}^n_H) \in W_h \times V_h \times \tilde{V}_h\) to satisfy the following linear system:

\[
\begin{align*}
\left( \frac{\tilde{p}^n_H - \tilde{p}^{n-1}_H}{\Delta t}, w_h \right) + (\nabla \cdot \tilde{\Psi}^n_H, w_h) &= (f^n, w_h), \quad w_h \in W_h, \\
(\tilde{\Psi}^n_H, \mathbf{v}_h) - (\tilde{p}^n_H, \nabla \cdot \mathbf{v}_h) &= 0, \quad \mathbf{v}_h \in \tilde{V}_h, \\
(\tilde{\varPsi}^n_H, \mathbf{v}_h) &= (K'(p^n_H) \tilde{\Psi}^n_H, \mathbf{v}_h + (\tilde{K}(p^n_H) \Psi^n_H, \mathbf{v}_h), \quad \mathbf{v}_h \in V_h.
\end{align*}
\]

\[ (4.4)-(4.6) \]
Step 3: On the fine grid $\Gamma^h$, compute \((\tilde{p}^n_h, \tilde{y}^n_h, \tilde{\psi}^n_h) \in W_h \times V_h \times V_h\) to satisfy the following linear system:

\[
\left(\frac{\tilde{p}^n_h - \tilde{p}^{n-1}_h}{\Delta t}, w_h\right) + (\nabla : \tilde{\Psi}^n_h, w_h) = (f^n_h, w_h), \quad w_h \in W_h, \tag{4.7}
\]

\[
(\tilde{y}^n_h, v_h) - (\tilde{p}^n_h, \nabla \cdot v_h) = 0, \quad v_h \in V_h, \tag{4.8}
\]

\[
(\tilde{\psi}^n_h, v_h) = (K(p^n_h)\tilde{y}^n_h, v_h) + (K'(p^n_h)\tilde{y}^n_h(p^n_h - \tilde{p}^n_h), v_h), \quad v_h \in V_h. \tag{4.9}
\]

Remark 4.1. Eq. (4.9) is motivated by the Taylor expansion

\[
K(p^n_h)\tilde{y}^n_h = K(p^n_h)\tilde{y}^n_h + K'(p^n_h)\tilde{y}^n_h(p^n_h - \tilde{p}^n_h) + \frac{K''(x^n)}{2}(\tilde{y}^n_h(p^n_h - \tilde{p}^n_h))^2, \tag{4.10}
\]

for some $x^n$ between $\tilde{p}^n_h$ and $\tilde{p}^n_h$.

Lemma 4.1. Suppose that assumptions (A1)-(A4) are fulfilled. Let \((\tilde{p}^n_h, \tilde{y}^n_h, \tilde{\psi}^n_h) \in W_h \times V_h \times V_h\) be the solution of the mixed finite element equations (4.1)-(4.6) for $n \geq 1$. If we choose the initial function

\[
\tilde{p}^0_h = Q_hp_0,
\]

for $1 \leq M \leq N$ and $k \geq 1$, then there exists a positive constant $C$, independent of $h$ such that

\[
\|p^M - \tilde{p}^M_h\| + \left(\sum_{n=1}^{M} \Delta t \|K(p^n_h)\|^{1/2} \|y^n - \tilde{y}^n_h\|^{2}\right)^{1/2} \leq C(\Delta t^k + H^{2k+1}).
\]

The proof can be found in [7].

Moreover, by an argument similar to that in Lemma 4.1, we can also prove the following result:

Remark 4.2. If we choose the initial function $\tilde{p}^0_h = Q_hp_0$, then for $1 \leq n \leq N$, $2 \leq q \leq \infty$, and $k \geq 1$, we have

\[
\|p^n - \tilde{p}^n_h\|_{0,q} \leq C(\Delta t^k + H^{2k+1}). \tag{4.11}
\]

Now, we can prove the following main theorem.

Theorem 4.1. Suppose that assumptions (A1)-(A4) are fulfilled. Let \((\tilde{p}^n_h, \tilde{y}^n_h, \tilde{\psi}^n_h) \in W_h \times V_h \times V_h\) be the solution of the mixed finite element equations (4.1)-(4.9) for $n \geq 1$. If we choose the initial function

\[
\tilde{p}^0_h = Q_hp_0, \tag{4.12}
\]

then for $1 \leq M \leq N$ and $k \geq 1$, we have

\[
\|p^M - \tilde{p}^M_h\| + \left(\sum_{n=1}^{M} \Delta t \|K(p^n_h)\|^{1/2} \|y^n - \tilde{y}^n_h\|^{2}\right)^{1/2} \leq C(\Delta t^k + H^{2k+1}). \tag{4.13}
\]
Proof. We now derive an estimate for this two-grid scheme. We set

\[ \alpha^n = Q_h p^n - \tilde{P}_h^n, \quad \beta^n = Q_h Y^n - \tilde{Y}_h^n, \quad \gamma^n = \Pi_h Y^n - \tilde{Y}_h^n. \]

Using (3.10)-(3.12), (4.7)-(4.9), and the definition of projection \( Q_h, \Pi_h \), we get the error equations:

\[
\begin{align*}
\frac{\alpha^n - \alpha^{n-1}}{\Delta t}, w_h) + (\nabla \cdot (\Pi_h \Psi^n - \tilde{\Psi}_h^n), w_h) &= (E_p, w_h), \quad w_h \in W_h, \quad (4.14) \\
(Q_h Y^n - \tilde{Y}_h^n, v_h) - (Q_h p^n - \tilde{P}_h^n, \nabla \cdot v_h) &= 0, \quad v_h \in V_h, \quad (4.15)
\end{align*}
\]

and

\begin{align*}
&\langle \Pi_h \Psi^n - \tilde{\Psi}_h^n, v_h \rangle \\
&= (\Pi_h \Psi^n - \Psi^n, v_h) - (K(p^n)(Q_h Y^n - Y^n), v_h) + (K(\tilde{P}_h^n)(Q_h Y^n - \tilde{Y}_h^n), v_h) \\
&+ \left( (\alpha^n + (p^n - Q_h p^n))K'(\tilde{P}_h^n)\tilde{Y}_h^n, v_h \right) + \left( K'(\tilde{P}_h^n)(p^n - \tilde{P}_h^n)(Q_h Y^n - \tilde{Y}_h^n), v_h \right) \\
&+ \frac{1}{2} \left( (p^n - \tilde{P}_h^n)^2 K'(x^n)Q_h Y^n, v_h \right), \quad v_h \in V_h, \quad (4.16)
\end{align*}

for some \( x^n \) between \( \tilde{P}_h^n \) and \( P_h^n \) and \( E_p = \partial_t p^n - \tilde{P}_h^n \).

Choosing \( w_h = \alpha^n \) in (4.14), \( v_h = \gamma^n \) in (4.15), \( v_h = \beta^n \) in (4.16), adding (4.14) and (4.15), and subtracting (4.16), yield

\[
\begin{align*}
\left( \frac{\alpha^n - \alpha^{n-1}}{\Delta t}, \alpha^n \right) + \left( K(\tilde{P}_h^n)\beta^n, \beta^n \right) \\
= (E_p, \alpha^n) - \left( \Pi_h \Psi^n - \Psi^n, \beta^n \right) + \left( K(p^n)(Q_h Y^n - Y^n), \beta^n \right) \\
- \left( (\alpha^n + (p^n - Q_h p^n))K'(\tilde{P}_h^n)\tilde{Y}_h^n, \beta^n \right) - \left( K'(\tilde{P}_h^n)(p^n - \tilde{P}_h^n)(Q_h Y^n - \tilde{Y}_h^n), \beta^n \right) \\
- \frac{1}{2} \left( (p^n - \tilde{P}_h^n)^2 K'(x^n)Q_h Y^n, \beta^n \right), \quad (4.17)
\end{align*}
\]

for some \( x^n \) between \( \tilde{P}_h^n \) and \( P_h^n \). Note that

\[
\begin{align*}
&\left( (p^n - \tilde{P}_h^n)^2 K'(x^n)Q_h Y^n, \beta^n \right) \\
= &\left[ (\tilde{P}_h^n - p^n)^2 - 2(\tilde{P}_h^n - p^n)(p^n - \tilde{P}_h^n) - (p^n - \tilde{P}_h^n)^2 \right] \cdot K''(x^n)(Q_h Y^n - Y^n) \\
+ &Y^n - Y^n, \beta^n \right) + \left( (p^n - \tilde{P}_h^n)^2 K''(x^n)(Y^n - Y^n + Y^n), \beta^n \right), \quad (4.18)
\end{align*}
\]

and using the Hölder’s inequality, we get

\[
\left( (p^n - \tilde{P}_h^n)^2 K''(x^n)Y^n, \beta^n \right) \leq C\|p^n - \tilde{P}_h^n\|_{\partial A}^2 \|\beta^n\|. \quad (4.19)
\]
Using the Hölder’s inequality and (3.23), multiplying by $\triangle t$ and summing $n = 1$ to $M$, $M \leq N$, using the approximation properties, we see that

$$
\|\alpha^M\|^2 - \|\alpha^0\|^2 + \sum_{n=1}^{M} \triangle t \|K(\tilde{p}_h^n)\|^2
\leq C(\triangle t)^2 + \delta \sum_{n=1}^{M} \triangle t \|\beta^n\|^2 + C \left[ \sum_{n=1}^{M} \triangle t \|\alpha^n\|^2 + h^{2k+2} \right]
+ C \sum_{n=1}^{M} \triangle t \left[ \| (p^n - Q_h p^n) \tilde{Y}_h^n \|^2 + \| (p^n - \tilde{p}_h^n) (Q_h Y^n - \tilde{Y}_h^n) \|^2 + T_1 \right]
\equiv C(\triangle t)^2 + \delta \sum_{n=1}^{M} \triangle t \|\beta^n\|^2 + C \left[ \sum_{n=1}^{M} \triangle t \|\alpha^n\|^2 + h^{2k+2} + T_2 \right], \quad (4.20)
$$

where

$$
T_1 = \| p_h^n - \tilde{p}_h^n \|_{0,4}^2 + \| p_h^n - \tilde{p}_h^n \|^2 \| Q_h Y^n - Y^n \|_{\infty}^2
+ \| p^n - \tilde{p}_h^n \|^2 \| Q_h Y^n - Y^n \|_{\infty}^2
+ \| p^n - \tilde{p}_h^n \|^2 \| Q_h Y^n - Y^n \|_{\infty}^2
+ \| p^n - \tilde{p}_h^n \|^2 \| Q_h Y^n - Y^n \|_{\infty}^2
+ \| p^n - \tilde{p}_h^n \|^2 \| Q_h Y^n - Y^n \|_{\infty}^2
+ \| p^n - \tilde{p}_h^n \|^2 \| Q_h Y^n - Y^n \|_{\infty}^2
+ \| p^n - \tilde{p}_h^n \|_{\infty}^2 \| Q_h Y^n - Y^n \|_{\infty}^2
+ \| p^n - \tilde{p}_h^n \|^2 \| Q_h Y^n - Y^n \|_{\infty}^2
\leq h^{-2} (h^{k+1} + H^{2k+1} + \triangle t)^4 h^{2k+2}
+ h^{-2} (h^{k+1} + H^{2k+1} + \triangle t)^2 (h^{k+1} + h^{2k+1} + \triangle t)^2 h^{2k+2}
+ h^{-2} (h^{k+1} + H^{2k+1} + \triangle t)^4 \| Y^n - Y^n \|_{\infty}^2 + (h^{k+1} + H^{2k+1} + \triangle t)^4. \quad (4.21)
$$

Using (4.11) and the Hölder’s inequality, we have

$$
T_2 = C \sum_{n=1}^{M} \triangle t \left[ \| (p^n - Q_h p^n) \tilde{Y}_h^n \|^2 + \| (p^n - \tilde{p}_h^n) (Q_h Y^n - \tilde{Y}_h^n) \|^2 + T_1 \right]. \quad (4.22)
$$

Combining (4.22)-(4.23), Lemma 4.1 and Theorem 3.1, we can obtain

$$
T_2 = C \sum_{n=1}^{M} \triangle t \left[ \| (p^n - Q_h p^n) \tilde{Y}_h^n \|^2 + \| (p^n - \tilde{p}_h^n) (Q_h Y^n - \tilde{Y}_h^n) \|^2 + T_1 \right]
\leq C \sum_{n=1}^{M} \triangle t \left[ h^{2k+2} + h^{-2} (h^{k+1} + H^{2k+1} + \triangle t)^4 h^{2k+2}
+ h^{-2} (h^{k+1} + H^{2k+1} + \triangle t)^2 (h^{k+1} + h^{2k+1} + \triangle t)^2 h^{2k+2}
+ (h^{k+1} + H^{2k+1} + \triangle t)^4 \right].
$$
\[ + h^{-2}(h^{k+1} + H^{2k+1} + \triangle t)^4 \sum_{n=1}^{M} \triangle t \| Y^n - Y_H^n \|_\infty^2. \]

From (2.4), (2.8)-(2.9), and Theorem 3.1, we have

\[ \sum_{n=1}^{M} \triangle t \| Y^n - Y_H^n \|_\infty^2 \leq C \sum_{n=1}^{M} \triangle t (\| Y^n - Q_H Y^n \|_\infty^2 + \| Q_H Y^n - Y_H^n \|_\infty^2) \]
\[ \leq C \sum_{n=1}^{M} \triangle t H^{2k+2} + C \sum_{n=1}^{M} \triangle t (H^{-1} \| Q_H Y^n - Y_H^n \|)^2 \]
\[ \leq C H^{-2} (H^{k+1} + \triangle t)^2, \] (4.24)

and
\[ \| Y^n - Y_H^n \|_\infty \leq \| Y^n - Q_H Y^n \|_\infty + \| Q_H Y^n - Y_H^n \|_\infty \]
\[ \leq H^{-1} (H^{k+1} + \triangle t). \] (4.25)

Then, we obtain
\[ T_2 \leq C \left( h^{2k+2} + H^{8k+2} + H^{10k+4} h^{-2} + (\triangle t)^2 \right), \] (4.26)
where \( h \leq H^{k+1} \), we have
\[ T_2 \leq C \left( h^{2k+2} + H^{8k+2} + (\triangle t)^2 \right). \] (4.27)

Noticing that \( a^0 = 0 \), and applying the discrete Gronwall’s Lemma, we deduce that
\[ \| a^M \| + \left( \sum_{n=1}^{M} \triangle t \| K(\bar{a}_H^n)^{1/2} b^n \| \right)^{1/2} \leq C (h^{k+1} + H^{4k+1} + \triangle t). \] (4.28)

Applying the triangle inequality and the approximation properties, we can prove the results of the Theorem.

5 Conclusion and future works

In this paper, we have presented a new two-grid scheme for expanded mixed finite element solution of nonlinear parabolic equations. The theory demonstrates a remarkable fact about the two-grid scheme: we can iterate on a very coarse grid \( \Gamma_H \) and still get good approximations by taking one iteration on the fine grid \( \Gamma_h \). We have derived
an error estimate for the algorithms of the two-grid method. It is shown that the algorithm achieve asymptotically optimal approximation as long as the mesh sizes satisfy

\[ h = \mathcal{O}(H^{\frac{d+1}{k+1}}). \]

The next step of our work is to carry out some the numerical experiment with the proposed two-grid scheme. Furthermore, we shall consider the two-grid schemes for expanded mixed finite element solution of nonlinear hyperbolic equations.

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